Introduction to Algebraic Topology (Part-II)
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Lecture - 34
Simplicial Homology – Continued

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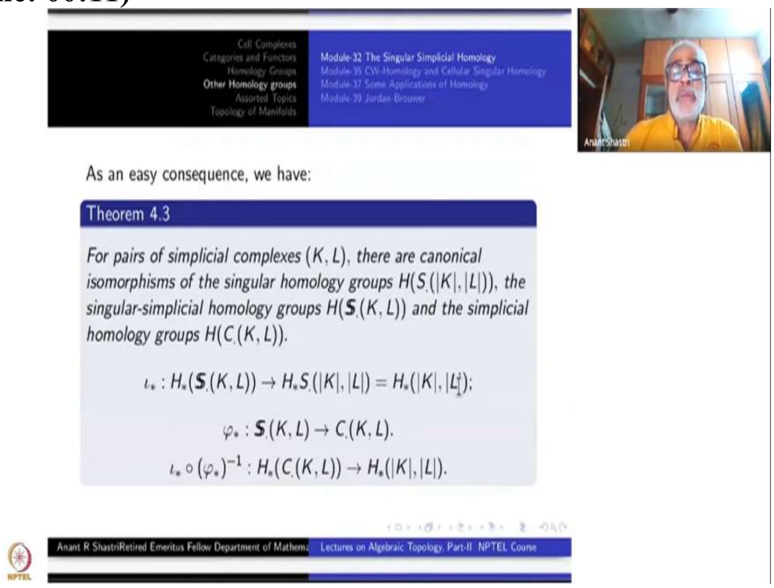

Last time we introduced the simplicial chain complex as a sub quotient of the singular chain complex of the underlying topological space. We stated the theorem namely the inclusion map from singular simplicial chain complex to the entire singular chain complex induces isomorphism i_* on the homology; and similarly the quotient map from the singular simpliial chain complex to ordinary simplicial chain complex also induces isomorphism ϕ_* on homology. Together if you take $\phi^{-1} \circ i_*$ that will be an isomorphism from the simplicial homology to the singular homology of the underlying topological pairs. This has many many consequences now. So, we will continue the study of this one today, postponing the proof of this theorem for a while.

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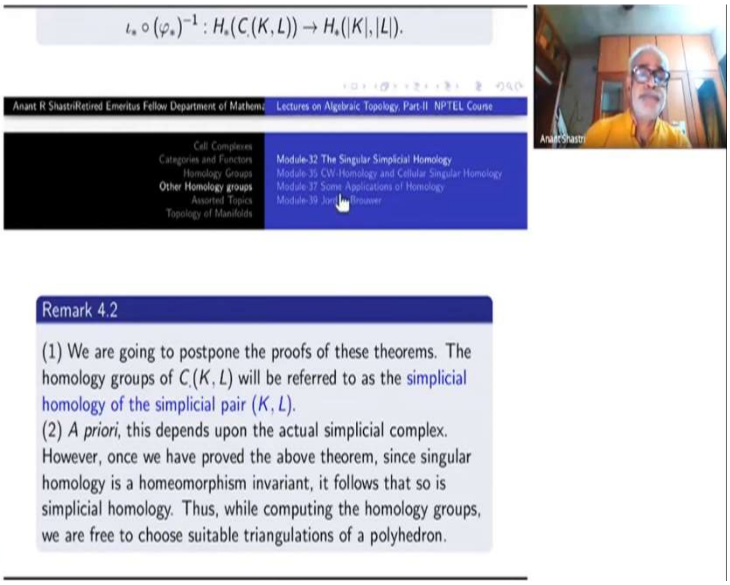

First of all the comment that I made last time, I would like to repeat it, namely, when you take a triangulation, the simplicial homology will seemingly depend upon the triangulation as such. However, once we have established the above theorem, we see that the simplicial homology is the same as the singular homology of the underlying space. Therefore, it is independent of what triangulation you have taken. The only thing that we have to be careful is that an arbitrary topological space may not have a triangulation at all. If it has a triangulation, then we can choose any triangulation and compute the associated simplicial homology. That will give you the singular homology as well. So this has a tremendous application now.

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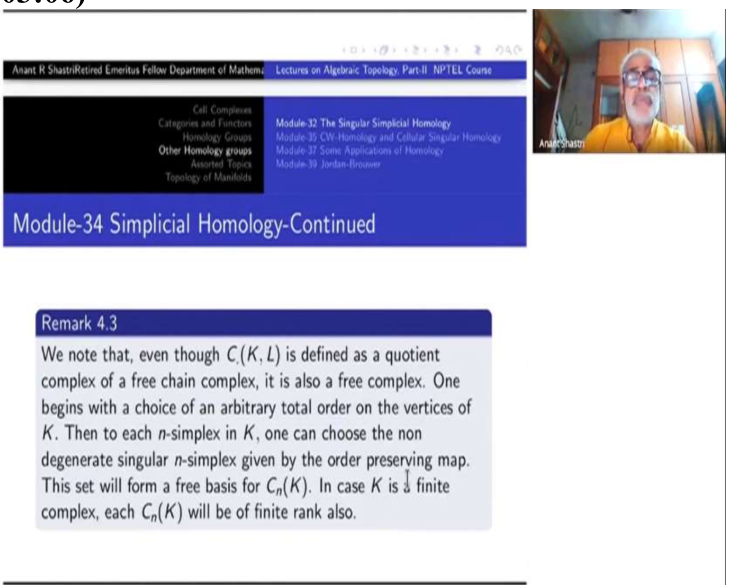

