Introduction to Algebraic Topology (Part-II)
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Lecture-38
Classification of CW-Chain Complex

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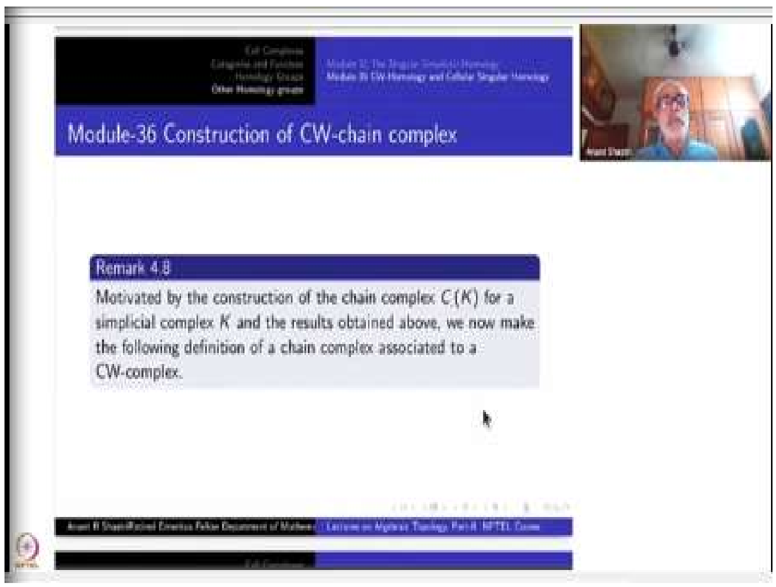

So, we would like to construct the CW-chain complex of a CW-complex. Let me recall what we did the last time, which is a very crucial for this construction. So, it will be worthwhile to recall the whole thing and also go through the proofs carefully. So, that you do not have any doubt left there. So, let me begin with this lemma that we did.

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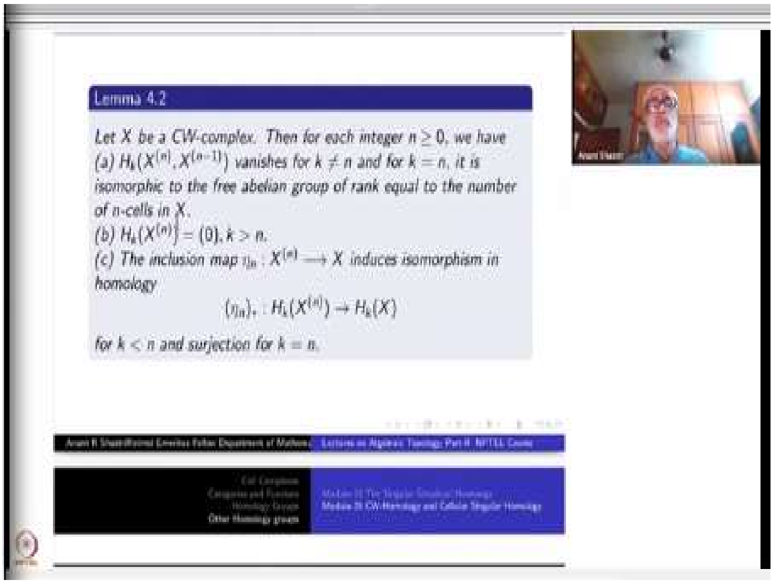

X is CW complex. Then for each n positive, we have the homology the k-th homology of the pair $(X^{(n)}, X^{(n-1)})$ vanishes for $k \neq n$. So, the homology is atomised; at one single k viz., $k = n$, it is isomorphic to the free abelian group of rank equal to the number of n-cells in X. This is a consequence of the previous lemma that we have done, I am summing it up here. The second thing and third statements are slightly more elaborate and require some poof. Of course, they follow from this one.

So, the second statement is that $H_k(X^{(n)})$ is 0 if $k > n$. That means, if you have a CW complex of dimension n, then all the homology beyond n, H_{n+1}, H_{n+2} etc, they are all 0.

The third statement is that the inclusion map, let us let us denoted by η_n from $X^{(n)}$ into the whole space X induces isomorphism in homology H_k to H_k for $k < n$ and a surjection for $k = n$. Ok?

