Introduction to Algebraic Topology (Part-II)
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Lecture - 33
Simplicial Homology

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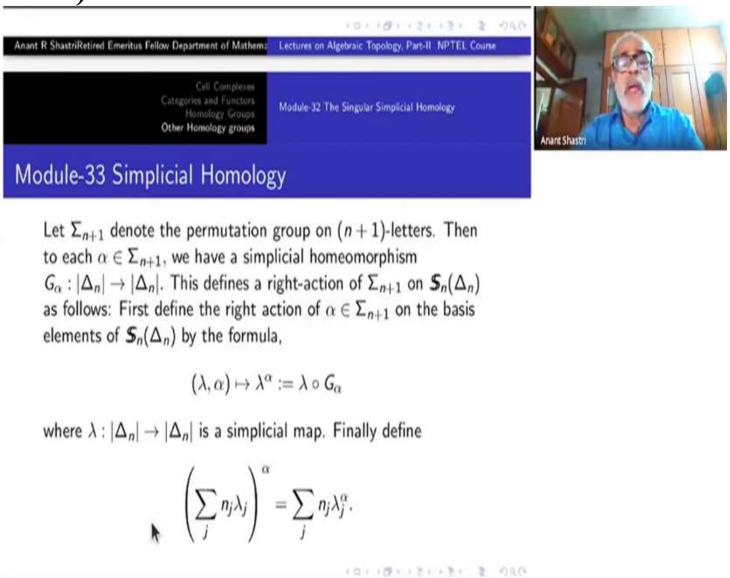

Having introduced the singular simplicial chain complex for a simplicial complex K as a sub complex of the singular chain complex of $|K|$, we shall now go for another simplification or further simplification. This time we are not going to get it as a sub complex but as a quotient complex which is the natural way of doing it. So, Σ_{n+1} denote the permutation group on $n+1$ letters. For each α inside Σ_{n+1}, that is for a permutation of $n+1$ letters, we will get a simplicial homeomorphism from Δ_n to Δ_n because any set theoretic map from the vertex set to the vertex set is a simplicial map on Δ_n. In particular, if you have permutation of vertices, that can be extended linearly to a unique simplicial map from mod Δ_n to $|\Delta_n|$ which will be automatically a homeomorphism. So, that we denote by G_α. So, this element represents an action of Σ_{n+1} on double of $S_n(K)$ by merely composing on the right, namely, take $(\alpha, \lambda) \mapsto (\lambda)^\alpha = \lambda \circ G_\alpha$.

So, that gives you a right action of Σ_{n+1} on double $S_n(K)$ as well because if λ is simplicial, since G_α is also simplicial, the composite will be also simplicial. Of course, we have only expressed this on the basis elements, but on a chain it is all obtained by linear extension.

This notation is standard notation in the many literatures, viz., writing the action as an exponent. Usually this notation comes from group theory wherein conjugation by an element ghg^{-1} was simplified by writing like this h^g, and most often it is the conjugation of the group on itself which is studied very thoroughly. So, that must be the motivation for writing like this. So, I am following that notation here.

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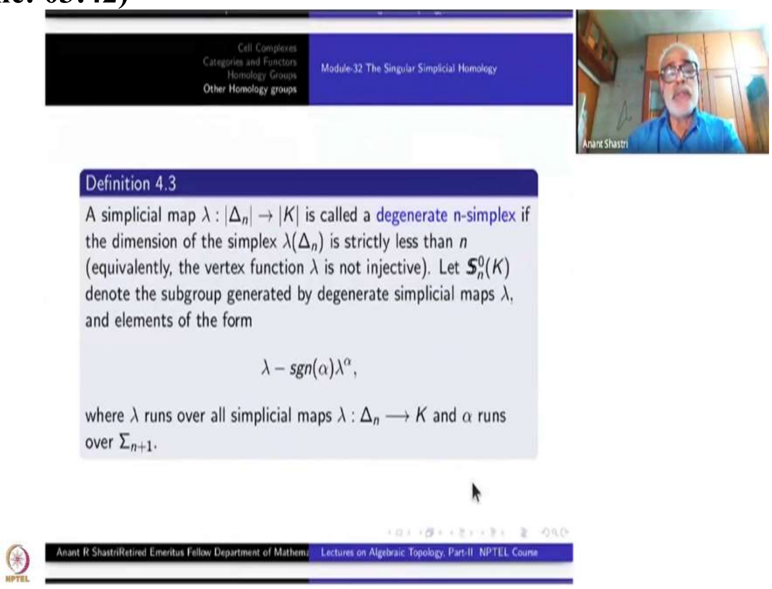