So one by one we will try to recap some of these. The first thing is $C.(K,L)$ is defined as a quotient complex of a free chain complex, namely, the quotient of singular simplicial chain compelx. However, even the quotient is also a free chain complex. So, one begins with a choice of an arbitrary order of vertices of the simplicial complex K. Then each n-simplex in K gets an induced order. One can choose the non degenerate singular n-simplex given by an order preserving map from Δ_n to the vertices of K. For each n-simplex, there is only one order preserving map. The collection of all such maps forms a free basis for $C_n(K)$. So, that way we get that $C_n(K)$ is also a free abelian group. If K is finite, then each $C_n(K)$ will be of finite rank also. In particular, if $|K|$ is compact, then we know that K is finite. So, these are the few things you have to keep in mind.

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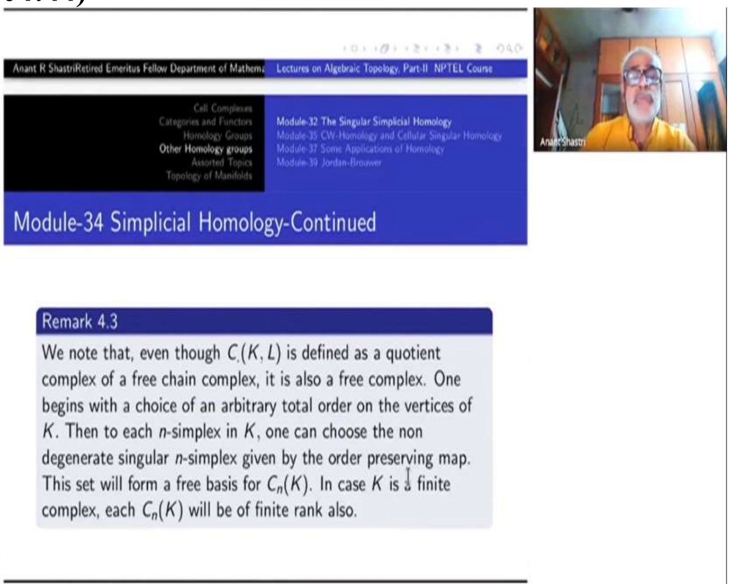

Moreover, what happens the boundary homomorphisms? They are given by just the incident matrices. These matrices are defined using the above bases as follows: First label the basic elements of C_n and C_{n-1} as, $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $\beta_1, \beta_2, \ldots, \beta_t$, respectively. Then ∂_n from C_n to C_{n-1} is given by a $t \times s$ matrix whose $(i, j)^{th}$ entry is 1 or 0 according as β_i is a face of α_j or not.

The boundary operator ∂_n is completely determined by this matrix. So, this way you can computerise the entire information on the simplicial complex and if programmed properly, in than five seconds, the computer will give you the homology groups of $|K|$.

So, this is the way computers have entered in topology to begin with during 80s. But now it has become a very huge industry.

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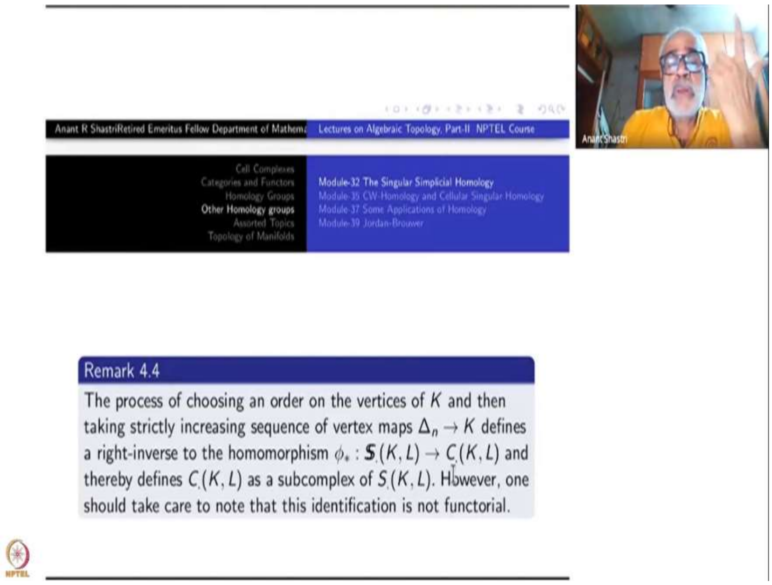

In the process of choosing an order on the vertices of K, you have to be cautious, because as soon as you choose an order it is your choice somebody else may choose something else and so on, functoriality will be lost. As far as computation is concerned, both will arrive at the same result, no problem. But funtorality will be lost. So, in applying the theories you have to be careful since you are choosing an order on the vertices.