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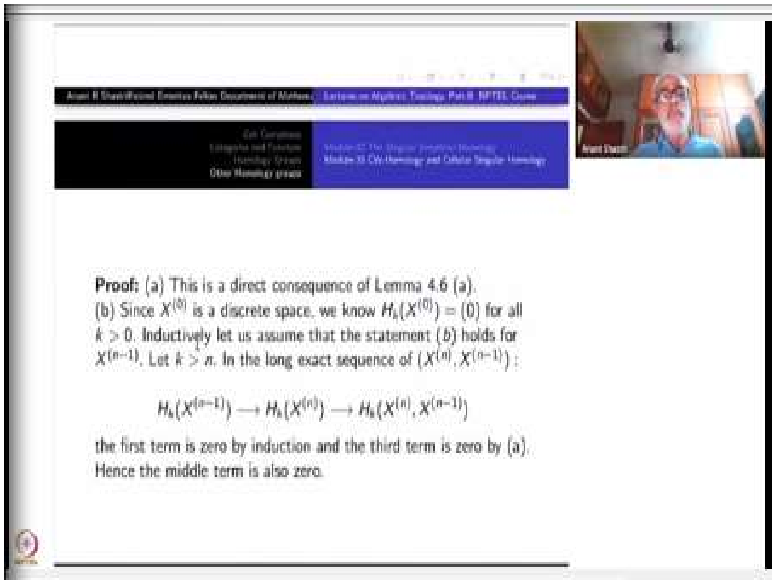

Let us look at the proofs. part (a) is actually a direct consequence of the previous lemma that we have proved. Namely, if you attach n-cells to a space Y to obtain X, then the relative n-th homology is precisely equal to the free abelian group over the number of cells you have attached, everywhere else $H_k(X, Y)$ is 0. You apply this to the special cases when Y is $X^{(n-1)}$ and $X = X^{(n)}$. Ok?

So, for (b) let us prove this by induction. Since $X^{(0)}$ is a discrete space $H_k(X^{(0)})$ is 0 for all k positive. Therefore we can start the induction. Suppose it is true upto $n - 1$. Then you want to prove it for n, ok? Let $k > n$. Then I want to show that $H_k(X^{(n)}$ is 0. Right?

In the long homology exact sequence of the pair $(X^{(n)}, X^{(n-1)})$, for $k > 0$, we have $H_k(X^{(n-1)}$ and $H_k(X^{(n)}, X^{(n-1)}$ are both 0 by induction hypotheses and (a). Therefore the term between then viz., $H_k(X^{(n)})$ is also zero. Now to prove (c): what does (c) say? This inclusion from $X^{(n)}$ to the whole of space X induces isomorphism in H_k for $k < n$ and surjection for $k = n$, ok? this is what we want to prove.

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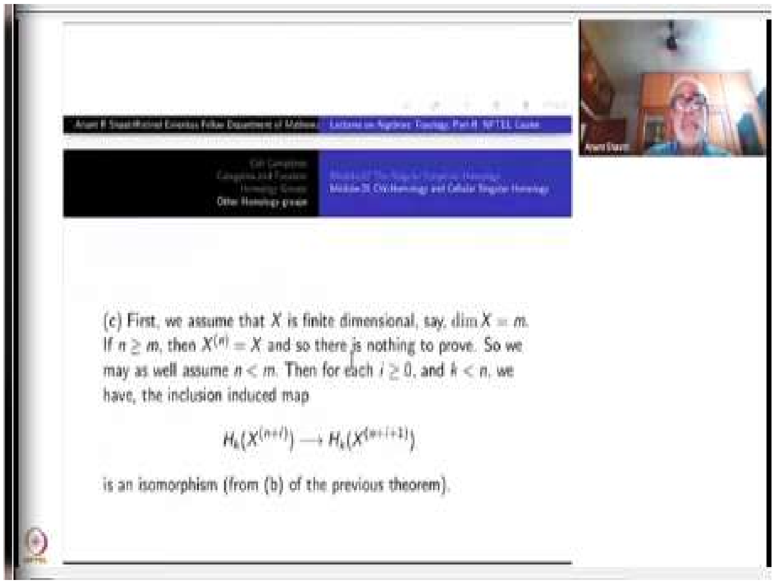

First you assume X is finite dimensional, say, dimension of X is m, if this $n > m$, then the n-th skeleton is the whole of X and inclusion map is the identity map and there is nothing to prove, ok? So, you may assume that n is smaller than m and $X^{(n)}$ is a proper closed subset of X, X is actually $X^{(m)}$, because dimension of X is m. ok? Then for each $i \geq 0$ and $k < n$, we have an inclusion needs to map from $(n+i)$-th skeleton to $(n+i+1)$-th skeleton. Ok?

Take composition of these inclusion to get the inclusion from $X^{(n)}$ to $X^{(m)} = X$. Ok? Since, all the inclusion maps induce isomorphisms for $k < n$ from (b). You have a composite of finitely many isomorphisms. Ok?