A simplicial map λ from Δ_n to any simplicial complex K is called a degenerate simplex if the dimension of $\lambda(\Delta_n)$ is strictly less than n, which is same thing as saying that the vertex function λ from vertices of Δ_n to vertices of K is not injective. If it is injective and it is a simplicial map then mod Δ_n would have been embedded as a subcomplex in K. So, the dimension would have been the same.

So, such things are called degenerate and anything which is not degenerate will be called non-degenerate. That means that the map λ is injective. So, let us look at the subgroup of double $S_n(K)$ generated by all those degenerate simplices and also of elements of the form λ minus the signature of $\alpha\lambda^\alpha$.

So, all these elements as well as degenerate elements, we are taking together. They may not form a subgroup, but take the subgroup generated by them. double $S_n(K)$ is an abelian group. So, we get an abelian subgroup double $S_n^0(K)$ subgroup of double $S_n(K)$.

The idea is to kill away all these things, go modulo all these things. Because the simplest thing that we want to say is that an edge like $[u, u]$ should not be counted as a line segment but as single point as a geometric object. Later we need to treat all contractible loops also as trivial things elements in the homology anyway. So, this is a very obvious first thing that you have to do, namely all the degenerate simplices should not be counted.

Similarly, if you have a 1-simplex $\lambda = [u, v]$ and you interchange the two vertices, i.e., consider the 1-simplex $[v, u]$, that should be treated as negative of λ. That is the kind of relation that we want to introduce here. If α is an odd permutation, signature of α is -1. So λ minus signature of $\alpha\lambda^{\alpha}$ should be killed to make λ equal to signature of $\alpha\lambda^{\alpha}$. So, I am taking such elements also in the subgroup double of $S_n^0(K)$.

Indeed, instead of doing this for all n separately, I can do them together simultaneously namely, take all degenerate n-simplexes and all elements of the form λ minus signature of $\alpha\lambda^{\alpha}$., in double $S.(K)$, they will generate a subgroup that subgroup is denoted by double $S_n^0(K)$.

Obviously, it is a chain subgroup. In other words, boundary(double $S_{\cdot}^0(K)$ subset of double of $S_{\cdot}^0(K)$ be contained inside $S_{\cdot}^0(K)$ if and only if we verify that, then it will follows that it is a sub chain complex. Whenever you have sub chain complex you can take the quotient and that will give you another chain complex.

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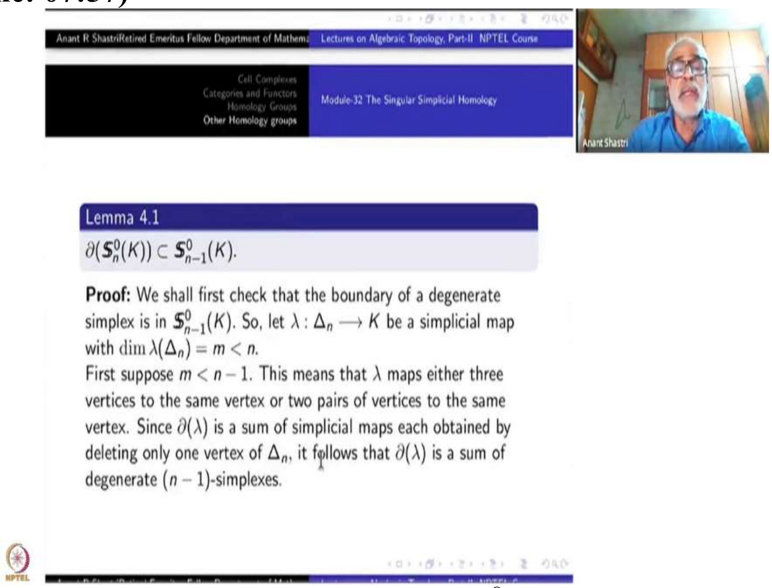

