Introduction to Algebraic Topology (Part - II)
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Lecture - 4
More Examples

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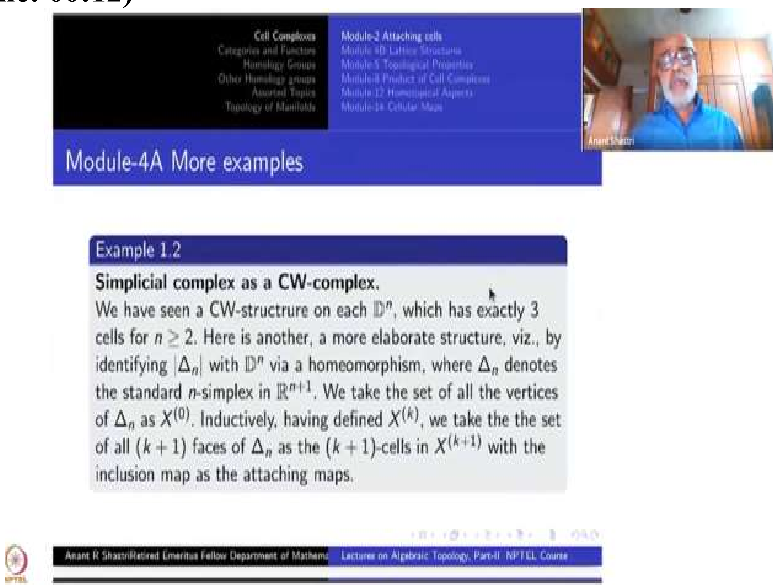

Last time we introduced the notion of relative CW complexes and CW complexes and checked a lot of examples, the standard examples, such as spheres, discs, and then later on projective spaces, both real and complex. We would like to study many more examples. So, in following two more modules which are denoted by module 4A and 4B we will only study more and more examples, okay? For example, $\mathbb{D}^n$ itself you know can be given several different CW-structures each of them can be used in different contexts.

One such thing is thinking $\mathbb{D}^n$ as the underlying space mod Δ_n, where Δ_n is a n-simplex in $\mathbb{R}^{n+1}$. Remember the n-simplex in $\mathbb{R}^{n+1}$ is convex hull of the standard basic elements $e_1, e_2, \ldots, e_{n+1}$. The endpoints of these vectors will be taken as the vertices, okay? Inductively having defined the k-skeleton $X^{(k)}$, we take the set of all $(k+1)$-faces of Δ_n as the set of $(k+1)$-cells in $X^{(k+1)}$ with the inclusion map as attaching maps.

The point is we are not building up the space here but that a space is already there and we are decomposing the space in to CW-complex, into a disjoint union of open cells. 0-cells, they are open simultaneously open as well closed cells, then 1-cells, with their boundaries inside of the union of all 0-cells, the 0-skeleton, then the 2-cells their boundaries inside the 1-skeleton and so on. Okay? So, this phenomenon, we will keep observing.

So, this happens to be a very special case, namely, of simplicial complexes. For any simplicial complex there is an associated CW-complex. The simplicial complex itself can be thought of as a CW-complex, namely, the 0-skeleton is the set of all vertices of the simplicial complex then, the one simplexes become 1-cells, etc. Only you are changing the terminology here that is all. Attaching maps are what? Attaching maps as well as the characteristic maps are the inclusion maps. Okay.

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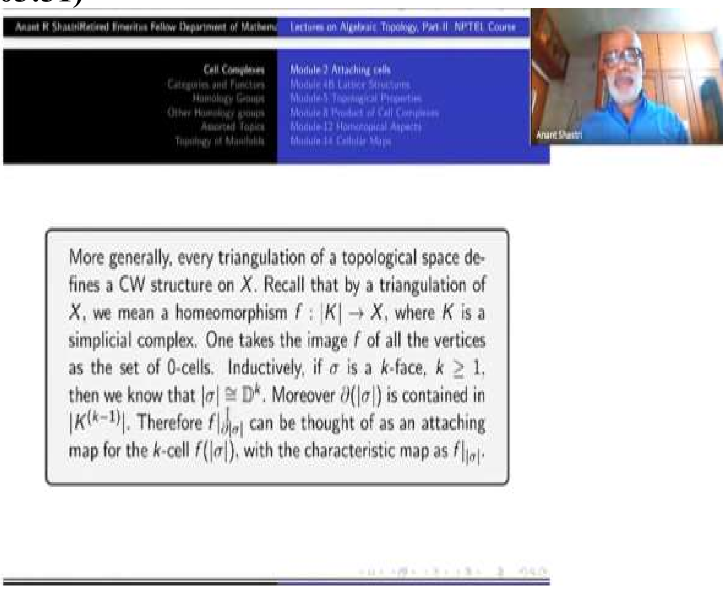

So, this is what we get. Take any triangulated space, say X is a triangulated space. Automatically, it acquires CW-structure. Okay? So, here all the characteristic maps are just the inclusion maps because we have already built up the space we don't have to build a space here. The inclusion maps happened to be injective, homeomorphisms onto their image, you may say embeddings. These are very special kind of CW-complexes okay?

