Introduction to Algebraic Topology (Part-II)
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Lecture-39
CW Structure and CW Homology of Lens Spaces

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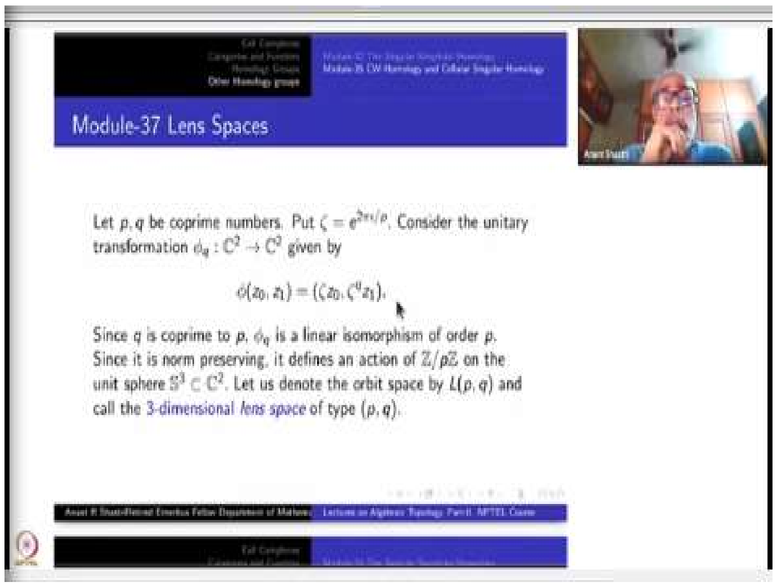

Today we shall study the lens spaces, their CW-structure and then use the CW chain complex associated to it to compute the homology. To begin with we shall define what is a lens space in dimension 3. Then you can extend this definition to higher dimensions. So, you begin with two natural numbers p and q which are coprime to each other, they need not be prime numbers, they are coprime to each other. That is all.

Put $\zeta = e^{2\pi i/p}$, so that ζ^p is 1. Actually, ζ is a primitive p-th root of unity. Ok? You could have taken any other primitive root also, it will not matter but let us fix one, no problem. Now consider the transformation ϕ_q on $\mathbb{C} \times \mathbb{C}$ given by (z_0, z_1) going to $(\zeta z_0, \zeta^q z_1)$, Ok? Clearly, this map from $\mathbb{C}^2$ to $\mathbb{C}^2$, is norm preserving, because ζ is of modulus one, ζ^q is also of modulus 1. Right?

So, this transformation is actually orthogonal. Since q is co prime to p, you can verify that this ϕ_q is actually an isomorphism and it is of order is p. That means, if I apply it p times, $\phi_q \circ \phi_q \circ \cdots \circ \phi_q$ (p times) is equal to identity, because $\zeta^p = 1$ and $(\zeta^q)^p$ is also 1. ok? Since it is norm preserving, it defines an action of $\mathbb{Z}/p\mathbb{Z}$, that is, the cyclic group of order p here ok, on the unit sphere $\mathbb{S}^3$ subset of $\mathbb{C}^2$.

Let us denote the orbit space by $L_{p,q}$, orbit space of $\mathbb{S}^3$, ok, under this action is denoted by $L_{p,q}$ and called the 3-dimensional lens space of type (p,q), ok? For each coprime pair of natural numbers (p,q), you have got a lens space here.

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Verify that the action is fixed point free. Since it is a finite group action which is point free, therefore it is an even action. Remember, the an even action by which we mean what? It is also called properly discontinuous action in some literature, old literature. In any case what you get is the quotient map ϕ from $\mathbb{S}^3$ to $L_{p,q}$, is a p-fold covering projection. In particular, since $\mathbb{S}^3$ is simply connected it follows that $\pi_1(L_{p,q})$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, the cyclic group of order p. So, these things you have seen in part I already.