Taking strictly increasing sequence of vertex maps defines a right inverse to the quotient homomorphism ϕ from double of $S.(K, L)$ to $C.(K, L)$ and thereby $C.(K, L)$ as sub complex of double of $S.(K, L)$. However, I want to tell you to take note that this identification is not funtorial you can identify it as a sub complex but somebody else may identify with some other sub complex and so on. There are problems of that nature.

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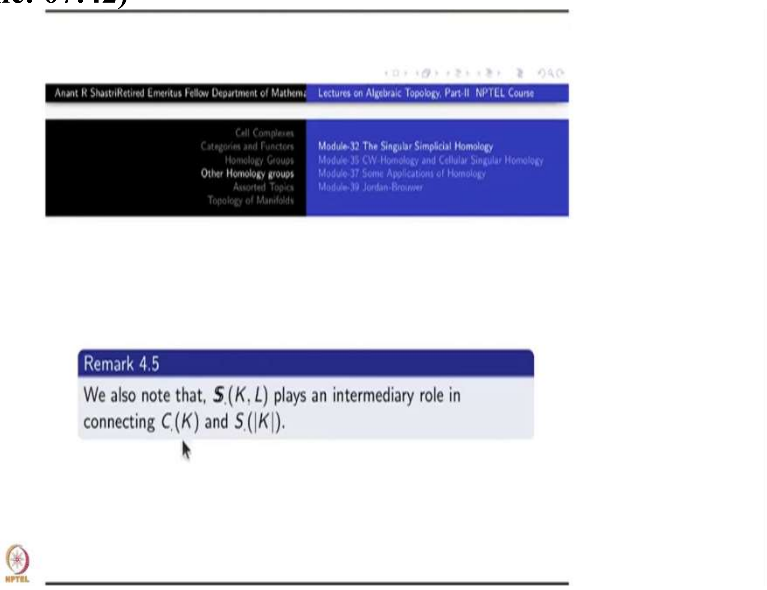

So we have to be careful we also note that double of $S.(K, L)$ plays an intermediary role in going from $C.(K)$ to $S.(|K|)$. We do not and cannot go directly.

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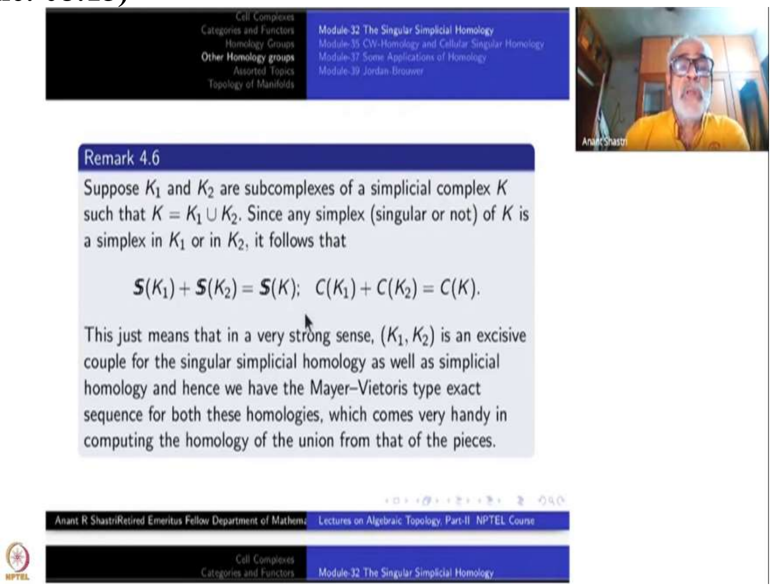

Now, suppose K_1 and K_2 to are sub complexes of a simplicial complex K, such that the union is the whole of K. Since any simplex (singular or not) of K is a simplex in K_1 or a simplex in K_2 because $K_1 \cup K_2$ is the whole of K, it follows that double of $S.(K_1)$ plus double of $S.(K_2)$ is the whole of double of $S.(K)$. Similarly, $C.(K_1) + C.(K_2)$ is the whole of $C.(K)$. For this much we do not need that K_1 and K_2 are sub complexes. But for what we want to say next, viz., about excision, we need to assume K_i are subcomplexes of K so that double $S_0(K_i)$ are actually sub-chain complexes. That is all.

Remember that we had to assume certain conditionsons on the subsets such as openness etc., for excision theorem for singular homology. With simplicial complexes, you do not need any such topological conditions. You have this subchain complex in the in general, but it is actually equal in this case $K_1 \cup K_2$ is the whole of K. What we wanted is that the inclusion induced morphism on H_* of this one to H_* is an isomorphism. But here the inclusion map itself is actually equality and hence on H_* we get the identity isomorphism here. So, the beauty is that immediately, without any verifying any conditions for excision. Therefore, Mayer-Vietoris etc., can be applied whenever you have two sub complexes spanning the whole of K. So, this will be very useful now, for both double S_* as well as C_*.