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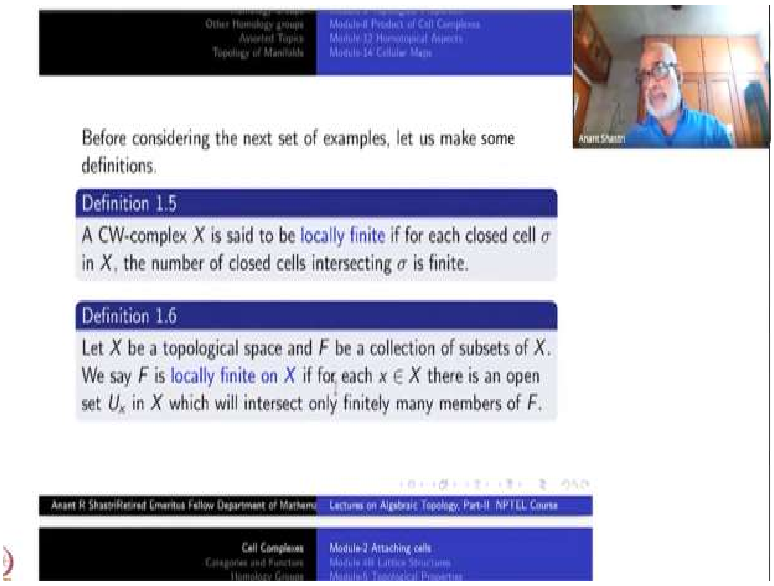

Before considering the next set of examples, let us make a few more definitions so that we will become familiar with the definitions also. A CW-complex X is said to be locally finite, if for each closed cell σ (closed cell is just the closure of a cell), the number of closed cells intersecting σ in X, that must be finite, Okay?

This condition implies quite a bit. For example, take a vertex, look at all 1-cells, which may intersect that vertex, okay, they must be finite many. Not only that, all 2-cells, 3-cells and so on in X, the codomains of whose attaching maps contain that point must be finitely many. Okay? So, this should happen for every closed cell okay, the closure of a cell should intersect only finite many other cells.

Here is a more general definition. Let X be topological space and F be a collection of subsets of X. We say F is locally finite on X, if for each $x \in X$, there is an open set U_x in X which will contain the point x and which would intersect only finitely many members of F. Okay? So, for each x, there is a neighbourhood U_x of x which will intersect only finitely many members of F. Okay?

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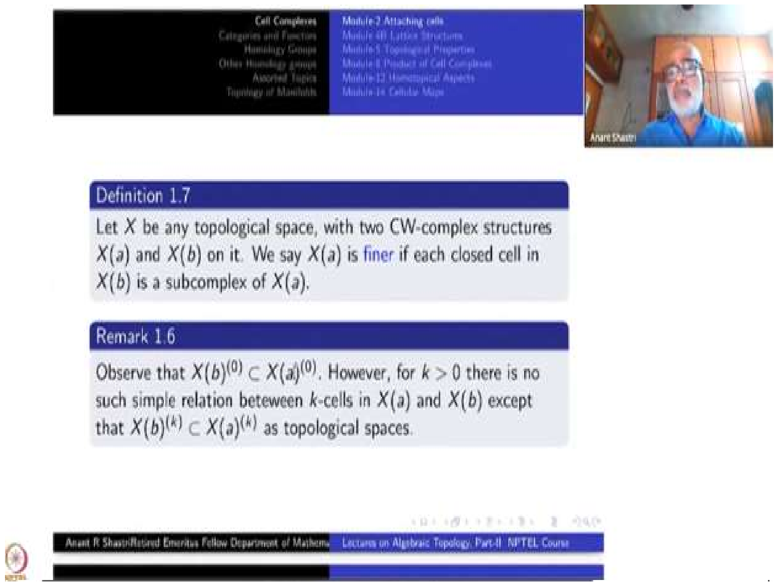

Let X be any topological space with two different CW structures on it. I will denote them by $X(a)$ and $X(b)$, just temporary notation just to distinguish between the two of them. We say $X(a)$ is finer than $X(b)$ if each closed cell in $X(b)$ is a subcomplex of $X(a)$. A subcomplex may have different structure than the just being a cell. Okay? If you take a closed cell along with all its faces then only it will become a sub complex in $X(a)$. That is the definition $X(a)$ is finer than $X(b)$. So, if you have each closed cell in $X(b)$ is a subcomplex of $X(a)$, okay?