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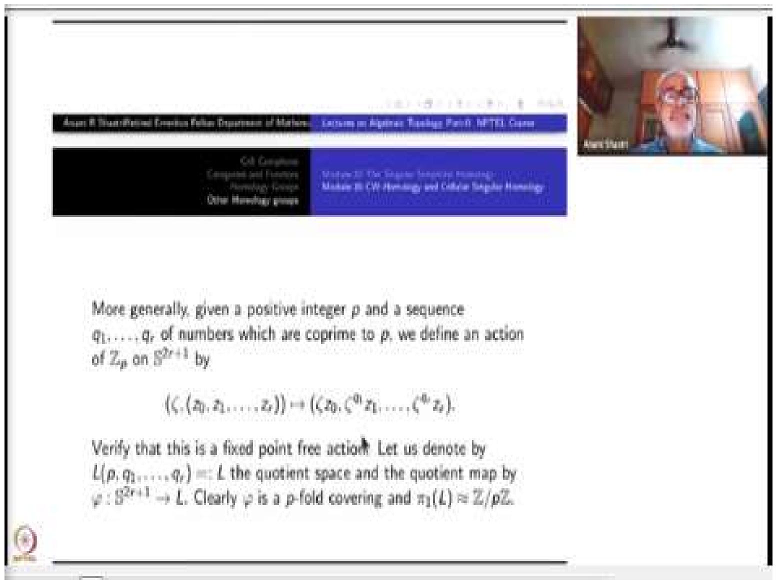

More generally, now take natural numbers, p and $q_1, q_2, \ldots, q_r$, all coprime to p. So, there I have just a pair p and $q = q_1$, but now I am taking r of them $q_1, q_2, \ldots, q_r$ each q_i coprime to p. So, fix such an unordered r-tuple of numbers. Then take the action of $\mathbb{Z}/p\mathbb{Z}$, given by $(\zeta, (z_0, z1, \ldots, z_r))$ going $(\zeta z_0, \zeta^{q_1} z_1, ..., \zeta^{q_r} z_r)$.

Earlier, we took a pair of complex numbers, Now I am taking an ordered $(r+1)$-tuple of complex numbers and each time I am multiplying by a corresponding power of ζ. Once again, this is also $\mathbb{C}$-linear and norm preserving action. Also it will give you a fixed point free action on $\mathbb{S}^{2r+1}$, using the fact that $q_1, q_2, \ldots, q_r$ are coprime to p.

So, let us have just have a simple notation L, for $L_{p,q_1,q_2,\ldots,q_r}$ for the orbit space. It is called the lens space of type $(p, q_1, q_2, \ldots, q_r)$, ok? So, this p has a different role to play than the numbers $q_1, q_2, \ldots, q_r$. This p tells you what is group, the order of the element ζ. ζ is a primitive p-th root of unitary. $q_1, q_2, \ldots, q_r$, decide the dimension and the twisted action, ok? You could have taken all of them equal to $1, 1, \ldots, 1$. Then this would be called the diagonal action. Ok?

So, the diagonal action is an interesting action, alright? So, there are various special cases here, ok? As before we have the quotient map ϕ from $\mathbb{S}^{2r+1}$ to L as a p-fold covering. It will tell you that the fundamental group of L is nothing but $\mathbb{Z}/p\mathbb{Z}$. So, the fundamental group has nothing to do with all these integers $q_1, q_2, \ldots, q_r$, ok? They are coprime to p is required to get a good action, viz., with no fixed points. Ok?

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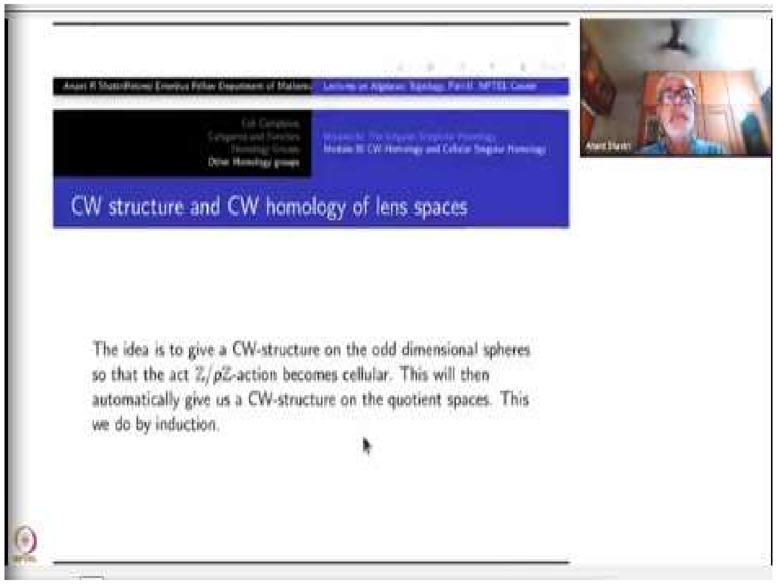

So, now we want to give a CW-structure to this lens spaces. So, lens space is defined as a quotient of something, therefore what you would like to do is to give a nice CW-structure on the spheres $\mathbb{S}^3, \mathbb{S}^5, \mathbb{S}^7$ and so on, ok? Nice means what? That the action by the group $\mathbb{Z}/p\mathbb{Z}$, in each case is cellular. That means the homeomorphisms permute the cells, so that on the orbits itself there will be a cell structure, Ok? This is what we want to do.