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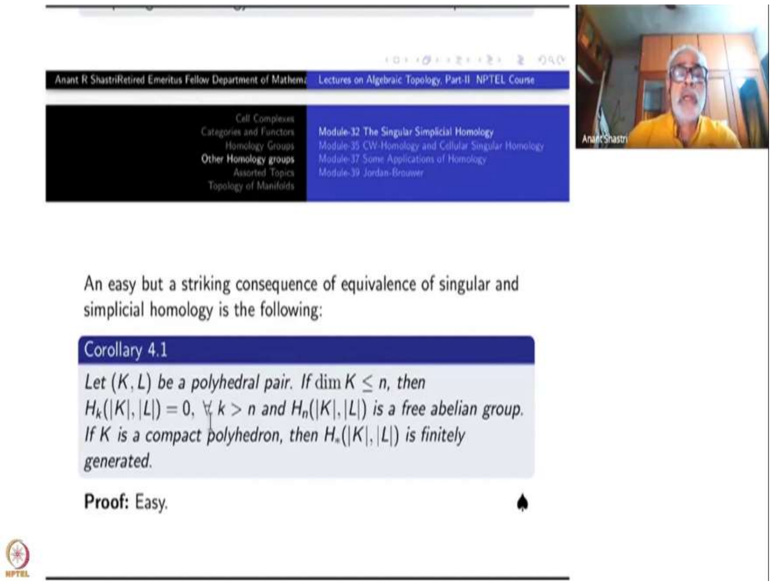

An easy but a striking and very useful information which can be derived from the equivalence of singular and simplicial homology is the following: Suppose you start with a polyhedral pair (K, L). That just means that K is a simplicial complex and L is a sub complex. Suppose the dimension of K less than or equal to n. So, this is the only condition, we are not assuming that K is finite or anything but just that the dimension of K is finite.

Then the homology of the pair $(|K|, L|)$ vanishes beyond the dimension. The n-th homology $H_n(|K|, |L|)$ may survive but is a free abelian group. Why is it free abelian. Because $H_n(C.(K, L))$ is the quotient of $ker\partial_n$ divided by the image of ∂_{n+1}, but $C_{n+1} = 0$ and hence H_n is actually a subgroup of the free abelian group $C_n(K, L)$. This is the beauty of using $C_*(K, L)$, rather than $S_*(|K|, |L|)$.

Further if K is a compact polyhedron then C_* will be finitely generated, that is what I already told you, and hence $H_*(K, L)$ is finitely generated. So, this information, though it is very easy, I mean it is not a difficult theorem, this will be very important.

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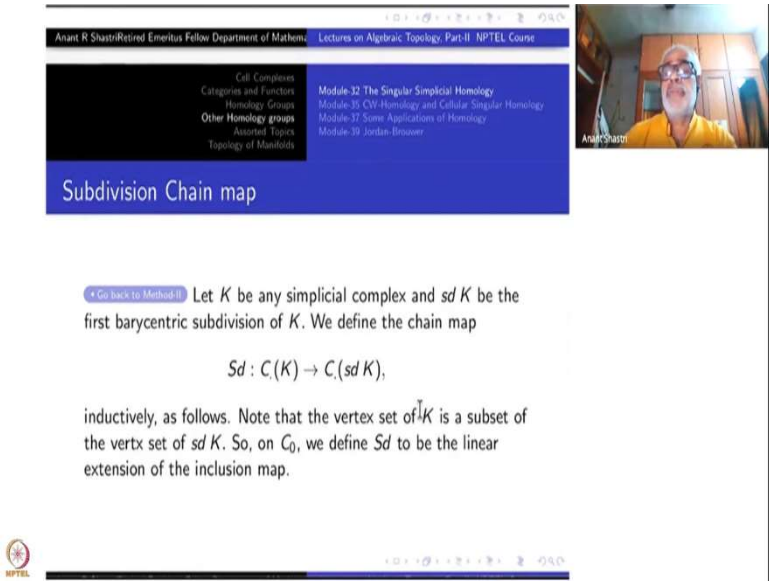

In this context let us study the role of the barycentric subdivision. Remember barycentric subdivision was used in simplicial complexes to get one of the very important theorems namely, simplicial approximation. So, so let us understand whether barycentric subdivision has any role to play in homology. So, start with a simplicial complex K and let $sd(K)$ denote its barycentric subdivision. This little sd denotes a subdivision as before.

Now capital Sd, I am going to define it as a chain map from $C.(K)$ to $C.(sdK)$. This is done inductively as follows. At 0-th level, C_0 is a free abelian group on the set of vertices of K and $sd(K)$ respectively. What are the vertices of $sd(K)$? It includes all the vertices of K and there are some additional vertices. They are all barycentres of various simplices in K. So, they will all be vertices is $sd(K)$. So, there are extra vertices here. So, from vertex set of K to vertex set $sd(K)$, there is the inclusion map. Extend this linearly to get Sd_0 from $C_0(K)$ to $C_0(sdK)$.