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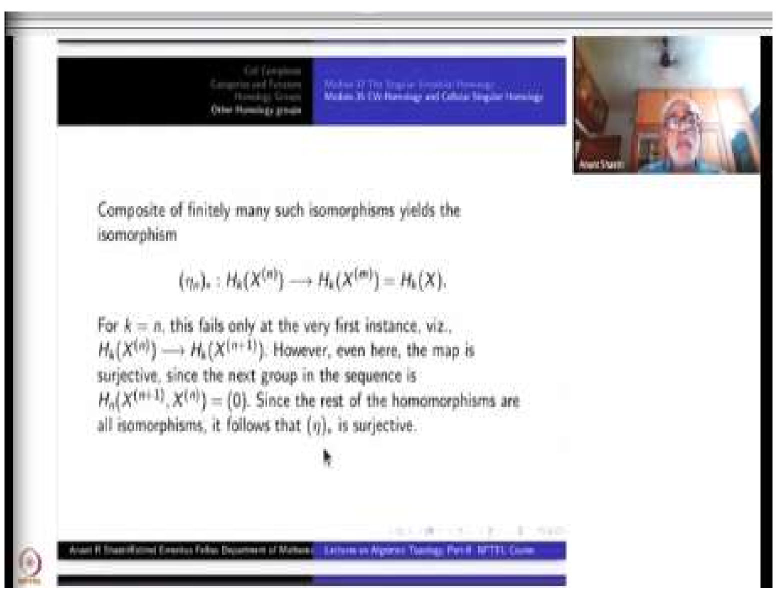

For $k = n$ what happens? The very first one is not an isomorphism, but is surjective. After that, all other things are isomorphisms. Therefore the composite will be also surjective. Ok? That takes care of the finite dimensional case. Ok?

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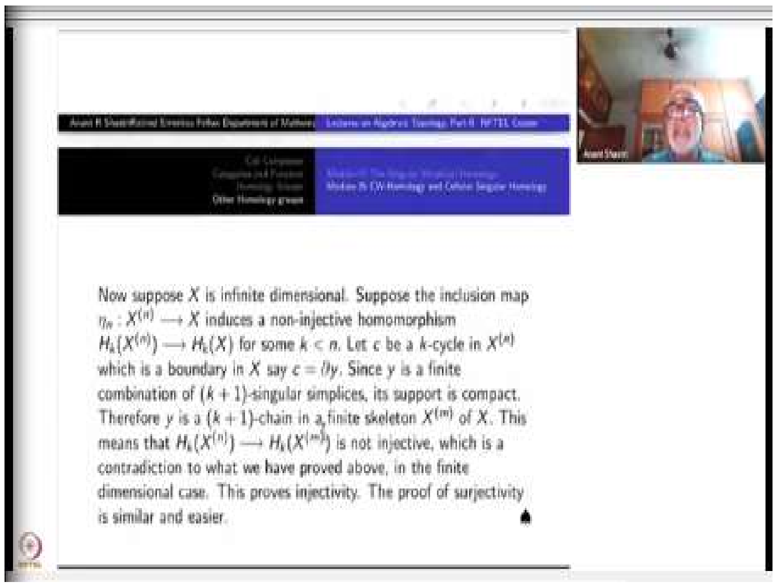

Now comes the infinite dimensional case. Suppose X is infinite dimensional. Suppose the inclusion indiced map $(\eta_n)_*$ from $H_k(X^{(n))}$ to $H_k(X)$ (for $n > k$) is not injective. (We want to show that it is injective first of all, then we want to show that it is an isomorphism suppose it is not injective) ok? Let c be a k-cycle in $X^{(n)}$ which is a boundary in X. That is the meaning of $\eta_*([c]) = 0$.

Say $y \in S_{k+1}(X)$ is such that $\partial(y) = c$. But y is a finite linear combination of $(k+1)$-singular simplices, each of them is compact. So, the entire support of the chain y is a finite union of compact sets. Therefore support of y is compact subset of X. ok? A compact subset of any CW complex is contained in a finite skeleton, ok? So it is contained in some $X^{(m)}$. alright? Now, what does this mean? This means that this c is the boundary of y, where y itself is in $S_{k+1}(X^{(m)})$. This means that the inclusion map from $X^{(n)}$ to $X^{(m)}$ itself is not injective and that contradicts whatever we have already proved. Ok? Therefore, η_* at the k-th level is injective.