Consider the simplest case which we have studied, for example $\mathbb{S}^1$, ok? If you take the 3-fold action on $\mathbb{S}^1$ by cube root of unity, then you could take $\{1, \omega, \omega^2\}$ as vertices and arcs from 1 to ω, ω to ω^2, ω^2 to 1 as the 1-cells, i.e., 3 vertices and three 1-cells, when you quotient out by this action, what you will get? You will get just one vertex and one 1-cell, so it is again $\mathbb{S}^1$. Ok?

So, more generlally, if ϕ from $\mathbb{S}^1$ to $\mathbb{S}^1$ is p-fold covering got by the action of $\mathbb{Z}/p\mathbb{Z}$, for$p = 3, 4$, or any other number, you can do a similar thing, namely cut the circle into p parts, along $1, \zeta, \ldots, \zeta^{p-1}$. Ok?

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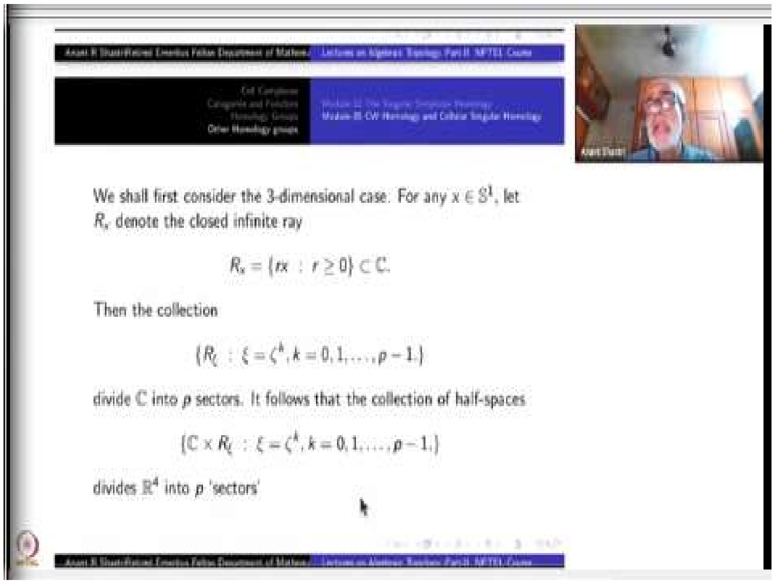

So, we shall do similar thing on all odd dimensional spheres, again in an inductive fashion, Ok? First consider the 3-dimensional case. For each x belongs to $\mathbb{S}^1$, let us have some notation. Let R_x denote the closed infinite ray, namely, $\{rx, r \geq 0\} \subset C$, ok? Consider the collection of all these rays R_ξ, where ξ is some power of ζ. For $k = 0$, we have just positive real axis, and $k = 1, 2$,... this ray is passing through ξ, ξ^2... so on.

So, I have taken i from 0 upto $p - 1$, so there will be p of these rays which will divide the entire plane into p sectors, ok? Each sector will be of angle $2\pi/p$, Ok? So, so you have these sectors inside $\mathbb{R}^2$. Now you take the product with $\mathbb{C}$. $\mathbb{C} \times R_\xi$'s. What are they? They are 3-dimensional half-planes in $\mathbb{R}^4$. ok?

Together, they will cut the whole of $\mathbb{C} \times \mathbb{C}$ (or $\mathbb{R}^4$) into p-sectors, I can call them 3-dimensional sectors. They are product of $\mathbb{C}$ with a sector in $\mathbb{C}$. Ok? I do not want to draw any pictures here because I am cannot. But we must do every thing rigorously. If you try to draw it correctly, which is a difficult task in any case, ok? You will understand it better.

In any case, the final output should be done without the help of pictures, just by rigorous arguments. So, argument should be converted into picture for getting familiarity, for your own sake that is what you have to do each time. ok? yeah. So, we have cut down the entire of $\mathbb{R}^4$ into p sectors. Ok? Closed sectors.