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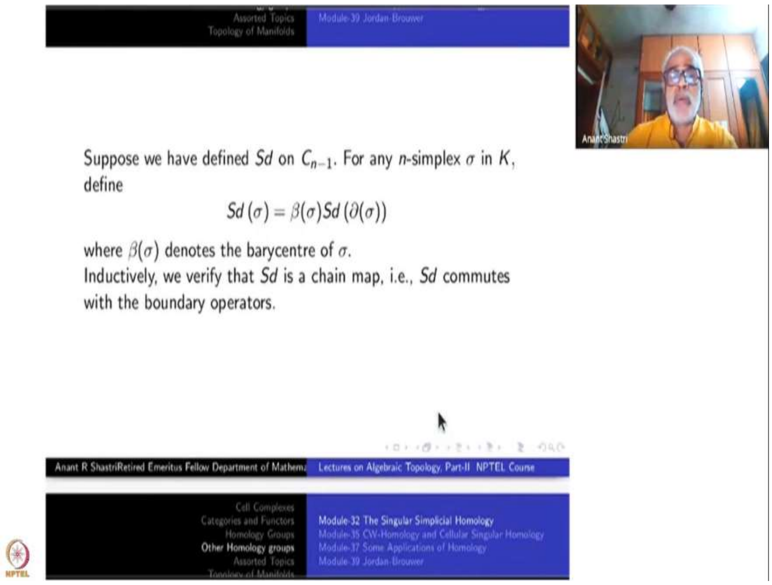

Now, suppose we have defined Sd_{n-1} on C_{n-1}. Then I want to define it on C_n. Sd is not only a homomorphism, but we want it to be a chain map. First let us define it has a homomorphism. So, if I define Sd on any n-simplex σ in K, then I can extend it linearly to whole of C_n, because C_n is a free abelian group over the n-simplexes of K.

Put $Sd(\sigma)$ to be equal to $\beta(\sigma)Sd(\partial(sigma))$, the cone construction. Note that by induction, Sd of the boundary of σ is a $(n-1)$-chain which is in C_{n-1} of the subcomplex σ. And $\beta(\sigma)$ is also a vertex in $sd(\sigma)$. Hence the cone construction make sense in $sd(\sigma)$ itself.

Next, we have to verify that Sd is a chain map. This will also be done inductively. I have to verify that it commutes with the boundary operators. So, at 0-th level there is nothing to bother. Because beyond that there all the groups are 0.

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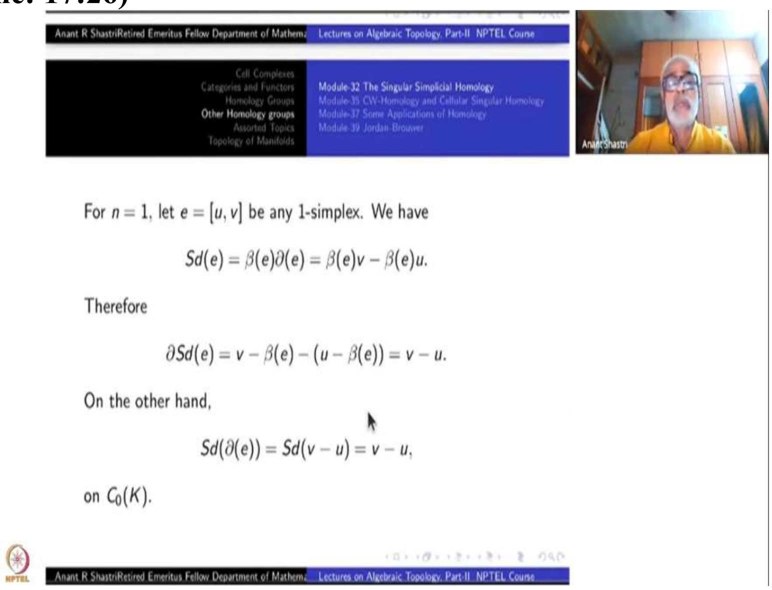

So, the induction starts at $n = 1$. So, let us just see how it looks like at $n = 1$. Let us look at one of the edges $e = [u, v]$, that is a 1-simplex. We have, by definition $Sd(e) = \beta(e)Sd(\partial(e)) = [\beta(e), v] - [\beta(e), u]$.Therefore,
$\partial(Sd(e)) = v - \beta(e) - (u - \beta(e)) = v - u$, the term $\beta(e)$ gets cancelled out.

What are definition of $Sd(\partial(e)) = Sd(v - u)$. But at the 0-th level, Sd is just the inclusion map. Therefore the chain-map condition is verified for $n = 1$. Now suppose we have to done it up to $n - 1$. Then we can verify it for n.