So, this is what we have to verify here, boundary of S_n^0 is contained S_{n-1}^0. So, if I show that boundary of generators of S_n^0 are in S_{n-1}^0, then the whole thing will follow because ∂ is also a linear map, i.e., $\partial(A+B) = \partial A + \partial B$ and $\partial(nA) = n\partial(A)$. Not that ∂ of a generator may not be a generator. That is not necessary either.

So, let us start with a λ from Δ_n into K such that dimension of $\lambda(\Delta_n)$ is less than n, say it is $m < n$. If $m < n-1$, that means that there are at most $n-2$ vertices in the image. That means, either 3 elements have gone to the same element or there must be 2 pairs of elements on which λ is not injective, say $\lambda(u_1) = \lambda(u_2)$ and $\lambda(u_3) = \lambda(u_4)$. This is purely set theoretic argument. Number of elements in the domain is $n+1$ and those in the image are less than $n-1$. If 3 elements have been mapped to same element, when you take the boundary of λ, what happens? Each term in the summation formula for $\partial(\lambda)$, you are deleting only on element and therefore each term is a sequence with at least one repetition. Therefore, it is a sum of degenerate simplices. Therefore it is in S^0_{n-1}.

In the second subcase, i.e., if 2 pairs of vertices are going to same elements then when you deleting 1 of them from one of the pairs, the other pair is still there other pair 2 of them are going to same element. Therefore, again each term in the summation is a degenerate simplex and hence $\partial(\sigma)$ is in S^0_{n-1}.

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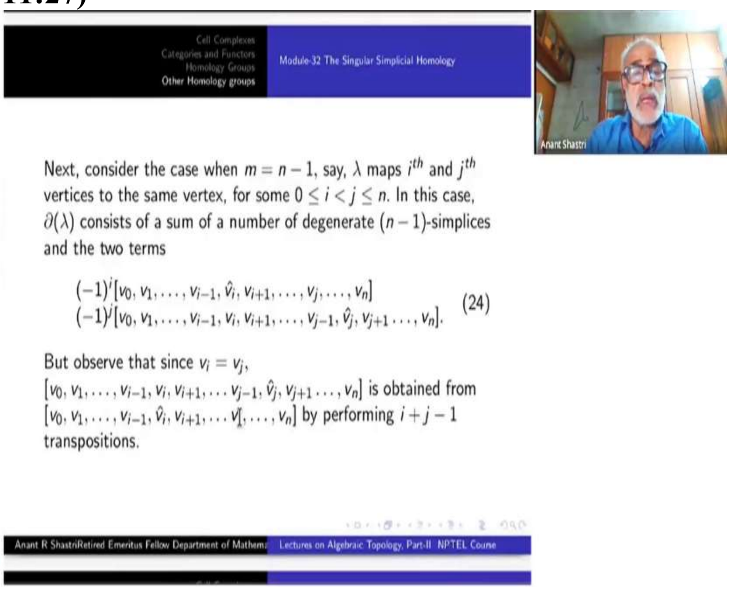

The second case; the second case is what $m = n-1$ itself. That means only a pair of vertices have been identified and all other vertices are mapped to distinct vertices. So, let us say, the i-th and the j-th vertices are mapped to the same vertex, for some $0 \leq i \leq j \leq n$. In this case boundary of ∂ consists of a sum of a number of degenerate $n-1$ simplices obtained when you are dropping neither the i-th element nor j-th element plus two more terms, namely, when you dropping the i-th element or the j-th element, whatever you get is a non-degenerate simplex. So, we need to check what happens in this case. So, this is the interesting case.