For example, this will imply that all 0-cells of $X(b)$ must be 0-cells in $X(a)$ also, there is no other way that a 0-cell will be a subcomplex. So, all the 0-cells must be contained inside the 0 -skeleton of $X(a)$.

But for k positive, there is no such simple relation between the k-cells in $X(a)$ and k-cells in $X(b)$, except that if you take the totality of all r-cells for $r \leq k$ of $X(b)$, that means the k^{th}-skeleton of $X(b)$ must be contained inside k^{th}-skeleton of $X(a)$ just as a topological space okay. So, this is the consequence of this definition, viz., one is finer than the other okay. So, you can think about these definitions. Think about some typical and easy examples and so on okay?

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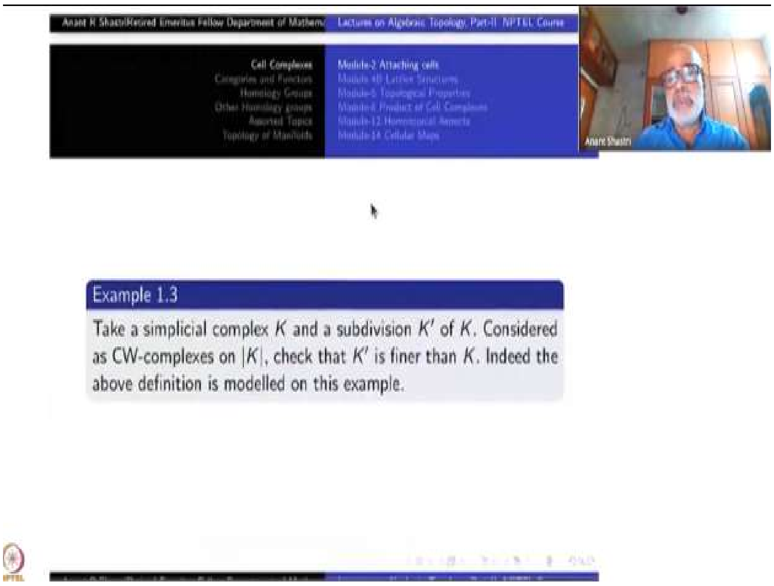

So, let us now take a more general example first and then just add a typical example. Take a simplicial complex K and a subdivision K' of K. We may consider them as CW complexes on the same underlying space $|K|$ because $|K|' = |K|$. Now check that K' is finer than K, okay. So, to figure out this, it will require that you know simplicial complexes and subdivsion well. okay? So, but that is what I am assuming in this course anyway. Indeed, the above definition is modelled on this example. Instead of calling $X(a)$ a subdivision of $X(b)$ which will be too much I am calling it a finer CW-structure, okay. So, just extracting certain properties of subdivision to call this as something is finer than that so that we can compare the two of them okay.

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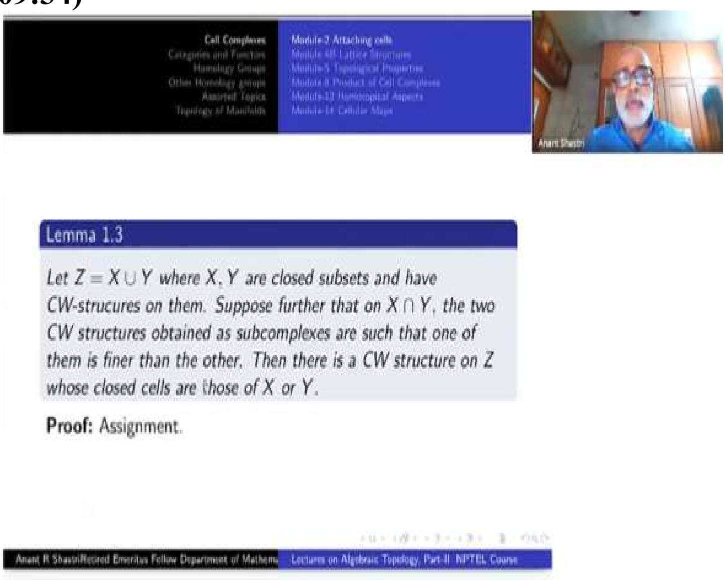

So, here is a lemma which will ensure how to patch up two different CW structures, on the union of two CW complexes. Suppose Z is the union of two topological subspaces X and Y, both of them are closed subsets of Z and have CW structures on them okay? Out of this data, we want to have a CW structure on Z. In order to achieve this, we have to assume one more condition. Suppose that on the intersection $X \cap Y$, the two CW structures obtained as subcomplexes from X and Y respectively, are such that one of them is finer than the other.