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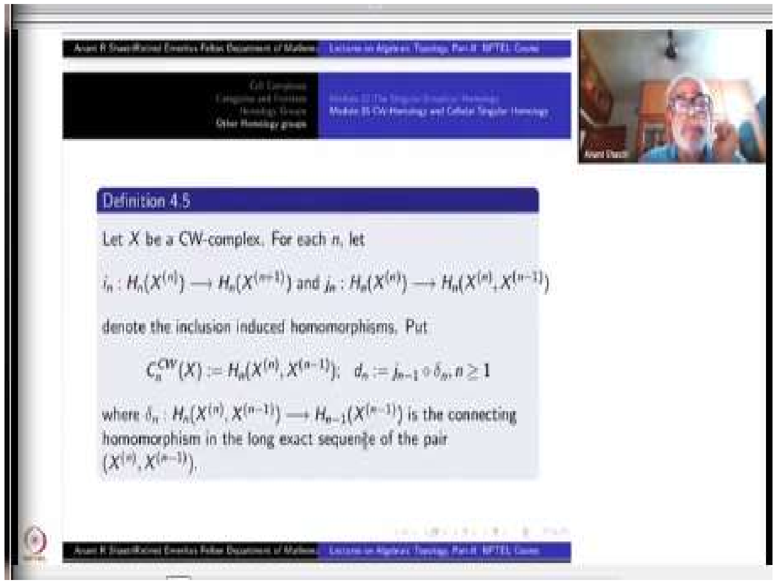

Exactly similarly, you can prove the subjectivity also for $k = n$. ok? The compactness of the support of any chain allows us to pass from infinite dimensional case to finite dimensional case, Ok? Now having recalled this one, we can now define the modules of the chain complex first and then the boundary map also. Ok? So, let us have some elaborate notations here, for this particular part only, we will have these notations fixed, Ok? So i_n is the inclusion induced morphism for $H_n(X^{(n)})$ to $H_n(X^{(n+1)})$. Actually i_n is the inclusion map at the space level, but I will denote the induced map at the homology also by i_n, instead of going on writing the suffix star and so on.

Similarly let j_n be the inclusion induced homomorphism from $H_n(X^{(n)})$ to $(H_n(X^{(n)}, (X^{(n-1)}))$, Ok? Think of $X^{(n)}$ as the pair $(X^{(n)}, \emptyset)$. That admits an inclusion map into the pair $(X^{(n)}, X^{(n-1)})$, and j_n is the induced homomorphism at the homology level. So, once you have this notation, just put the n-th (module or whatever just abelian) group of the chain $C_n^{CW}(X) := H_n(X^{(n)}, X^{(n-1)})$. Recall that this is actually a free abelian group over the number of n-cells attached to $X^{(n-1)}$ to get $X^{(n)}$.

And the boundary operator $d_n := j_{n-1} \circ \delta_n$, where δ_n itself is the connecting homomorphism in the long homology exact sequence of the pair, $(X^{(n)}, X^{(n-1)})$. So d_n is a homomorphism of degree -1, alright? What we have to verify? We have to verify that $d_n \circ d_{n+1}$ is 0. Once you do that this will be a chain complex. Ok?

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So, the following lemma tells you $d_n \circ d_{n+1}$ is 0 for all n. And hence, this becomes a chain complex, ok? This chain complex is called the cellular chain complex associated to X. So, we will just write the superfix CW, both on the chain complex ans the associated homology which is called the cellular homology of X and temporarily denoted by $H_*^{CW}(X)$.

We will prove that this homology is the same as the singular homology of X. And therefore you will not need an extra notation for this one at all. You can directly use the singular homology and its description provided by this theorem. Ok? The homology of this this chain complex is called a cellular homology of X. Alright.

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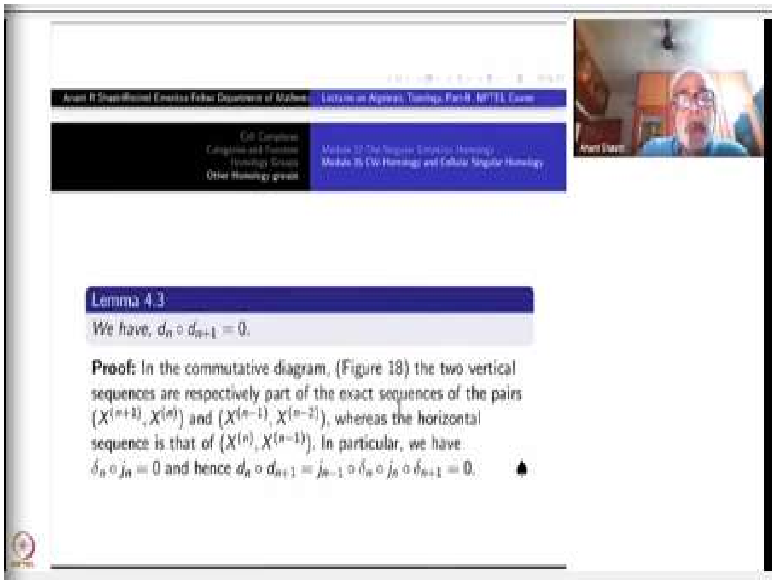

So, first let us prove that these composites are 0; d^2 is 0 is what we have to show. Ok?