Write $\sigma = [v_0, ..., v_n]$. What happens is all other terms have been degenerate you can forget about them. What are the two terms remaining here. One is $(-1)^i\tau$ where $\tau = [v_0, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_j, \ldots, v_n]$ and the other term is $(-1)^j\rho$, where $\rho = [v_0, \ldots, v_{j-1}, \hat{v}_j, v_{j+1}, \ldots, v_n]$. But now use the fact that $v_i = v_j = u$, say. So, this u is in the $(j-1)$-th place in τ and it is in the i^th place in ρ. So, you have to bring u to the i-th place without changing the order of other entries. So, this can be done by performing $(j-1-i)$ transpositions. If α is the composite of these transpositions, it follows that $\rho = \tau \circ \alpha$.

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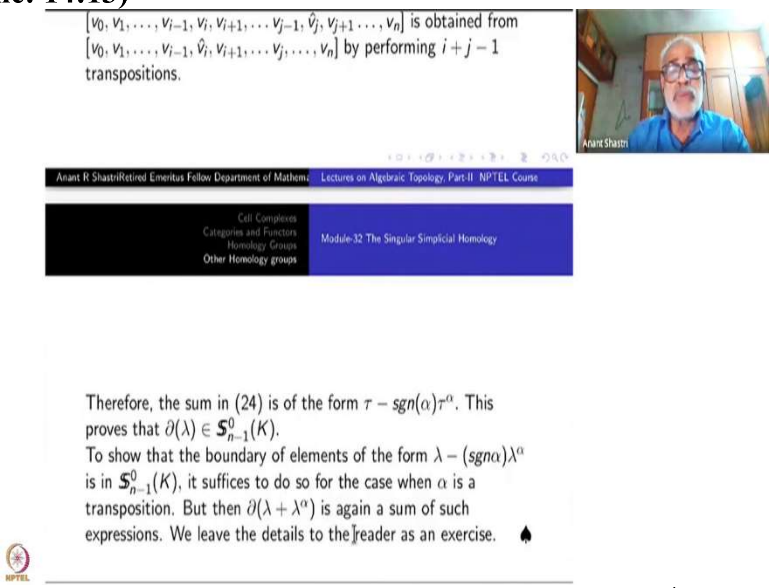

Therefore the sum of these two remaining terms is of the form $(-1)^i$(τ-(-1)^{signature of α} τ^α, which belongs to S^0_{n-1}. So, boundary of λ is in double of $S^0_{n-1}(K)$.

It remains to verify that ∂(λ-(-1)^{signature of α}λ^α) is also in S^0. Since each α can be expressed as a product of transpositions, such a term can be written as a finite sum of elements of the above form wherein each α is a transposition, (by adding and deleting). Therefore, we need to do this for the case when α is a transposition. Thus for a transposition alpha, we gave to show that $\partial(\lambda + \lambda^\alpha)$ is also a sum of such terms.

We leave this for you as an exercise. You have to do that. This will give you a good practice. Unless you toil that much you would not get familiar with these symbols, meaning of the boundaries and so on. So, this part we leave it as an exercise to you.

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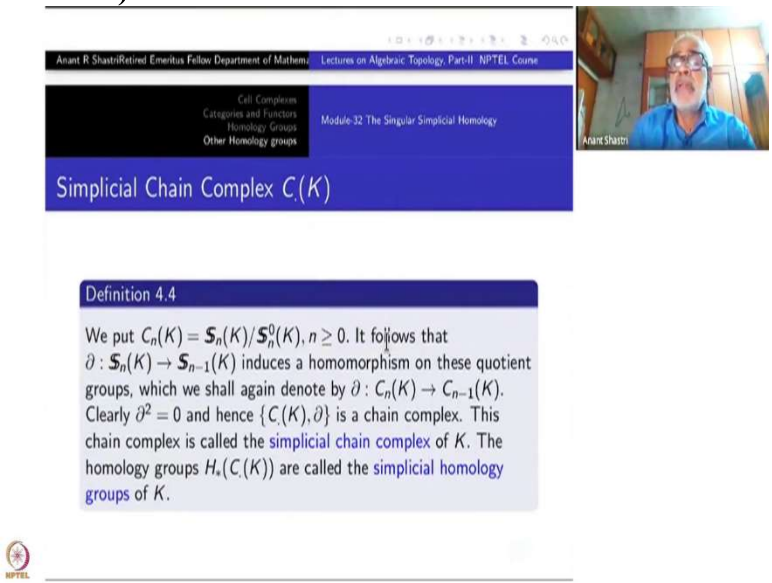