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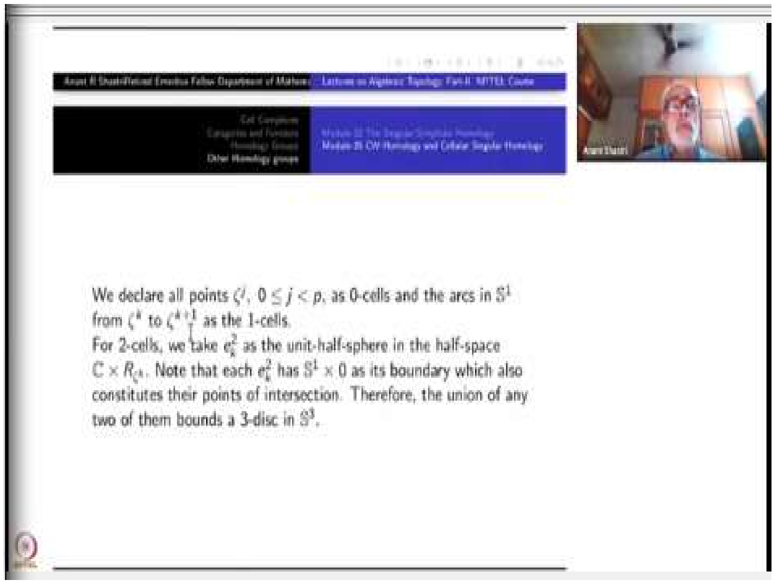

Now only, we shall start constructing the cell structure on the sphere $\mathbb{S}^3$. Ok? So, we declare all the points, namely, $1, \zeta, \zeta^2, \ldots, \zeta^{p-1}$ in $\mathbb{S}^1$, as 0-cells. And the arcs of $\mathbb{S}^1$ from ζ^k to ζ^{k+1} as 1-cells, similar to the way we cut $\mathbb{S}^1$ in to three arcs when $p = 3$.

So, so you have p 0-cells and p 1-cells, ok? So, this already describes the 1-skeleton of the CW-structure. Now, we have to attach 2-cells. So, what are the 2-cells and 3-cells? I am telling you now. So, they are all subspaces of $\mathbb{S}^3$, so finally they will constitute the entire of $\mathbb{S}^3$. So, we are trying to build up $\mathbb{S}^3$ here. So, for 2-cells we take e_k^2, where k ranging from 0 to $p-1$, each e_k^2 as the unit half sphere in the half space $\mathbb{C} \times R_{\zeta^k}$. These half-spaces are there, right? They are in some 3-dimensional subspace, in which they are half-spaces, ok? In that you take the half unit spheres, that means you are intersecting these half-spaces with the unit sphere $\mathbb{S}^3$, Ok? They are sets of points $(z_1, t\zeta^k)$, where $|z_1|^2 + t^2 = 1$ and $t \geq 0$. So, when fact t in the second coordinate ranges from 0 to 1, the circle becomes smaller and smaller and ultimately when $t = 1$, k is itself equal to 1, the first coordinate will be just 0.

So, that is the picture for the 2-cell e_k^2, Ok? So, it has the boundary $\mathbb{S}^1 \times 0$. And this common boundary is precisely the intersection of any two of them, that is e_k^2 and e_{k+1}^2 intersect is precisely in $\mathbb{S}^1 \times 0$. Ok? A point is in the intersection only if the second coordinate are equal i.e, $t\zeta^k = s\zeta^{k+1}$ iff $t = s = 0$ and then the converse is also true.

Hence union of any two of them constitutes a surface homeomorphic to a 2-sphere embedded inside $\mathbb{S}^3$. Ok? Therefore, the inside of that is actually a $\mathbb{D}^3$. But if you do not like to use such general arguments here, you can directly show that the bounded part between any two consecutive 2-cells, e^2_k and e^2_{k+1} is actually homeomorphic to $\mathbb{D}^3$ as follows. Ok