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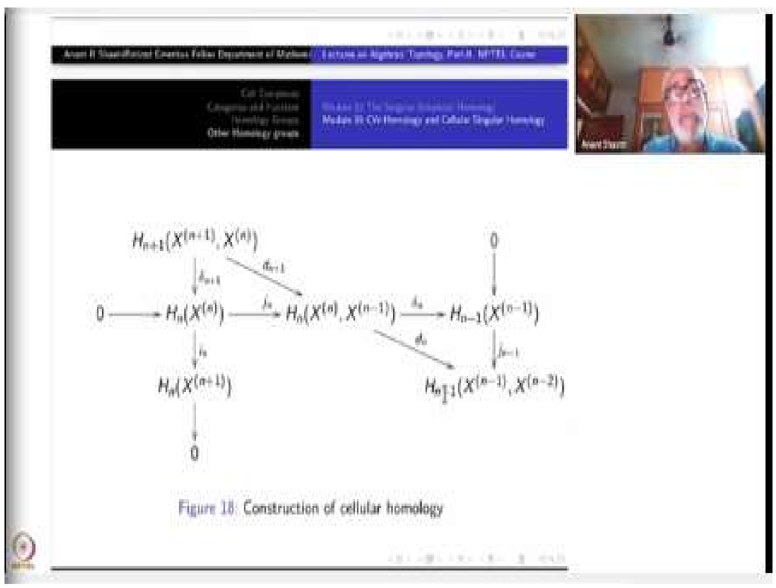

So, I have displayed here a commutative diagram from which all these things will become clear. This is the diagram. Here, this horizontal sequence is the homology exact sequence of the pair $(X^{(n)}, X^{(n-1)})$. Only a relevant portion of it is taken, the four terms $H_n(X^{(n)}), H_n(X^{(n)}, X^{(n-1)})$ (this is, by definition C_n^{CW}, now) and then the connecting homomorphism δ_n to $H_{n-1}(X^{(n-1)})$. What is the term here? It is $H_n(X^{(n-1)})$ which is 0 as shown in the previous lemma.

So, these four terms are in this horizontal line. There are two vertical lines which are also coming from homology exact sequence of the pairs, $(X^{(n+1)}, X^{(n)})$ and $(X^{(n-1)}, X^{(n-2)})$. The bottom) corresponds to $H_n(X^{(n+1)}, X^{(n)})$ and the top 0 corresponds to $H_{n-1}(X^{(n-2)})$.

By definition, the diagonal sequence is a part of our $C^{CW}_{\cdot}(X)$. C_{n+1} to C_n to C_{n-1}, with $d_{n+1} = j_n \circ \delta_{n+1}$ and $d_n = j_{n-1} \circ \delta_n$. Therefore the composite of these two homomorphisms is nothing but start from here, go vertically down here and then go to the right twice and then go down again to the right go to the right and come down. By the exactness of the horizontal line it follows that this composite is zero; $d_n \circ d_{n+1} = j_{n-1} \circ \delta_n \circ j_n \circ \delta_{n+1} = 0$ because $\delta_n \circ j_n = 0$.

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Now next thing is to prove that the kernel of this by the image of that is actually isomorphic to $H_n(X)$, ok? That is the statement here, right? $H_n(C^{CW}_{\cdot})(X) = H_n(X)$. Now look at this one, what is d_n? d_n is this composite this, I am taking the kernel of this but this is an injective mapping. Therefore kernel of d_n is the same thing as kernel of δ_n. And kernel of δ_n, by the horizontal exact sequence, is equal to image of j_n. ok? Therefore my group on the left hand side is the quotient of image of j_n by the image of d_{n+1}, right? See, j_n is an injective mapping. Now the image of δ_{n+1} is the same as $j_n(Im(\delta_{n+1}))$. Therefore, j_n induces an isomorphism of $H_n(X)/Im(\delta_{n+1})$ with $Im(j_n)/Im(d_{n+1})$. By the exactness of the first vertical sequence, it follows that the former group is isomorphic to $H_n(X^{(n+1)}) = H_n(X)$.

So, I repeat, ...

So, those things form a chain complex, the homology of that chain complex is the singular homology of X itself. ok? It is very easy to understand these chain modules you do not have to work out all continuous functions so on. ok? So, this is quite similar to what we have done for simplicial complexes, the simplicial homology. Ok? But we had to work a little harder here because we are not having the luxury of linear maps and so on. It is not just combinatorial, it is not completely combinatorial here. It retains a little bit of combinatorial nature of simplicial complexes, but it combines a lot of topological information. The most difficult thing here is to compute the boundary operators δ_n. The groups themselves are very easy to compute.

The boundary operators are determined by the attaching maps, so that is still purely topological. So, so you may say that is a catch. But even then it helps by bringing a lot of simplification from the singular chain complex of an arbitrary topological space. Ok?