Now we can define whatever we have promised, namely, a further simplification of the singular simplicial chain complex. So, what we do we take this subgroup S^0, go modulo that. To go modulo those which are degenerate or which correspond to under permutation lambda minus signature of $\alpha\lambda^{\alpha}$, go modular that. This will be denoted now by $C.$, a chain complex. The same boundary operator of double $S.$ will induce the boundary operator for this quotient complex. This chain complex is called the simplicial chain complex of K, the homology group of this chain complex is called simplicial homology group of K.

This is the ultimate simplification that we saw. This $C.(K)$ as given is not a sub chain complex of the singular chain complex of $|K|$. It is the quotient of a sub. So, this is called a sub-quotient. It is the quotient of a sub chain complex. Nevertheless this was our goal, a lot of simplification in the singular homology of a $|K|$, where K is a simplicial complex.

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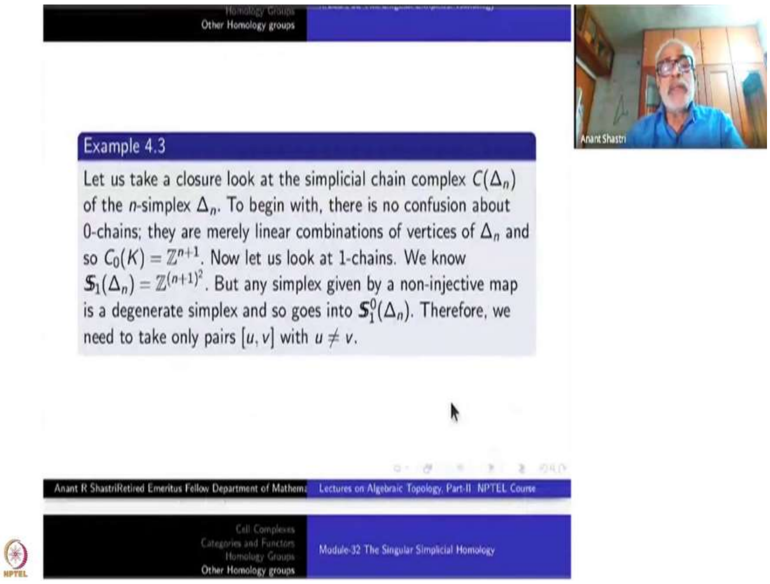

So, let us take a closer look at it. How things look like, in the very special case, namely, when K is just Δ_n itself. If you do not understand this case, then you cannot do much. So let us understand what is $C.(\Delta_n)$ to begin with. There is no confusion what $C_0(K)$ is. Since there are no degenerate 0-simplexes nor there is any non trivial permutation of a single element set, it follows that $S^{0(K)=0}$ and hence $C_0(K)$=double $S_0(K)$ for all K. In particular, $C_0(\Delta_n) = \mathbb{Z}^{n+1}$.

Let us now look at $C_1(\Delta_n)$. Since any non injective map defines a degenerate element which goes into S^0, we need to consider only edges $[u, v]$ where $u \neq v$. Since $[v, u]$ can be obtained from $[u, v]$ by a transposition, in the quotient group C_1, they represent the same element. Observe that how we are bringing the geometry in to play its role, namely, an edge traced in the other way direction is treated as the negative of that edge. Once you have $[u, v]$, you do not need to take $[v, u]$ in the generating set. This just means that we need not count an edge in Δ_n only once in any one of the two directions and other direction we do not count. What is the meaning of that? Therefore, taking the natural order on the vertices of Δ_n, we take all edges $[e_i, e_j]$ where $i < j$.

