Introduction to Algebraic Topology (Part - II)
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Lecture - 35
CW Homology and Cellular Singular Homology

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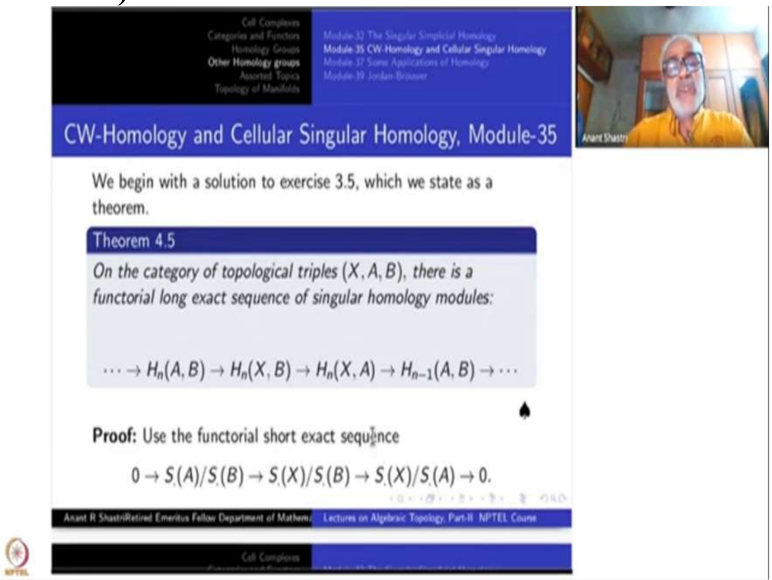

Let us introduce a new homology today and then relate it to the singular homology. This is called CW homology, just like we had simplicial homology. This is applicable only for CW complexes of course. Because simplicial complexes are also CW complexes, it will be applicable for simplicial complexes also but not for arbitrary spaces. So, before going further, let me recall a result that we have put as an exercise earlier, exercise 3.5. Now, I will state it as a theorem and indicate a proof also, which is one line proof actually.

On the category of topological triples (X, A, B) there is a functorial long exact sequence of singular homology modules, $H_n(A, B)$ to $H_n(X, B)$ to $H_n(X, A)$ and then the connecting homomorphism δ to $H_{n-1}(A, B)$ and so on. So, I have not bothered to write down what are the maps explicitly. Slowly, you will follow this practice. The context itself will tell you what the homomorphisms are.

Remember what is the meaning of a topological triple; it is just, like topological pairs, consists of two subspaces, B subset A subset of a space X. So, X is the topological space and then you are taking two subsets one inside the other, with the convention that smaller things are written later (X, A, B), means X contains A contains B. So, that is a topological triple.

They form a category in which a morphism f from (X, A, B) to (Y, A', B') is a continuous function f from X to Y taking A into A' and B into B' correspondingly.

That is the meaning of a morphism inside the category of triples. Once you have that you take the singular homology of (X, A), singular homology of (X, B), singular homology of (A, B). So these three things will fit together into a long homology exact sequence.

So this is not any mystery or anything it's a one line proof. Whenever you have a short exact sequence of chain complexes we have assigned, through snake lemma, a long exact sequence of homologies and that's what you have to do. Namely, look at $S.(B)$ subset S_.(A) subset $S.(X)$. One sitting inside other, as sub chain complexes okay? Therefore the quotient $S.(A)/S.(B)$ becomes a chain complex on its own. Similarly, $S.(X)/S.(B)$ and $S.(X)/S.(A)$. But by the usual isomorphism theorem of abelian groups $S.(A)/S.(B)$ will be a sub chain complex of $S.(X)/S.(B)$ given by the inclusion map A to X. Similarly, there is quotient map $S.(X)/S.(B)$ to $S.(X)/S.(A)$ which is induced by the inclusion A to B.

So you have such a short exact sequence. Now if you take the associated long exact sequence of homology modules, what do you get is the one stated above. So, the proof of this theorem is over. But this is what we are going to use now quite heavily. So, I did not want to leave it as unproved statement, okay?