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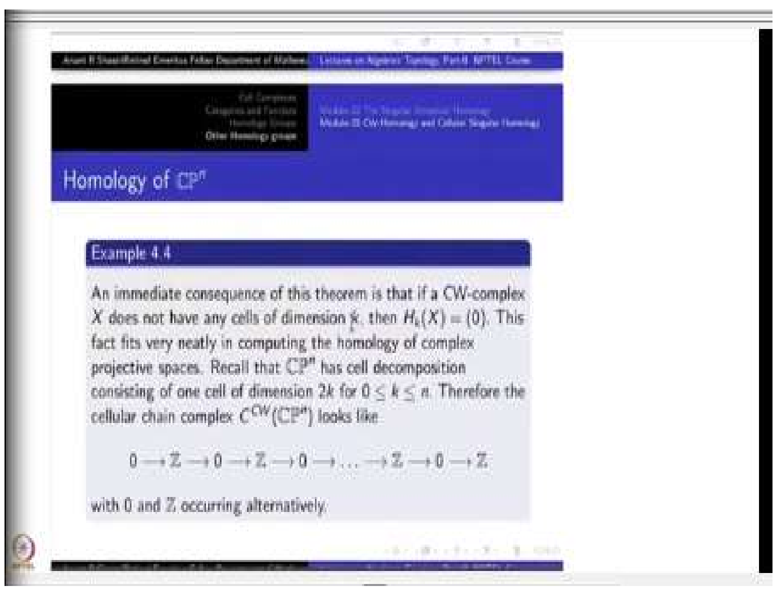

So, as an illustration let us study this example now, the homology of the complex project spaces. We have already given the complex project space a CW structure consisting of one 0-cell, one 2-cell, one 4-cell and so on, right? The odd dimensional cells are missing there, there are no odd dimension cells, alright. And in each dimension, there is exactly one cell. So what does it tell? One of the simplest thing is, more generally suppose that a CW complex X does not have any n-

simplexes for a fixed n. Then $C_n^{CW}(X) = 0$. Hence $H_n(X)$ is also zero. So, in particular, immediately you can tell that all the odd dimensional homology groups of $\mathbb{C}P^n$ are all 0. ok? There is more to come, what happens is even dimensions? In fact look at the chain complex, $C^{CW}(\mathbb{C}P^n)$. How does it look like?

At 0, there is 1 cell, so it is $C_0 = \mathbb{Z}$, there is no 1-cell, so $C_1 = 0$, there is one 2-cell that is why $C_2 = \mathbb{Z}$ and so on. Alternatively you have 0 terms and infinite cyclic groups. Therefore all the boundary maps are automatically 0. ok? The kernels will be always the whole domain group, which is alternatively 0 or $\mathbb{Z}$. The images are always 0. Therefore the homology groups are 0 or $\mathbb{Z}$ alternatively.

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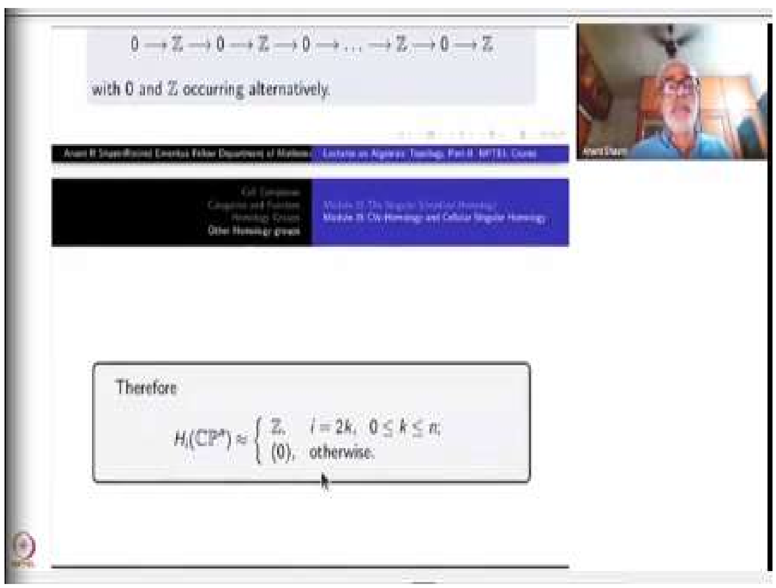

So, to sum up, we have $H_i(\mathbb{C}P^n) = \mathbb{Z}$, if $i = 2k$, and equal to 0 otherwise. A very neat result.