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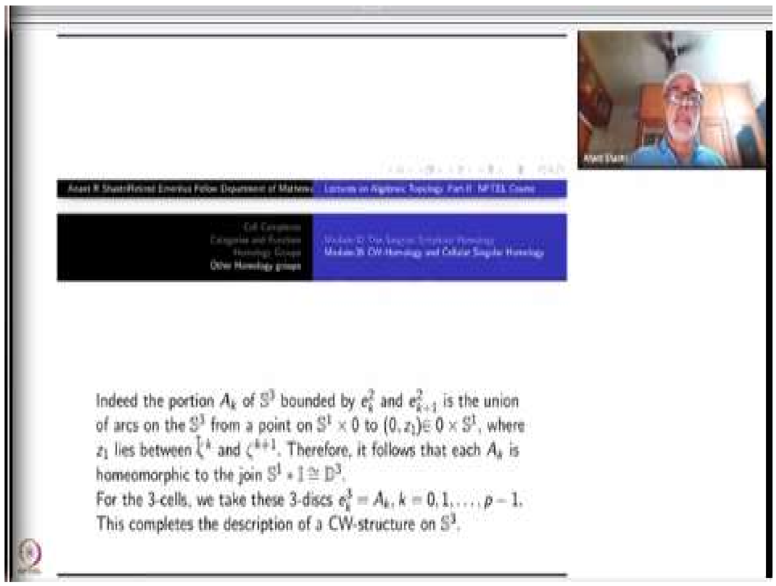

Indeed, let me write this part of $\mathbb{S}^3$ bounded by e^2_k and e^2_{k+1} as A_k. Each point of A_k lies on a portion of the great arc on the sphere $\mathbb{S}^3$, ok, from a point $\mathbb{S}^1 \times 0$ to $(0, z) \in 0 \times \mathbb{S}^1$, where z itself lies on the arc between ζ^k and ζ^{k+1} on $0 \times \mathbb{S}^1$. That gives you a homeomorphism from A_k to the joint of $\mathbb{S}^1$ with a closed interval and hence a homeomorphism to $\mathbb{D}^3$. So, we take A_k as a 3-cells now, as k ranges from 0 to $p-1$. Ok?

So, that will cover the entire of $\mathbb{S}^3$. So, the partitions that we have made inside $\mathbb{R}^4$ restrict to partitions inside for $\mathbb{S}^3$, the unit sphere inside $\mathbb{R}^4$. Ok? So, the largest parts are the 3-cells A_k, their boundaries are $e^2_k \cup e^2_{k+1}$ respectively, and the boundary of any two 2-cells is the circle $\mathbb{S}^1 \times 0$, which itself is the union of p one cells and boundaries of these one cells are in the vertex set $\{\zeta^k, k = 0, \ldots, p-1\}$. That completes the description of a CW structure on $\mathbb{S}^3$.

Now you observe that if you take the action the group $\mathbb{Z}/p\mathbb{Z}$ as described earlier, what happens? The points on the circle $\mathbb{S}^1 \times 0$, will get permuted, the arcs will get permuted, the 2-cells will get permuted, and then 3-cells will get permuted. That means that the action is actually cellular. And the action is transitive on each skeleton.

If you take any one 0-cell, all other 0-cells can be got by the action, take any one of the arcs, all other arcs are got by the action, take any 2-cell, other 2-cells are got by the action, take any 3-cell, you will get all other 3-cells by the action. That is why the action is transitive on 0-cells, 1-cell, 2-cells and 3-cells. Therefore in each dimension you have exactly one orbit.

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So, in the orbit space $L_{p,q}$, you get one k-cells for $k = 0, 1, 2$, and 3. Now what are the attaching maps? That is what you have to understand. Ok? Attaching maps have to be described.

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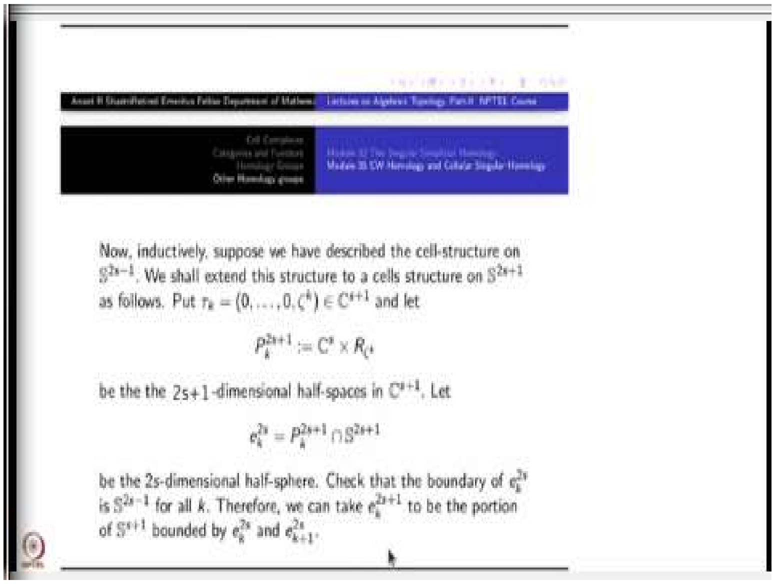