Note that on $X \cap Y$, there are two structures which are subcomplexes of X and Y respectively, you may call this one A and other one B, and one of them should be finer than the other. So, this is the condition I am assuming on the intersection. Then, there is a CW structure on Z whose closed cells are those of X or Y. You do not have to do any more work to get Z, all those cells needed are in X or in Y (or both).

So, this I am putting it as an assignment just because I want you to participate in this. So, that if you start thinking and working on them, these things will become more familiar to you. Okay?

So, in this lemma, I have already used the definition that I have made here okay. So, while proving that lemma you will automatically become more familiar with the definition. Okay?

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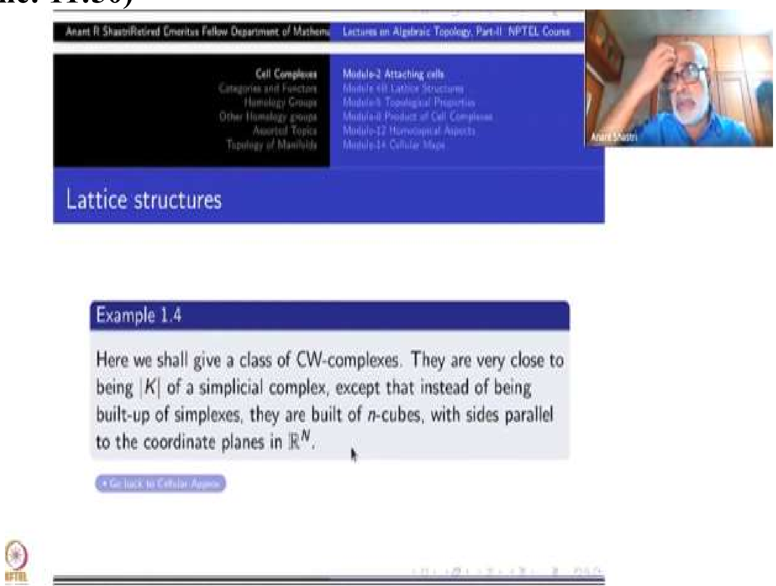

Now, I come to a class of structures, which are very nice CW-structures, which fall short of being a simplicial complex is just a little bit okay. So, here are those examples they are very close to being a simplicial complex, except that, instead of being built up of triangles and tetrahedrons and so on, we use squares and cubes, cubes of higher and higher dimensions and so on. Only in 0 dimension and 1 dimension, the simplicial as well as cells coincide. Okay. So all these n-cubes are inside $\mathbb{R}^n$, everything is happening inside $\mathbb{R}^n$, n-cubes with their sides parallel to the coordinate planes. Okay? So these things will be very useful in analysis all the time. You cut $\mathbb{R}^n$ into smaller and smaller subdivisions. And that is precisely what we are going to do here and then put them to produce CW-complexes.

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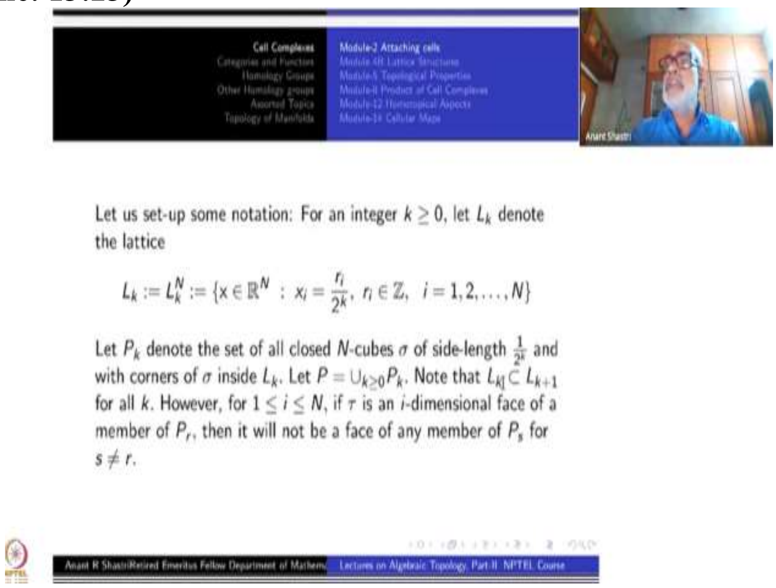

So, I will start with these subdivisions, namely, what are called lattice points. Fixing integer $N > 0$ and I am working in $\mathbb{R}^N$. In the notations below, sometimes I am writing that N but whenever there is confusion I may not write it. Now for each non negative integer k, I am writing L_k^N or just L_k. This is the notation for the set of all points x in $\mathbb{R}^N$ all of whose coordinates look like some integer $r_i/2^k$; this k is fixed here.