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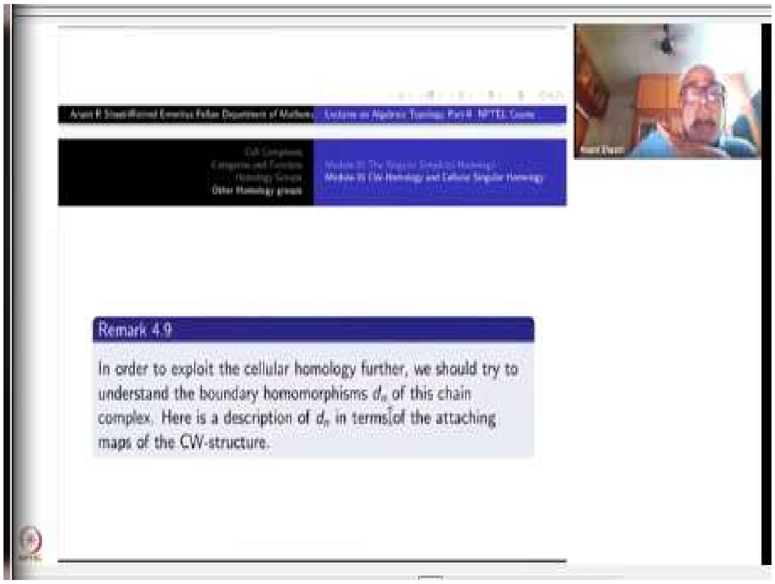

In order to exploit cellular homology further, we should try to understand the boundary operators, ok? So, here is a description, a complete description of it. But I do not have any examples right now to illustrate the use of it. My examples were too simple like complex projective space or a sphere and so on. But we will have more examples when we study the lens spaces. So let us make a preparation for that. Ok?

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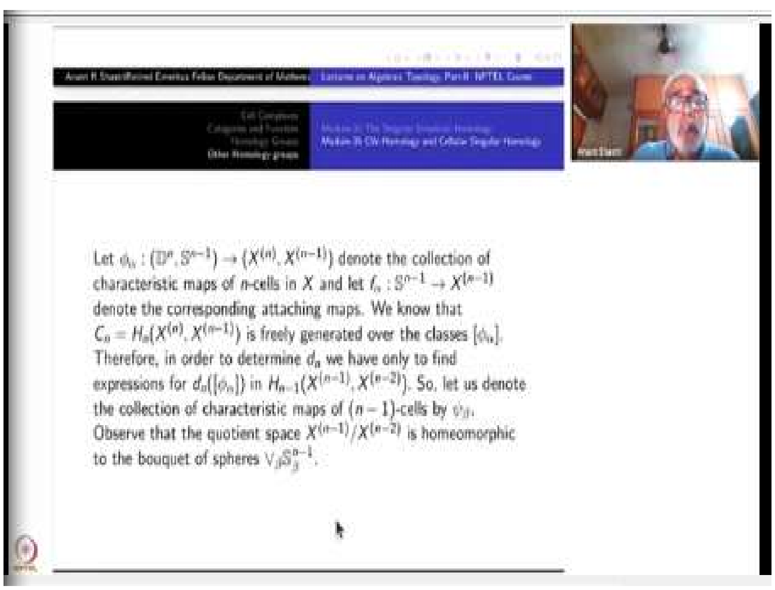

Let ϕ_α from $(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $(X^{(n)}, X^{(n-1)})$ denote the collection of all characteristic maps of n-cells. Restricted to the boundary you will get the corresponding attaching maps f_α's. I am now just writing C_n, for the group $C_n^{CW}(X)$, ok? it is $H_n(X^{(n)}, X^{(n-1)})$, the relative homology, which is freely generated over the basis $\{\phi_\alpha\}$ where ϕ_α's are maps, so they can be thought of singular n-simplexes.

An n-chain will be a finite linear combination of these things, ok, with integer coefficients. Therefore in order to determine d_n we have only to find what happens to $d_n(\phi_\alpha)$ as an element in $H_n(X^{(n-1)}, X^{(n-2)})$. So, this codomain is again a free module over the characteristic maps of $(n-1)$-cells in X.

So, let us denote the collection of all characteristic maps of $(n-1)$-cells by $\{\psi_\beta\}$. Observe that in the quotient space, $Y_{n-1} := X^{(n-1)}/X^{(n-2)}$, when you collapse the $(n-2)$-th skeleton to a single point, then all the attaching maps of $(n-1)$-cells, become the constant function. Therefore each cell $(n-1)$-cell will become a sphere, ok? The entire boundary of the $(n-1)$-cell, $\mathbb{S}^{n-2}$ is collapsed to a single point, which is the same for all β. What is that single point? It is the image of $X^{(n-2)}$, right? So, the quotient space looks like a bouquet of $(n-1)$-spheres, $\mathbb{S}_\beta^{(n-1)}$. So, bouquet of spheres indexd by β. Right?