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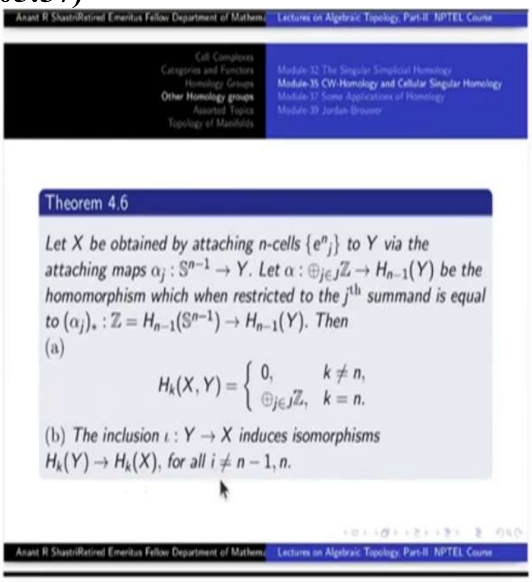

Now, as usual, instead of directly attacking CW complex, let us start with the first stage, namely, the case when X is obtained from Y by attaching only cells of a single dimension n.

The family of n-cells which could be finite or infinite, we don't care, this space X be obtained by attaching the n-cells $\{e_j\}_{j\in J}$ to Y, with the attaching maps $\{\alpha_j\}$ which are defined on the boundary of $\mathbb{D}^n$ to Y, okay?

Once you have this collection, let me put α, a homomorphism from the direct sum of copies of $\mathbb{Z}$, how many copies I have taken? As many as the indexing set here $j \in J$, okay? Take the direct sum of copies of the infinite cyclic group, as many as the indexing set J to $H_{n-1}(Y)$. Note that a homomorphism can be defined on this direct sum by defining it on each component here right? Namely, on a generator of each of these summands.

Whatever the homomorphism, which when restricted to the j-th summand here is equal to the α_{j_*} from $H_{n-1}(\mathbb{S}^{n-1})$ to $H_{n-1}(Y)$. Since α is from $\mathbb{S}^{n-1}$ to Y, when you pass on to the homology at the $(n-1)$-th level, $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$ is infinite cyclic. I am identifying this with the j-th copy here. So, $\mathbb{Z}$ which is $\widetilde{H}_{n-1}(\mathbb{S}^{n-1})$, take that homomorphism on the j-th copy. So, once you define these things α gets defined completely as the sum of all these α_{j_*}'s. So, now, this α is going to play a crucial role in the homology of X.

The first statement is that the relative singular homology $H_k(X, Y)$ is zero accepted when $k = n$, and at $k = n$, it is just the direct sum of infinitely many copies of $\mathbb{Z}$. How many copies? As many as the indexing set J, okay? That is the first part (a). The second part (b) is that the inclusion map from Y to X induces, when you pass to the homology, an isomorphism for all $i \neq n-1$ and $i \neq n$. And at $k = n-1$ and $k = n$, there are some disturbances because of the attaching cells, we have some other information, given below in the third and the fourth statements.

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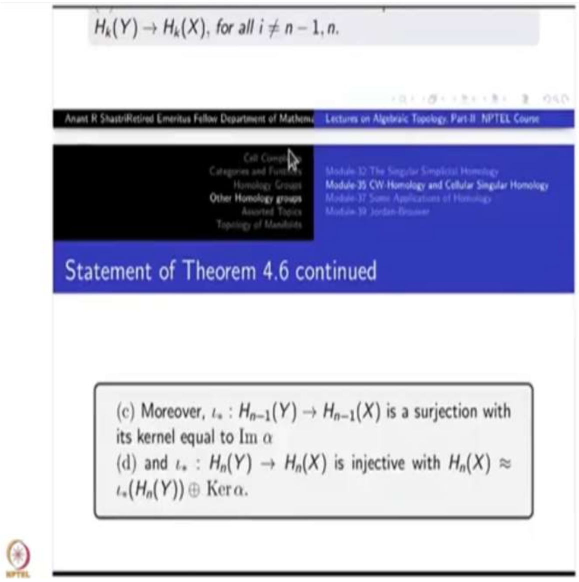