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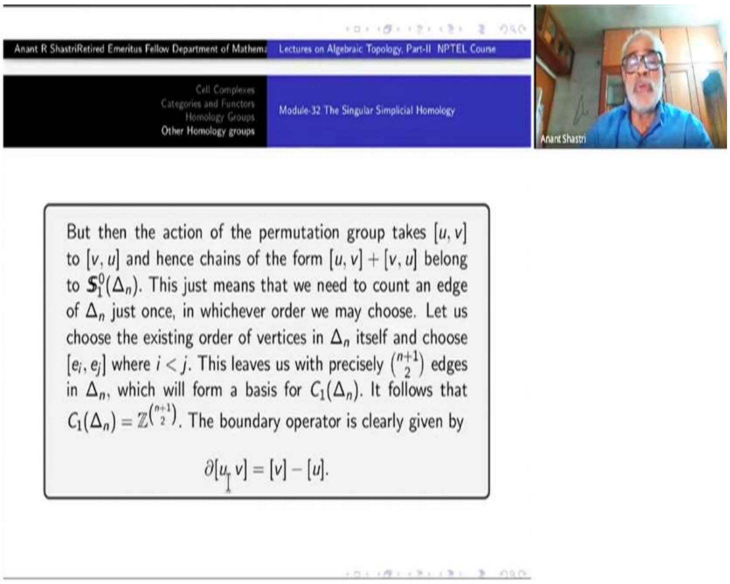

This gives us precisely $n+1$ choose 2 non-degenerate 1-simplexes in Δ_n. Let B_1 denote the set $\{[e_i, e_j], i < j\}, B_2$ denotes the set $\{[e_i, e_j - [e_j, e_i], i < j\}$ and B_3 denote the set $\{[e_i, e_i] : 0 \leq i \leq n\}$. Check that $B_1 \cup B_2 \cup B_3$ forms a basis for double $S_0(\Delta_n)$. From this, it follows that the quotient map ϕ restricted to B_1 is an isomorphism onto $C_1(\Delta_n)$. It follows that $C_1(\Delta_n)$ is isomorphic to $\mathbb{Z}$^$\{n+1$ choose $2\}$. The boundary operator ∂ from C_1 to C_0 is clearly given by $\partial([u, v]) = v - u$.

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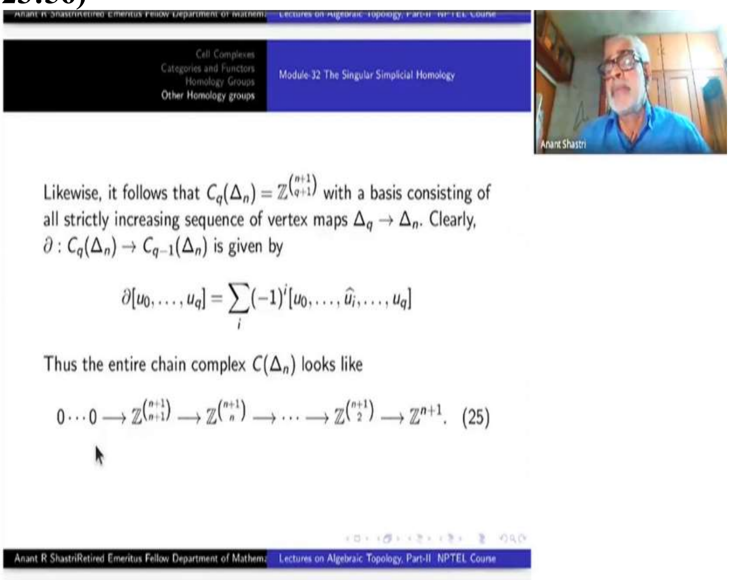

Likewise, the same argument will gives you that $C_q(\Delta_n)$ is ismorphic to $\mathbb{Z}$^$\{n+1$ choose $q+1\}$ and boundary of this map ∂ from C_q to C_{q-1} is given by this formula. Remember that we are actually working in the quotient group $C.$ though we have used the elements of double $S.$ to represent the elements of $C.$

It is important to notice that we are not using the order of vertices in the definition of the boundary operator. The order is used in obtaining a copy of C_0 as a subgroup of double $S.$ This is how the entire chain looks like: starting with $\mathbb{Z}$^{$n+1$ choose one}, this is C_0 then $\mathbb{Z}$ ^{$n+1$ choose 2} which is C_1 and so on. This will denote $C_n(\Delta_n)$, generated by only one n-simplex, namely, you have take the entire set of vertices and write them is some order. If you change the order of vertices by an odd permutation then it will be a negative of the former generator and if it is an even permutation, then it represents the same generator of C_n. Is that clear?