So, the same k should come here; 2^k. Okay? The numerator is an integer denominator is 2^k, Okay? For example, it may be 2^0, which means denominator is 1. So, the integer coordinates are now allowed, then the denominator could be 2^1, etc. Next, for example something like $2/4, 3/8, 6/8$ is same thing as $3/4$ all those points are allowed here okay. So, those are the coordinates of these points x.

Let now P_k denotes the set of all closed N-dimensional cubes. So, generally I am denoting them by σ. All N-cells have side length $1/2^k$. Remember that k is fixed here, each cube σ has its corners of inside the lattice L_k, okay? So, that is P_k:

Take all the P_k for all $k \geq 0$, the union will be denoted by P. This is just a notation again. and again we will have to use this notation I am setting it up that is all, what you want to remember is that L_k consists of only points with coordinates and coordinates are like this, this is what we call lattice.

Note that each L_k is contained is L_{k+1} contained L_{k+2} and so on. okay? However, for i between 1 and N, if τ is an i-dimensional face of one of one cubes (you know face means what? You take a square, the square has all the sides and vertices as its faces, a 3-cube has 2-faces that are 6 in number, 1-faces that are 12 in number and 0-faces that 8 in number.) An i-dimensional face of a member of P_r, start with P_r, take any face of dimension between 1 and N.

It will not be a face of any other member of P_s, where $s \neq r$. Because $s \neq r$ just means that its size, the side length, volume etc, will differ by multiplication factor of some n_0 power of 2. Only in case of vertices, by subdivisions, more and more vertices will come and the old vertices still remain there. for example the origin $(0, 0, 0, \ldots, 0)$ is in P_r for all r. Similarly vertices with integer coordinates are there in are in all of them, That is what you have to remember. Okay?

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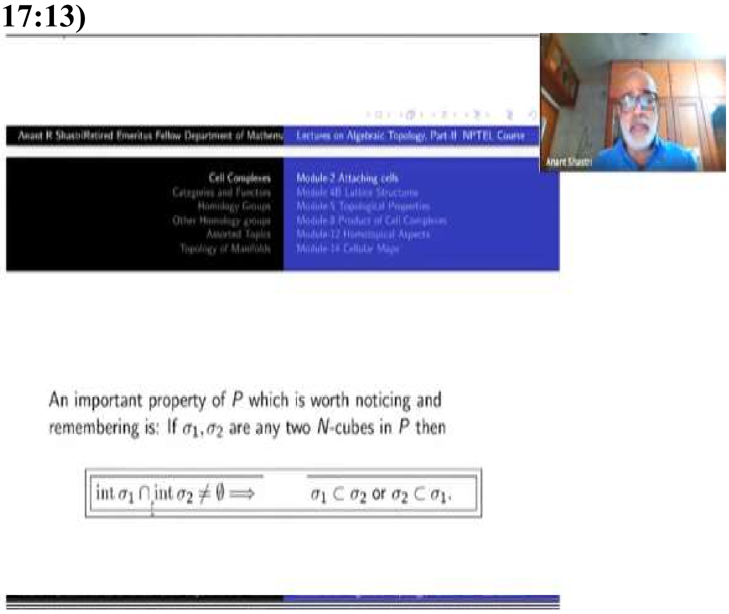

An important property of P worth noticing and remembering is the following. Take two cells σ_1 and σ_2, two N-cubes. I am taking then in P, that means their sizes could be different, okay? Then the interior of σ intersection interior of τ is non empty would imply one of them is contained entirely in the other. This is what I am saying.

So, first of all, if both of them are in P_r for the same r, then of course it is easy to see the interiors of σ_1 and σ_2 are disjoint, Okay? So, this case does not occur. So the intersection is non empty means that σ_1 and σ_2 are in P_r and P_s for $r \neq s$. Say $r = 1$ and $s = 2$ or 5 and so on. Then it can happen only this way it can happen okay? The intersection is non-empty would imply σ_1 contains σ_2 or σ_2 contains σ_1, okay?

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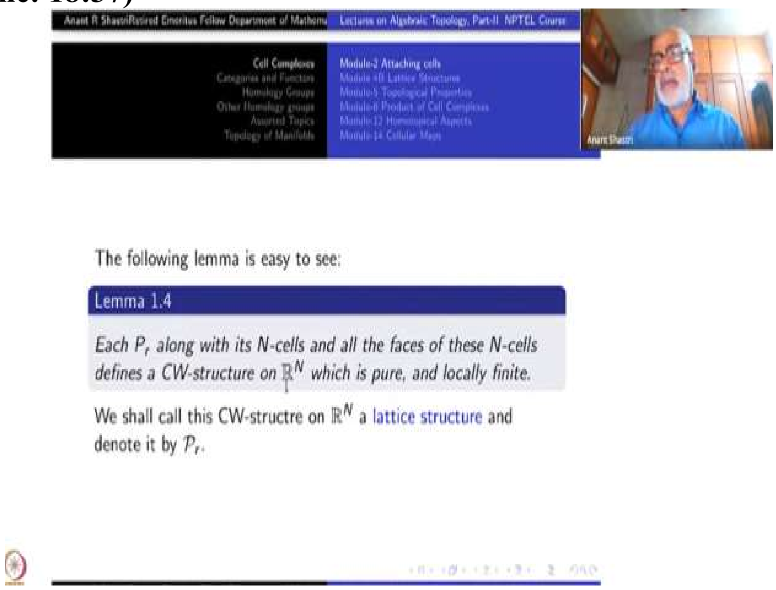