The third statement (c) is that i_* from $\tilde{H}_{n-1}(Y)$ to $\tilde{H}_{n-1}(X)$ is a surjection and its kernel is precisely equal to the image of α. This kernel of i_* will be sub group of this this one which is equal to the image of α.

Similarly, statement (d) says that for $k = n$, i_* from $H_n(Y)$ to $H_n(X)$ is injective, Okay? and $H_n(X)$ itself is isomorphic to the direct sum the image of i_* with the kernel of α.

So, this gives you a complete picture of the homology of X okay, in terms of the homology of Y and the attaching functions. Since we are attaching cells of a fixed dimension n, attaching maps have a role to play here in the homology at exactly at $n-1$ and n-th level. Elsewhere it does not change. That is because the reduced homology of $\mathbb{S}^{n-1}$ is non-trivial only as $(n–1)$-th level, okay? So, look at the induced maps α_{j_*}. They will have some say in the homology of X, finally, okay?

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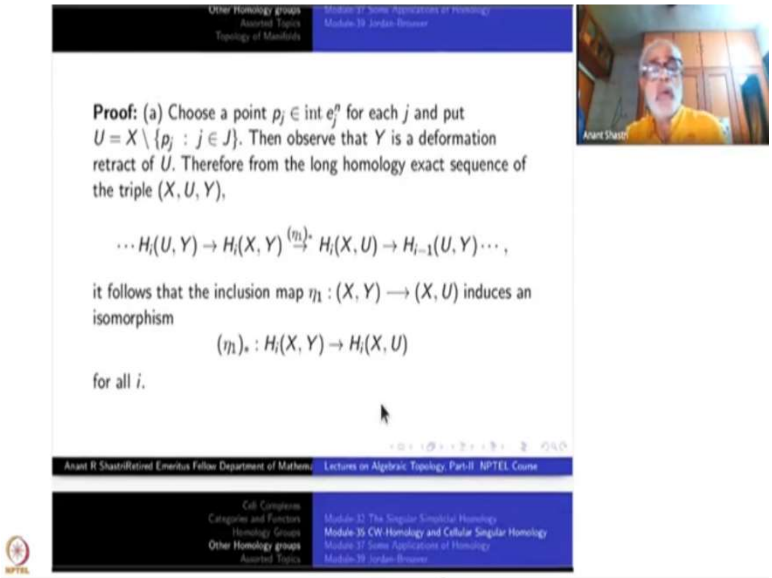

So, let us go to the proof of this one. Proof of (a): For each j, choose a point p_j in the interior of the cell e_j^n, for example, p_j may correspond to 0 inside $\mathbb{D}^n$, under a homeomorphsim from $\mathbb{D}^n$ to e_j^n, okay? For each j, you take that point and put $U = X$ minus all these points. $X \setminus \{p_j, j \in J\}$. Throw away all the points p_j from X. Then U can be strongly deformed to Y, i.e., Y subset U is a strong deformation retract, because each of these $e_j^n \setminus \{p_j$ can be deformed to the image of $\alpha_j \in Y$. So, Y is a deformation retract of U.

Now look at the long exact sequence of the homology of the triple (X, U, Y). We have studied this for the triple (X, A, B) right? Instead of A and B, put U and Y. What do you get? $H_i(U, Y)$ to $H_i(X, Y)$ to $H_i(X, U)$, these are inclusion induced morphisms. If η_1 from (X, Y) to (X, U) is the inclusion map then $(\eta_1)_*$ is an isomorphism for on all H_i. This is so because, Y is a strong retract retract of U and hence the inclusion induced morphism $H_i(Y)$ to $H_i(U)$ is an isomorphism for all i and hence we can use the long homology exact sequence of the pair (U, Y) and the five lemma. Therefore, $H_*(X, Y)$ can be computed by computing $H_*(X, U)$. Alright?