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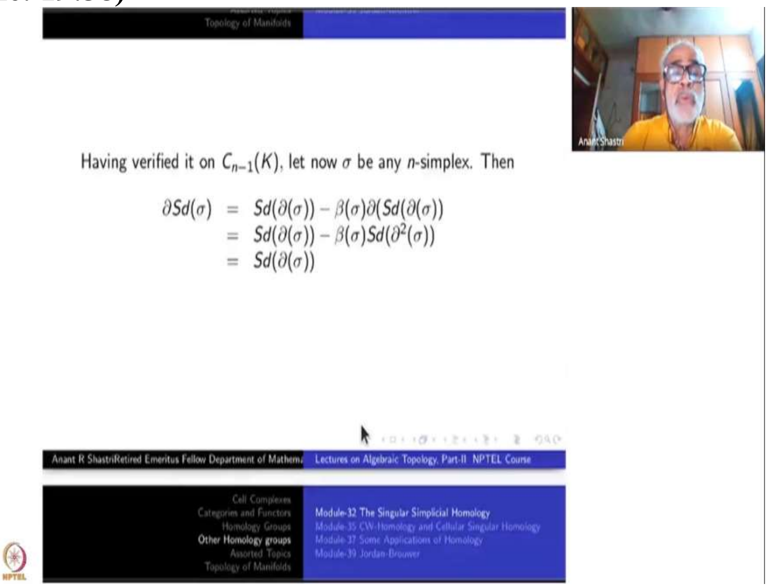

Let now σ be any n-simplex. Then, boundary of $Sd(\sigma)$ is the boundary of $\beta(\sigma)Sd(\partial(\sigma))$ which is equal to $Sd(\partial(\sigma)) - \beta(\sigma)\partial(Sd(\partial(\sigma)))$. This by the definition of ∂ on the cone, we first drop $\beta(\sigma)$, what you get is this first term. Next, you keep this one and take the boundary of that one.

But in this second term, by induction hypothesis, $\partial \circ Sd = Sd \circ \partial$, therefore you get Sd of boundary square of σ and the boundary square of anything is 0. Therefore, this drops out what you are left with is Sd of boundary of σ. So that verifies the chain-map condition at n. So, Sd from $C.(K)$ to $C.(sdK)$ is a chain map.

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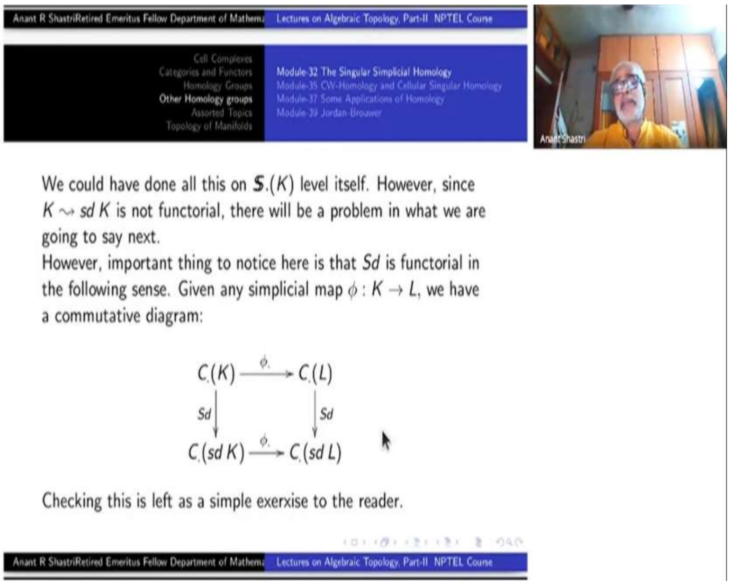

All this can also be done with the singular simplicial chain complex as well. An important thing to notice is the functoriality. If we have a simplicial map ϕ from K to L, recall that we have simplicial map $sd(\phi)$ from $Sd(K)$ to $sd(L)$ defined by sending barycenter $\beta(\sigma)$ to the barycenter $\beta(\phi(\sigma))$. Therefore we have a commutative diagram as shown when you pass to $C.(K)$ to $C.(L)$ to $C.(sdK)$ to $C.(sdL)$. On the top it is $\phi.$, and in the bottom it is $sd(\phi)$.

Details are left to the reader as an exercise, so that you will carefully go through what is happening here.

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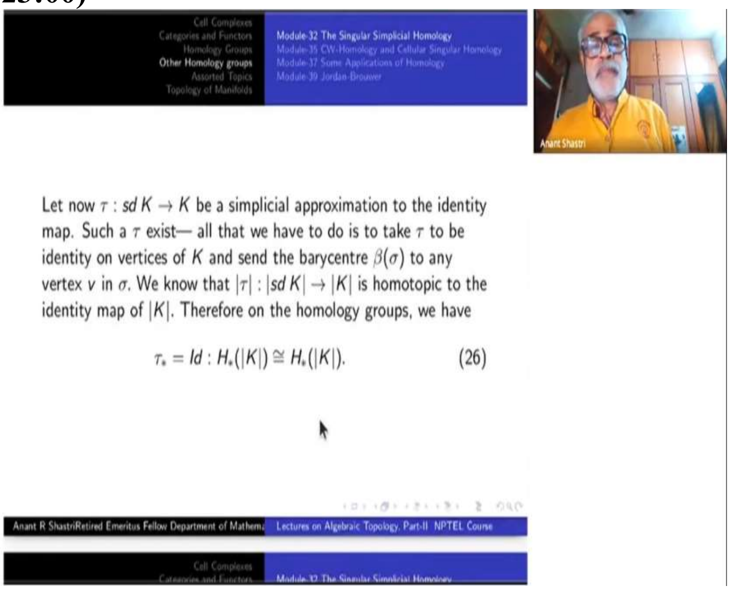