Now, following lemmas is easy to see. Each P_r (here r is fixed) along with its N-cells and all the faces of these N-cells defines a CW-structure on $\mathbb{R}^N$ which is pure and locally finite. Okay?

The local finiteness is obvious because around a point in P in inside $\mathbb{R}^N$, how many edges you will have? How many edges will be incident? How many squares will be incident there? You just think about that. The same picture at every other point. These numbers just depend upon N only and not r. For instance, the number of edges will be $2N$.

To see the CW decomposition all that we have to use is the fact that interiors of all the i-cells are disjoint and they cover the entire $\mathbb{R}^N$. First of all, the N-cells you know, their interiors do not overlap. Next their boundaries are covered by several $N-1$ cells whose interiors do not over lap and so back down to tie 0-cells. So, this is what the structure CW structure on $\mathbb{R}^N$. If you change the integer r, you will get a different structure on $\mathbb{R}^N$, okay? So, we refer to them in short, as lattice structures on $\mathbb{R}^N$, okay?

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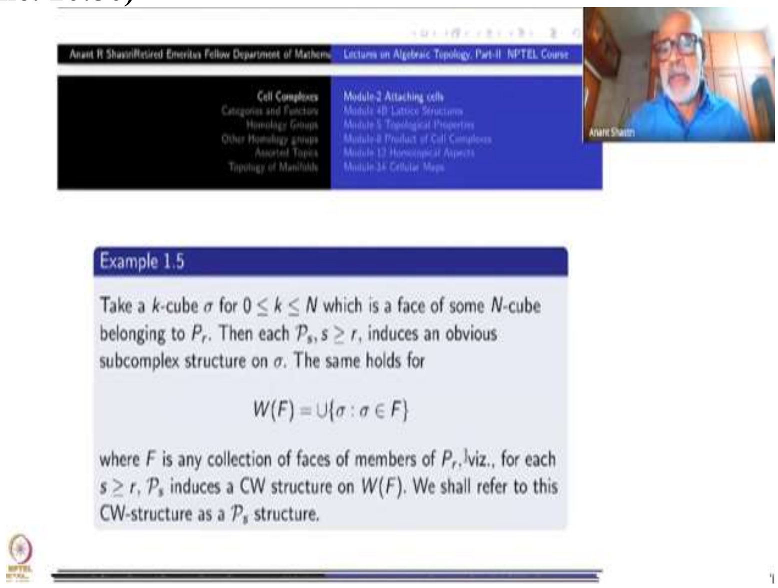

So, out of these, you can get several interesting examples of CW complexes. The first example is the following: Take a k-cube σ where k is between 0 and N (k equal to 0 is too simple an example; it is just a singleton right? So, that is also a nice thing all right, but you may assume $k > 0$); this σ should be a face of some N-cube belonging to P_r for some r. So, take a σ a k-cube, which is a face of some member of P_r.

Then for $s \geq r$, each CW structure P_s induces an obvious subcomplex structure on (the closure of) σ. For example, suppose σ is some edge in in P_2. then in P_3, this edge will get divided into 2 edges, in P_4, it will get divided into 4 edges and so on. Okay? You start with P_0 for example, an edge of length 1. In P_0 itself, it is just one single edge attached to two vertices. In P_1, it gets divided into 2 edges on three vertices. So, this way, you will get CW-structures coming from P_s on each k-cube of P_r. The same holds a little more generally.

Let us look at $W(F)$ okay? What is the $W(F)$? Here F is any collection of collection of cells; I am taking the union of all of them. For instance, F may have some points of L_r, some edges, some cubes then some higher dimensional k-cubes all belonging to P_r, where r is fixed. F can be infinite also. Then each P_s will induce a CW structure on the $W(F)$, okay? Why? On each of this cells, and on the intersection of any two cells, the two CW-structures will be comparable, because the two CW-structures on the intersection will be comparable because both of them belong to P_s. So, this generalises the earlier example.