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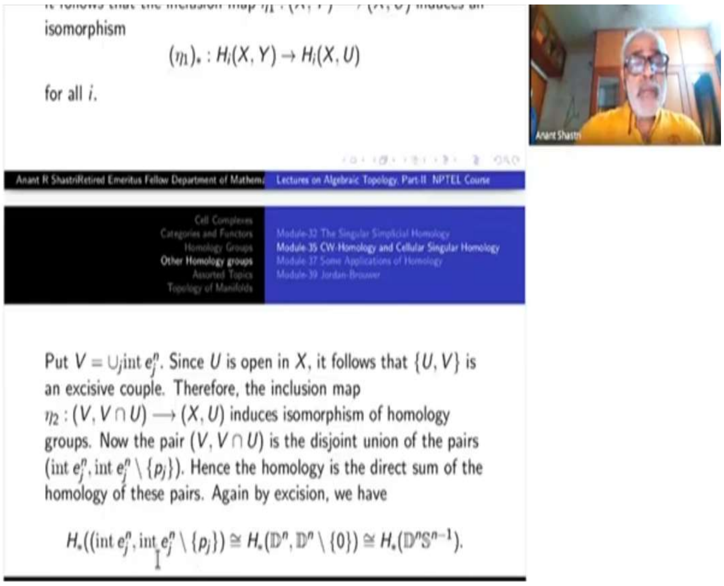

Now, let us put V equal to the disjoint union of the interior of all the n-cells. Then U, V are open in X and $X = U \cup V$. So, $\{U, V\}$ is an excisive couple. Therefore, the inclusion map $(V, V \cap U)$ to (X, U) induces isomorphism in homology groups. This is from the excision theorem okay.

Now, look at the pair $(V, V \cap U)$. V is a disjoint union of all these open cells and $V \cap U$ is obtained by throwing away all the center points p_j's, one of the interior points okay? So, this is a disjoint union of pairs (interior of e_j, interior $e_j \setminus p_j$). Therefore, the homology of $(V, V \cap U)$ is the direct sum of homology of $(inte_j^n, inte_j^n \setminus p_j)$. But of these pairs is homeomorphic to $(int\mathbb{D}^n, int\mathbb{D}^n \setminus p_i)$. Again by excision, these are isomorphic to the homology of pair ($\mathbb{D}^n, \mathbb{S}^{n-1}$), which we have computed earlier. They are 0 except for $i = n$ and $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ is infinite cyclic, okay? So, that completes the proof of (a).

So, let's complete the other three statements which are all easy now; (b), (c) and (d) are easy consequences of long exact sequence of the homologies of the pair (X, Y), okay? (I have not written down this on the slides). If you look at $H_i(X, Y)$, it is 0 unless $i = n$ and therefore, it follows that the inclusion induced homomorphism $i_* H_i(Y)$ to $H_i(X)$ is an isomomorphism for $i \neq n - 1$ and $i \neq n$. Moreover, the same reason also tells you that for $i = n - 1, i_*$ is surjective and for $i = n$ it is injective. The remaining parts of (c) and (d) follow from the observation that the connecting homomoprhism δ from $H_n(X, Y)$ to $H_{n-1}(Y)$ can be idenitified with the homomorphism α via the statement (a).