Now, let us continue to examine what is this good for, this the barycentric chain map Sd. Let τ from sdK to K be a simplicial approximation to the identity map. See $|SdK|$ is the same space as $|K|$, so there is the identity map. It can be simply approximated by just taking any subdivision of the domain, all that you have to do is to send the extra vertices v of the subdivision to one of the vertices of the old simplex F such that v is in $|F|$. In particular, such a τ exists. All that you have to do is to take τ to be identity on the vertices of K and send $\beta(\sigma)$ to any one of the vertices of σ. That is all. There are many choices of course. Such a map will be a simplicial approximation to the identity map.

Therefore it follows that the induced continuous map $|\tau|$ is homotopic to the identity map. In particular, when you pass to the homology, the induced morphism will be actually the identity map. So, this information has something to play with the chain map Sd which is from this here to here on $C.(K)$ to $C.(sdK)$. That is what we are after.

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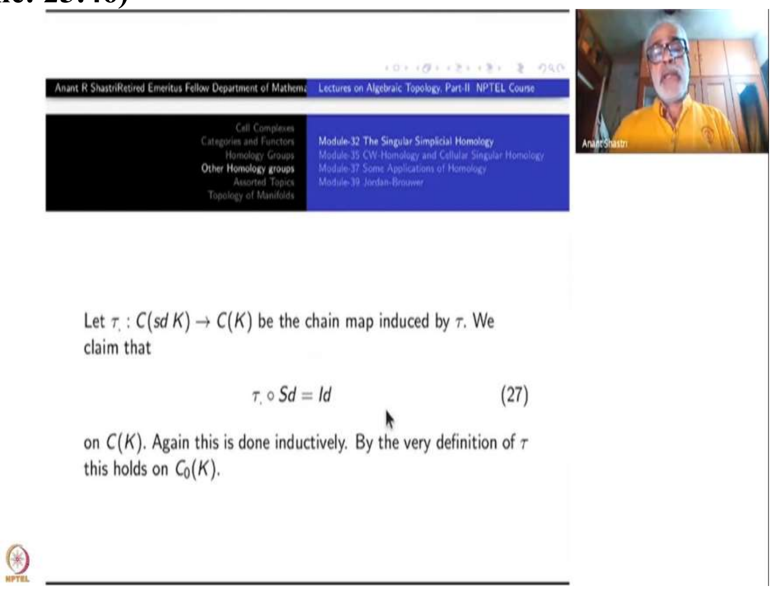

Let $\tau.$ from $C.(sdK)$ to $C.(K)$ be the chain map induced by τ. After all, τ is from $Sd(K)$ to K is a simplicial map. We actually claim that $\tau. \circ Sd$ is the identity map of $C.(K)$. Thus the barycentric subdivision chain map is a right inverse to $\tau.$ or barycentric subdivision chain map has a lot of left inverse.

So, this also will be done by induction. On C_0, Sd is just an inclusion map and τ restricted to vertices of K is identity map. So the claim is obvious at C_0 level.

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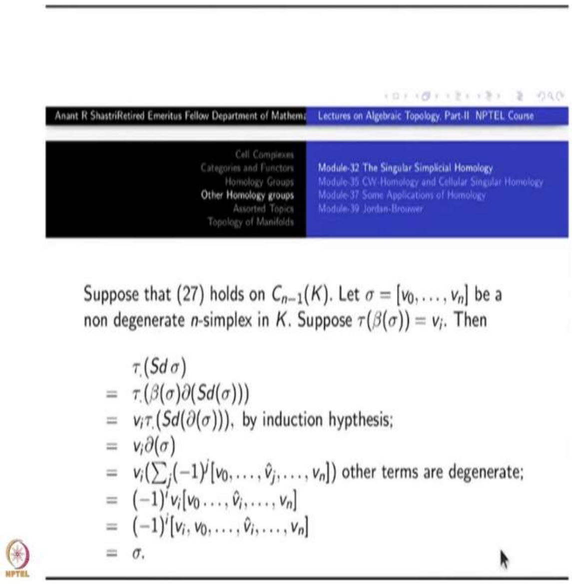

Suppose this claim holds on $C_{n-1}(K)$. Let $\sigma = [v_0, \ldots, v_n]$ be a non degenerate simplex in K. I want to verify $\tau(Sd(\sigma)) = \sigma$. We have, $Sd(\sigma) = \beta(\sigma)(Sd(\partial(\sigma))$. Now where does $\beta(\sigma)$ go under τ? To one of these vertices, say v_i. I do not know which vertex because that depends upon the choice of τ.