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A little more generally, let F be family of finitely many faces of n-cubes in P, of various dimensions and various sizes. (N is fixed). Okay? Faces of N-cubes of some P_r, there may be 0 dimension, 2 dimension, 3 dimensions and so on. Take all this, but finitely many of them. Okay? So what I am doing here? Earlier, there was no restriction on the number of cells, but r was fixed. Now, I am ranging r indefinitely, but I put a condition that the family F must be finite, okay? Then $W(F)$ will again get a CW structure by some P_s. What is that s? Any number bigger than equal or equal to something that I have to tell you, namely, choose r to be the maximum of all t such that one of the N cubes containing a member of F belongs to P t, look at all the integers t such that there is a member of F, which is a face of some member of P_t. This collection is finite because F is finite. So, take the maximum of this collection that you call r. Then if $s \geq r, P_s$ will divide all of them very nicely. That is the whole idea that will give you a CW-structure $W(F)$ itself.

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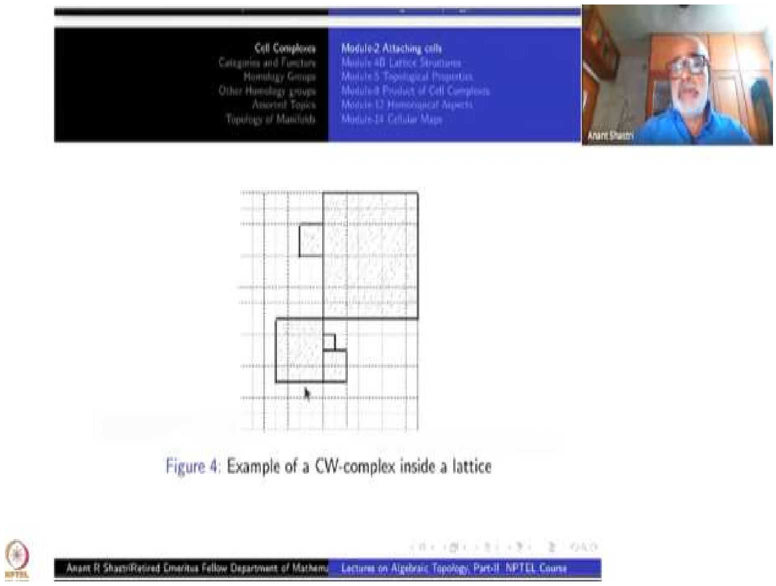

So, here is a picture of an example. You see that I have divided all of them into squares of smallest size. The actual squares from the collection F are shown by heavy lines here okay? Union of this large square here this small square here, this square is that square small is 1 so, you cut them like this. So, that will be CW structure on this entire union. It is very easy to see okay begin a finite case very easy to see alright.

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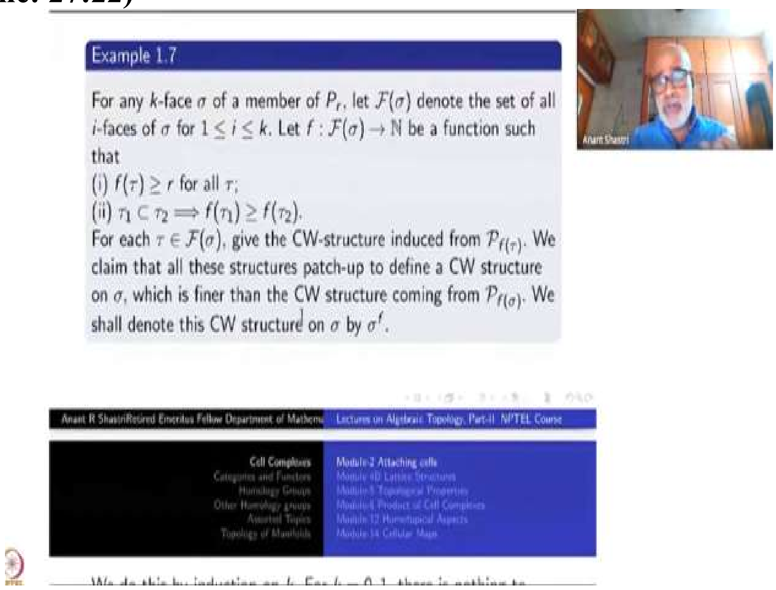

Slowly, we are going to build up on this theme, making it more and more meaningful and more and more useful. For any k-face sigma of a member P_r, i.e., take a member of P_r and then take a k-face of that. And let $\mathcal{F}(\sigma)$ denote the set of all i-faces of σ, $i \leq k$, okay? But I do not want to take 0-cells, so, $1 \leq i \leq k$. Denote this collection by $\mathcal{F}(\sigma)$. On $\mathcal{F}(\sigma)$, take a positive integer function f such that:

(i) $f(\tau)$ is bigger than equal to r for all τ, and

(ii) τ_1 is contained in τ_2 should imply $f(\tau_1)$ is bigger than or equal to $f(\tau_2)$, i.e., f is order reversing, okay?... smaller the face, larger the integers that is what I want. Alright.