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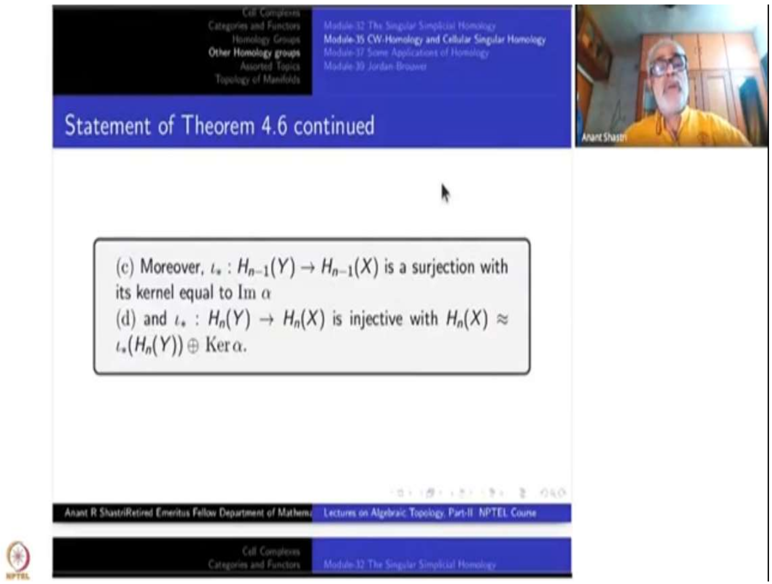

Indeed, for each j, we have take the composite of the connecting homomorphism δ from $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $H_{n-1}(\mathbb{S}^{n-1})$ and α_{j_*} from $H_{n-1}(\mathbb{S}^{n-1})$ to $H_{n-1}(Y)$ and take their sum to get the connecting homomorphism δ from $H_n(X, Y)$ to $H_{n-1}(Y)$.

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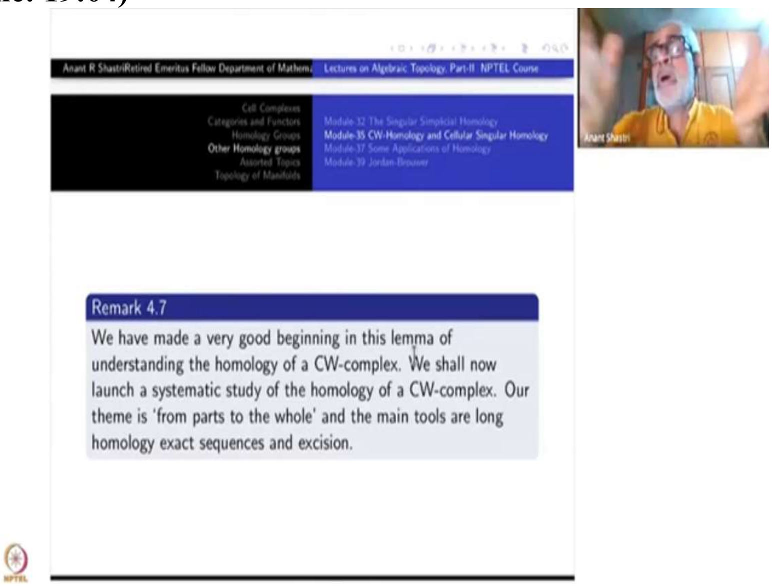

So what we have done so far is the first step in computing the homology of CW complexes. We shall now launch a systematic study of homology of CW complexes. Recall our theme of building thing from parts to the whole. The main tool so far was just the long homology exact sequences, and the excision theorem, okay? We will keep using them again and again. Then we shall have the definition of CW chain complex. May be will not be doing all that today. Let us proceed with a lemma first.