So, before that let me complete now, what I am going to do for higher dimension? So, inductively, having given a CW structure for $\mathbb{S}^3$, suppose we have done it for $\mathbb{S}^{2s-1}$, ok? Then I want extend the cell structure of $\mathbb{S}^{2s-1}$ to that of $\mathbb{S}^{2s+1}$. So I am taking product with one more factor $\mathbb{C}$ and doing the same trick as I did from $\mathbb{S}^1$ to $\mathbb{S}^3$. ok? So, put $ta_k = (0, 0, ..., \zeta^k)$, ok, $\mathbb{C}^{s+1}$. And let P_k^{2s+1} equal to $\mathbb{C}^s \times R_{\zeta^k}$, just like what we have done for $s = 1$. Ok? So, these will be define a partition of $\mathbb{R}^{2s+2}$, into p sectors.

I have to now introduce the $2s$-cells and $2s + 1$ cells. Ok? what are they? So, e_k^{2s} is by definition, P_k^{2s+1} the half space intersected with the sphere $\mathbb{S}^{2s+1}$, exactly same way as we did it in the 3-dimensional case. Check that its boundary is precisely the sphere $\mathbb{S}^{2s-1}$, for all k.

The same argument as before, will give you that the portion of the sphere $\mathbb{S}^{2s+1}$ bounded by two consecutive e^{2s} cells, is homeomorphic to $\mathbb{D}^{2s+1}$. This picture is exactly the same thing. Ok.

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This extends the cell-structure of $\mathbb{S}^{2k-1}$ to a cell structure on $\mathbb{S}^{2k+1}$. Verification that it is invariant under the $\mathbb{Z}/p\mathbb{Z}$ action is straightforward. It follows that the quotient space has a cell-structure with precisely one cell in each dimension $r, 0 \leq r \leq 2s+1$. Ok?

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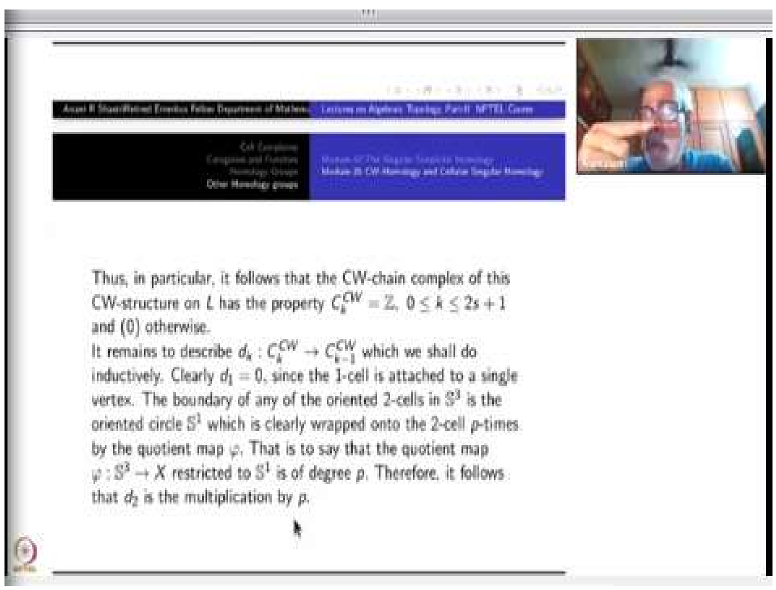

Thus, in particular, the CW-chain complex of this CW-structure on L has the property that all C_k isomorphic to the infinite group, for k ranging from 0 to $2s+1$, and there after that everything is 0. Ok?

So, what remains is to describe the boundary operators, d_k from C_k to C_{k-1}, and that as we have pointed out, will depend upon the attaching maps. What are the attaching maps here? If you describe them correctly, then we shall be able to determine d_k, ok? So, let us do that one now, again inductively one by one. ok?

The boundary operator d_1 from C_1 to C_0: What is C_0? It is just $\mathbb{Z}$, the infinite cyclic group generated by a single 0-cells. What is C_1. There is only one 1-cell and both the boundary points of this 1-cell are identified with this single 0-cell.