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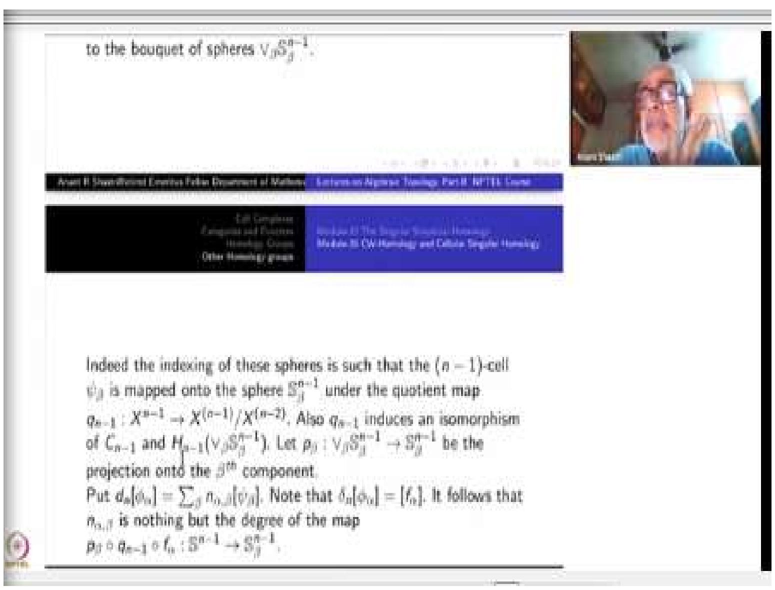

So, as you may recall from an earlier exercise, what is the homology of this bouquet? Of course $H_0(Y_{n-1})$ is $\mathbb{Z}$, since Y_n is a path connected space, so H_0 is infinite cyclic and all other homologies are 0 except $H_{n-1}(Y_{n-1})$ is again the free module over this indexing set $\{\beta\}$. And what are the generators? Take $H_n(\mathbb{S}^{n-1})$ which is infinite cyclic, take a generator g_{n-1} for it and take its image under the inclusion map $\mathbb{S}^{n-1}$ to Y_n corresponding to the β-th factor. Infact, if q_{n-1} from $X^{(n-1)}$ to Y_n is the quotient map then $q_{n-1} \circ \psi_\beta$ factors down to define a map $\tilde{\psi}_\beta$ from $\mathbb{S}^{n-1}$ to Y_n. You can take the collection $\{[\psi_\beta] = \tilde{\psi}_{\beta_*}(g_{n-1})\}$ as a set of generators for $H_{n-1}(Y_n)$. Also note that the quotient map q_{n-1} induces an isomorphism of C_{n-1} with $H_{n-1}(Y_{n-1})$. Both of them are free abelian groups with bases indexed over the same set $\{\beta\}$.

Now you look at the projection map p_β here from the bouquet Y_{n-1} to one of the spheres $\mathbb{S}^{n-1}_\beta$, p_β is identity map restricted to the component $\mathbb{S}^{n-1}_\beta$ and is the constant map on all other spheres. Ok? That is the projection map, ok? I am call them components or factors, because Y_{n-1} can be actually thought of as a subspace of the product of all these $\mathbb{S}^{n-1}_\beta$ indexed over $\{\beta\}$. Details will be left to you, Ok?

Then $d_n[\phi_\alpha]$ is anyway, a linear combination of these $[\psi_\beta]$'s, right? they are generators. So, $d_n[\phi_\alpha] = \sum n_{\alpha,\beta}[\psi_\beta]$ (sum over β), where $n_{\alpha,\beta}$ are integers. Take the sum over all β, but it is a finite sum, most of the $n_{\alpha,\beta}$'s will be 0.

So, it looks like this, but we do not know what are these integers coefficients. However, what I can tell you now is the following: If you take $\delta_n([\psi_\alpha]$, ok? Tthis will be equal to $[f_\alpha]$, the boundary of ϕ_α, ok? f_α is the attaching map of the n-cell e^n_α. Therefore, what happens to $[f_\alpha]$ under j_{n-1} is precisely what you have to understand. It follows that the coefficients. $N_{\alpha,\beta}$ is nothing but the degree of this $p_\beta \circ q_{n-1} \circ f_\alpha$, starting from $\mathbb{S}^{n-1}$ into $X^{(n-1)}$ then followed by the quotient map q_{n-1} and then followed by the projection p_β to the one single factor $\mathbb{S}^{n-1}$.

So, computing degrees of some specific maps from $\mathbb{S}^n$ to $\mathbb{S}^n$ is the make thing that will help you in understanding more complicated spaces such as CW-complexes. It will help you a little bit is knowing the singular homology of CW complexes. So, we shall give an illustration of this in computing the singular homology of lens spaces. That is our next topic. We shall do it next time, Thank you.