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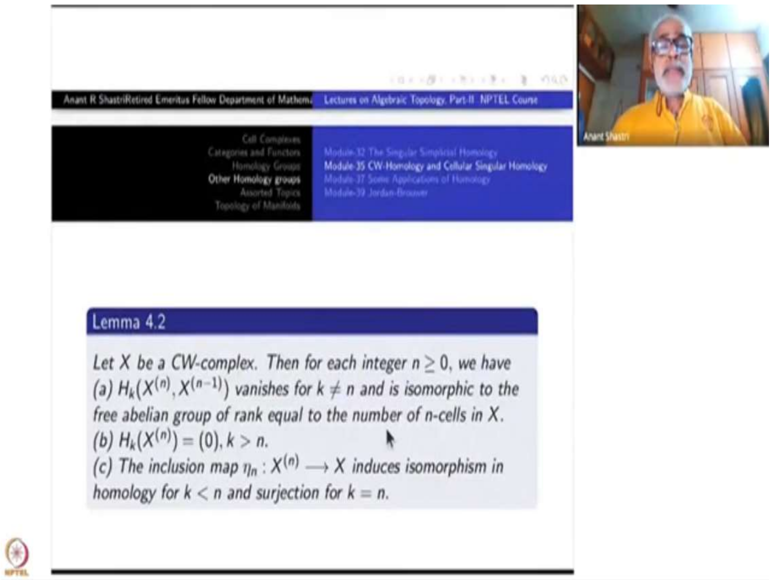

Let X be CW complex.

(a) For each integer $k \geq 0$, we have the k-th homology of the pair (n-th skeleton of X, $(n-1)$ -th skeleton of X) vanishes for $k \neq n$, and for $k = n$, it is isomorphic to the free abelian group of rank equal to the number of n cells in X.

(b) $H_k(X^{(n)}) = 0$ for $k > n$.

(c) The Inclusion map X^n to X induces an isomorphism in homology for $k < n$ and surjection for $k = n$.

All these things are easy consequences of what we have done in the earlier lemma, wherein the cells attached were of a single dimension. For each fixed n, $X^{(n-1)}$ plays the role of Y here, okay $X^{(n)}$ plays place lays the role of X, because $X^{(n)}$ is obtained by $X^{(n-1)}$ by attaching n-cells. Therefore, the k-th homology of the pair will be 0 for $k \neq n$, okay. And for $k = n$, it is isomorphism is the free abelian group of rank equal to the number of n-cells in X. That is also that's also part (a) here which is part (a) of the previous lemma, which actually says more.

Second one says, $H_k(X^{(n)})$ is 0 for $k > n$. This is an easy consequence of part (a) by induction. For $n = 0$, since $X^{(0)}$ is a discrete we know that H_k vanishes for $k > 0$. Assuming this result for $(n-1)$-th skeleleton you employ the long exact sequence for the pair $(X^{(n)}, X^{(n-1)})$ to conclude the result for $X^{(n)}$.

The third statement says that the inclusion of map η from $X^{(n)}$ to X induces isomorphism in homology for $k < n$ and a surjection for $k = n$. Let us take the case $k < n$ first. What happens inclusion map induces an isomorphism from $H_k(X^{(n)})$ to $H_k(X^{(n+1)})$. Same holds for all inclusion maps $X^{(n+i)}$ to $X^{(n+i+1)}$ as well.

Since composite of injective maps is injective you may just say the η_* is also injective. This will be valid if X is finite dimensional. However, in the general case you have to argue a little more, because composite of infinitely many maps will be involves. Suppose η_* is not injective. That means there is some element $[c] \in H_k$ here, represented by a k-cycle, such that $\eta(c)$ is a boundary inside X, i.e., there is $(k+1)$-chain sigma in X such that $\partial(\sigma) = c$. Being a finite sum of $(k+1)$-singular simplexes, it follows that sigma is contained in $X^{(n+r)}$, for some large r. But then the relation $\partial[\sigma] = c$ is valid inside $X^{(n+r)}$ which is finite dimensional. So, that is not possible. Thus for $k < n, \eta_*$ is injective.

For $k = n$, this injectivity fails only at the very beginning $H_n(X^{(n)})$ to $H_n(X^{(n+1)})$. Even here, this map is a surjection, from part (c) of the previous lemma. Therefore arguing similarly as above, we can conclude (c). So, let us stop here. Let us use this one next time to give a complete picture of the homology of CW complex. And then out of that, we are cooking up another chain complex, of course, for this X itself, which is CW complex. Okay, The homology of that will be isomorphic to the singular homology of X. That is the beauty of this new chain complex. Okay? Thank you.