Therefore, δ of the 1-cell is $[u, v]$ is $v - u$, right both v and u are going to the same point, so what will be the effect? The map d_1 will be 0. So, the map d_1 is 0 from $\mathbb{Z}$ to $\mathbb{Z}$, C_1 to C_0, ok? So this is a picture for the CW structure for $\mathbb{S}^1$, ok?

By the way, you can choose any one of the generators for each C_k. The corresponding k-cell will called positively oriented and minus of that generator will be called negatively oriented. So, the oriented 2-cell just means that I have already chosen the generator there. So, 2-cell is in CW structure $\mathbb{S}^3$ has its boundary the oriented circle $\mathbb{S}^1$, which could be anticlockwise or clockwise, there is no problem. It clearly wrapped on to the 1-cell in L, p-times under the restriction of the quotient map ϕ from $\mathbb{S}^{2s+1}$ to L. It follows that d_2 is just multiplication by p, the generator goes to p times that.

If you have understood this one, the story just repeats now. Ok? Let us go through this one more round.

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Now look at the oriented 3-cell, remember how the 3-cell in $\mathbb{S}^3$ is got. It was got by as a loon, namely, lying between two 2-cells both of which themselves have their boundary as $\mathbb{S}^1 \times 0$. So, when you take the boundary of e_k^3, for any k, it is $e_{k+1}^2 \cup e_k^2$. Both of them are mapped onto the unique 2-cell in L. However, with any choice of compatible orientation it follows that $\delta_2(e_k^3) = e_{k+1}^2 - e_k^2$ and hence $d_2(e^3) = 0$.

So, both of them are mapping the same way to the 2-cell in the quotient map one of them which is the choice here and equivalence class or whatever way you want to think of that. Therefore when you say e_k^2 goes to some e^2 in the in L which also comes to e^2, this minus is will be 0. Therefore, d_3 on each 3-cell is 0, the whole d_3 is 0, Ok whichever cell you take as an exam as a representative, there is only 1-cell after all when you come to the quotient.

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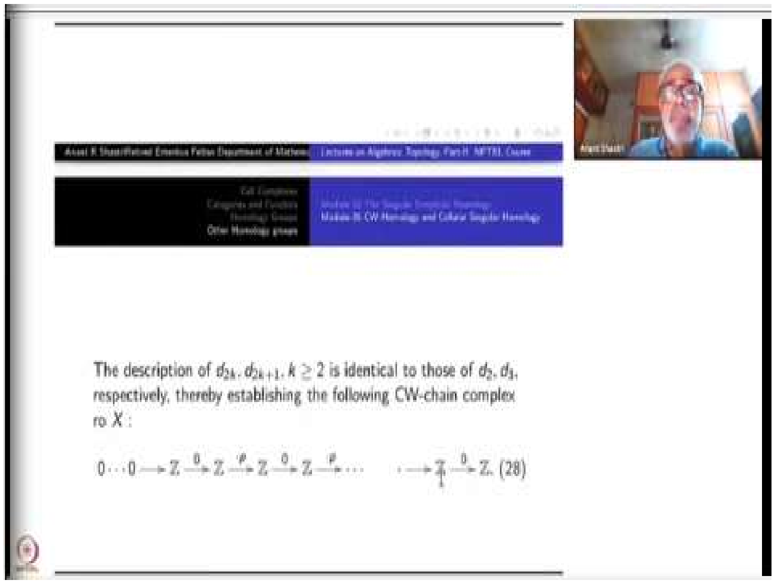

Repeating this argument, it follows that d_{2k} is multiplication by p and $d_{2k-1} = 0$ for all k. Therefore, $H_0(L) = Z$. (This can be directly seen because this is connected.) At the top dimension, $H_{2s+1}(L) = Z$, because $C_{2s+s} = 0$ and $td_{2s+1} = 0$. Like this you can conclude that $H_{2k+1}(L) = \mathbb{Z}/p\mathbb{Z}$ and otherwise in the range $0 < k < s$. In odd dimension it is $\mathbb{Z}/p\mathbb{Z}$ and in even dimension, it is zero.