Take such a function f. For each τ in $\mathcal{F}(\sigma)$, give the CW-structure induced from $P_{f(\tau)}$, okay? We claim that all these structures patch-up to define a CW-structure on σ. σ is what? σ is a k-face of some member of P_r. Alright. So, on all faces of σ, we have a CW structure and these together define a CW-strucutre on σ itself. This structure is finer than the CW structure coming from $P_{f(\tau)}$. It will be denoted by σ^f.

In the interior of the N-cell σ, we will have the CW-structure coming from $P_{f(\tau)}$, but for the smaller faces τ of σ, the division may be even finer because the numbers $f(\tau)$ are larger or at least that much. That is important okay? For instance, the subdivision on the boundary of a 2-cell may be a pentagon or a hexagon or just a square okay? We can attach the 2-cell to get topologically the same picture, yet the CW -structures may be different. So this is what you have to keep in mind okay? So, we get a CW structure on sigma itself depending on this function, the function should have these two properties that is all, okay? So, this is the germ of this idea, how to how to cut a cell into a union of finer cells. This will be of some use soon.

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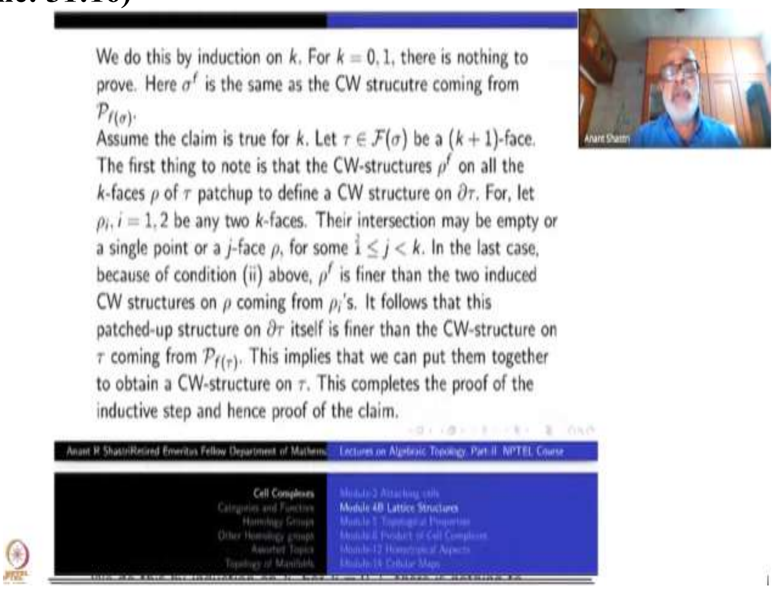

So, we prove this by induction on k. I have already explained that how it is working, the way it is done, this is what you have to understand, to patch up CW-structures for the more complicated spaces from that of simpler ones, okay? So, let us do it by induction on k, k is what? k is the dimension of σ itself.

Suppose $k = 0$ that means, σ is just a single point. Then there is nothing to do, a singleton never gets divided at all, no matter what f you choose for it that is what one observation I have made earlier, namely, once a point is in L_k, it is there in L_{k+1}, L_{k+2} etc, okay? So, the case $k = 0$ is no problem.

For $k = 1$, what does it means? It is an edge okay. So, what do you have to do? You have to look at the 0-cells of that namely the two endpoints which will never get divided further, only the interior gets divided depending upon what f you have chosen and that is all. So, there is no trouble patching up the subdivisions of two such edges to the union, intersection of two edges being either empty or a single point, the CW-structures coming from f on the two edges simply gets extended on the union. Okay?

Now, assume the claim is true for k, then we will do it for $k + 1$ okay. So, now σ be a $(k + 1)$-face of some member of P_r, Okay? Well, the first thing to note is that the CW-structure ρ^f on all the k-faces ρ of τ, all k-faces of τ will patch up to a finite CW-structure on the boundary of τ: suppose ρ_1 and ρ_2 are any two k-faces of τ. Okay? Then their intersection may be empty or maybe a single point or a j-face ρ, for some $j < k$. ρ_1 and ρ_2 are k-faces, their intersection ρ has to be a j-face of some lower dimension. In the first and second case, there is no problem. In the last case, because of condition (ii), ρ^f is finer than the two CW-structures on ρ coming from ρ_1 and ρ_2. Okay? Therefore, all these structures patch up to define a CW-structure on the boundary of τ itself and this structure is finer than the CW structure on τ coming from $P_{f(\tau)}$.

This implies that we can put them together to obtain a CW structure of tau, okay? This completes the proof of the inductive step for $k + 1$ faces okay? Remember these things, we will use them and we will try to do some examples of when we are taking in infinite families of these F's. Okay, Thank you.