Because you cannot choose any $n+k$ subsets out of $n+1$ elements, $k>1, C_{n+k}$ are all 0-groups beyond this. This is the biggest simplification coming here which was not possible in the case of double of $S.(K)$ which has infinitely many non zero groups. $C.$ stops at n-th stage itself, $C_n(\Delta_n)$ is $\mathbb{Z}$ and then we have $\mathbb{Z}^{n+1}$, $\mathbb{Z}$^{$n+1$ choose 2} and so on $\mathbb{Z}^{n+1}$. If you put one more $C_{-1}=\mathbb{Z}$ and take the augmentation morphism epsilon from C_0 to C_{-1}, this chain complex becomes perfectly symmetric, just like binomial expansion and for that beauty at least, it is better to consider the augmented chain complexes. Also, for combinatorial considerations, the augmented chain complex is simpler.

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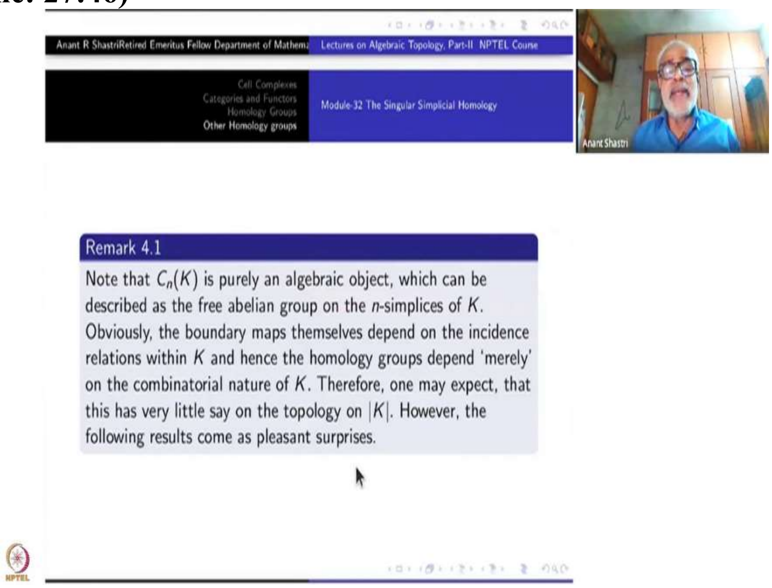

Given a simplicial complex K, $C.(K)$ is purely an algebraic object. There is no continuity, no topology, nothing. Once the simplicial complex K is given which is an abstract combinatorial object, we have the definition of $C.(K)$ as a purely algebraic object. Obviously, the boundary maps themselves will depend on the incidental relations within the simplicial complex K, which simplex is sitting where, I mean relations such as what are the vertices of an edge?

What are the vertices of a triangle, which are the edges of a triangle, etc, only the incidence relations. So, that is the only the thing that matters and that is nothing but the combinatorial information. That has the full control of topology of $|K|$ is what we have seen already while studying simplicial complexes to some extent. It comes again here too. You may wonder that this pure algebra may say only a little bit about the topology. However, the following results come as a pleasant surprise.

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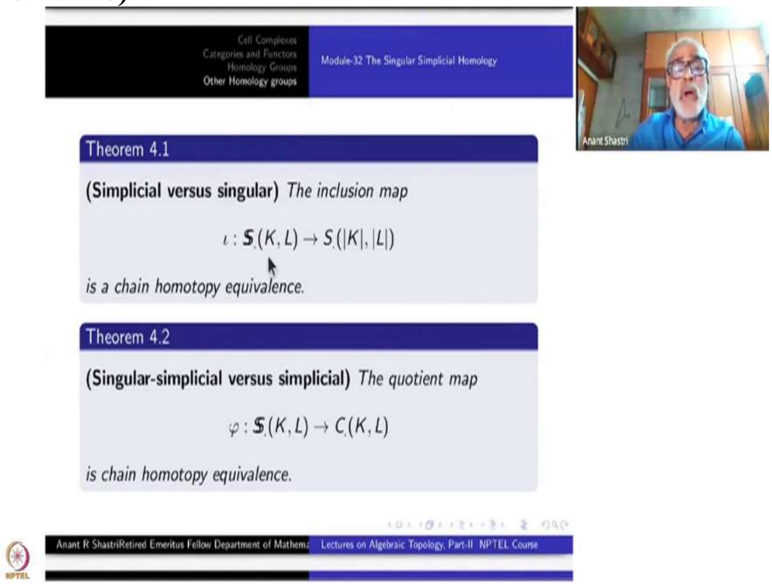