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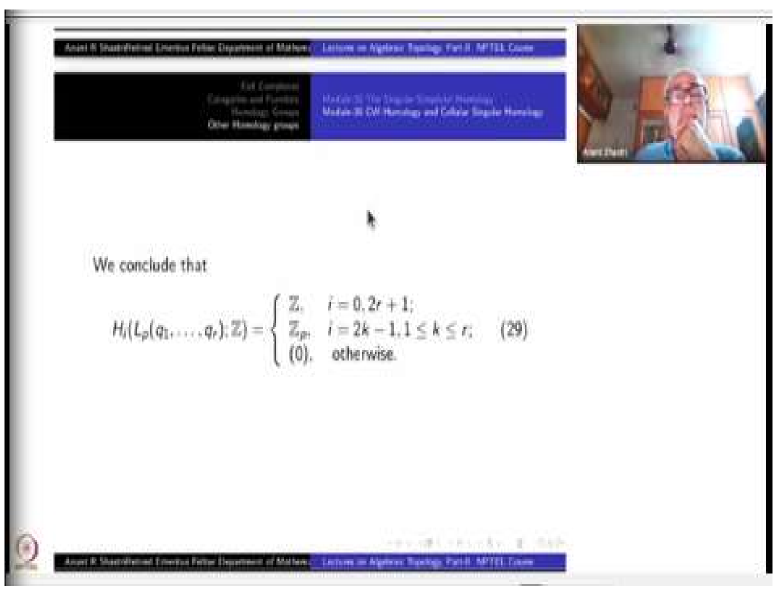

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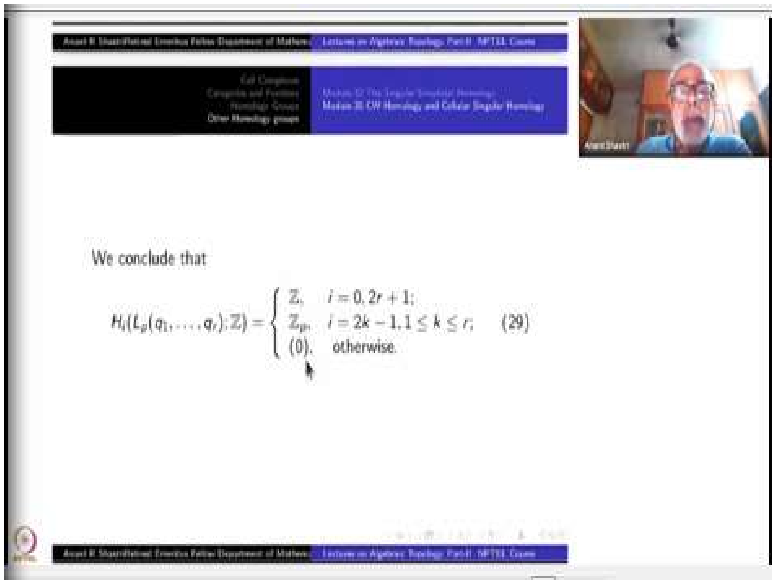

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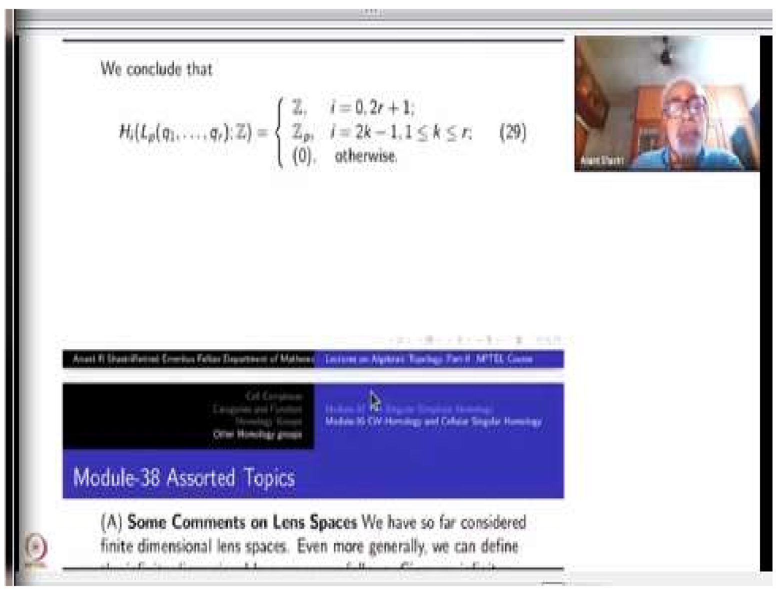

So, this brings us to something very satisfactory: CW chain complex has been used to compute at least something non trivial. Ok? So, next time we will take care of a few assorted things that we have missed earlier, because of time constraints. And after that we will start applications of this homology, thank you.