Since τ is a simplicial map, and $\tau.$ is a linear extension of τ, it follows that $\tau.(\beta(\sigma)(Sd(\partial(\sigma))) = v_i\tau.(Sd(\partial(\sigma)))$.By induction, this is equal to $v_i\partial(\sigma)$. Now, I am writing the formula for boundary of σ, it is $\sum(-1)^j[v_0, \ldots, \hat{v_j}, \ldots, v_n]$. Now look at what happens if $j \neq i$.

Then, v_i will be repeated and so that will be a degenerate simplex. We can throw all of them away. In $C.$ they are always 0. So, the only term that survives is when $j \neq i$. That term is equal to $(-1)^i v_i[v_0, \cdot, \hat{v_i}, \cdot, v_n]$ which is nothing but σ.

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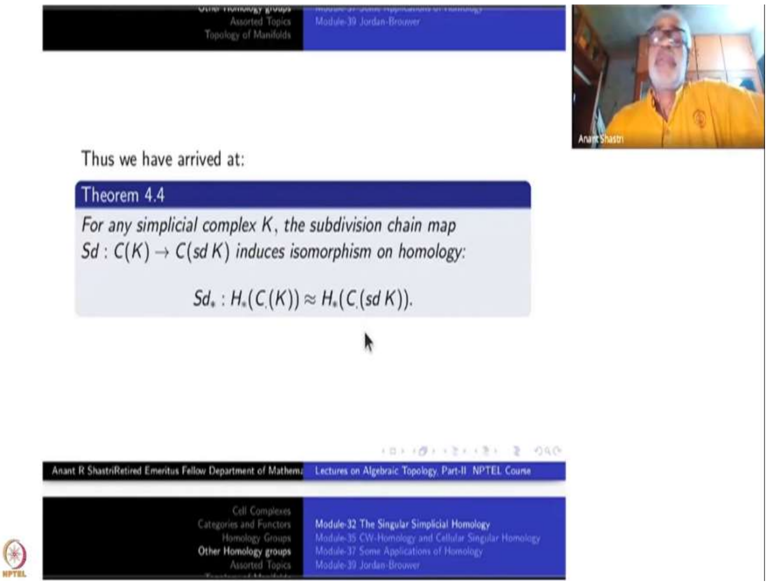

Thus we have arrived at one of the important result. Here, namely, the subdivision chain map induces an isomorphism on H_* how do you get this one? Tell me. Because Sd is the right inverse of τ here when you pass on to the homology $\tau_* \circ Sd_*$ will be identity, but we know τ_* is an isomorphism so, any right inverse of it is also an isomorphism.

At the chain level, there is no isomorphism as such except at the 0th level. This right inverse τ_0 is the only isomorphism. But when you pass to the homology, τ_* is an isomorphism being homotopic to the identity map is actually the identity isomorphism on singular homology. Note that at the chain level, the domain and the codomain of Sd are quite different chain complexes. But at the homology, we have seen that they are the same. homology, the singular homology of $|K|$. To say that Sd_* is actually the identity map requires a number of identifications here.

What do you know is this is an isomorphism, and it is a canonical isomorphism because Sd itself is functorial.

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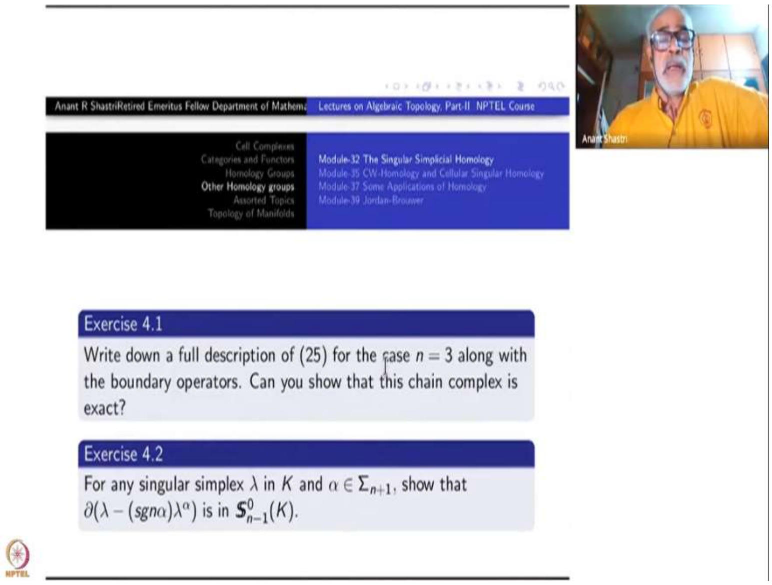

So we have some exercises here for you. These are not very difficult exercises. I hope you try them before we proceed further, so that you get familiar with the stuff covered so far. Let us stop here today.