These are two statements here. Consider the inclusion map from the full singular simplicial chain complex double $S.(K, L)$ to the singular chain complex $S.(|K|, |L|)$. The codomain is purely topological, and this domain is purely combinatorial. This inclusion map is a chain homotopy equivalence.

Exactly the same way, the quotient map from double $S.(K, L)$ to $C.(K, L)$ is a chain homotopy equivalence. You already know that chain homotopy equivalence induces isomorphism on the homology groups. Thus combining these two results, what you get is that the homology of the topological space $|K|$ is isomorphic to the homology of $C.(K)$.

So that is the big result that we get. So I am going to state that one. For pairs of simplicial complexes, there are canonical isomorphisms of singular homology $H_*(|K|, |L|)$ to the singular simplicial homology $H_*(S.(K, L))$ and the simplicial homology $H_*(C.(K, L))$.

What are these isomorphisms? The first one is inclusion induced map. Note that $H_*(|K|, |L|)$ is a simplified notation in which we are not writing this $S.$ at all. We are taking it for granted. But here we are writing double of $S_*(K, L)$. I have to write it because this is something different from this one. Similarly, the isomorphism is induced by the quotient map ϕ from double $S_*(K, L)$ to $C_*(K, L)$. Now you can take $i_* \circ \phi_*^{-1}$, from $H_*(C.(K, L))$ to $H_*(|K|, |L|)$. That gives a canonical isomorphism.

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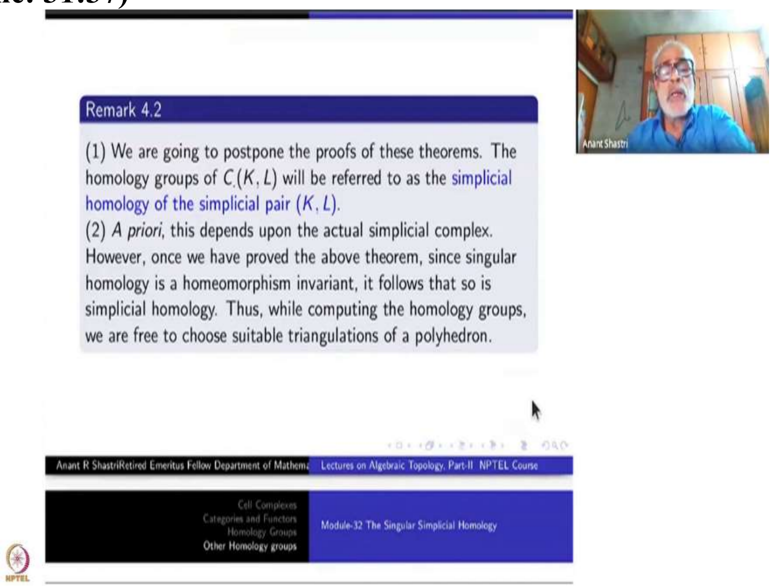

We are going to postpone the proof of this isomorphism theorem here. The homology of $C.(KL)$ will be referred to as the simplicial homology of the simplicial pair (K, L). When L is empty, it is purely $C.(K)$, the simplicial homology of a simplicial complex K. It seems to depends upon the actual simplicial complex structure on the underlying topological space $|K|$, that much is obvious. But however, once we have proved the about theorem, it means that it will depend only on the underlying topological space. That means, if I take a simplicial complex, take its underlying space, and put another simplicial complex structure on that and then conclude that the two simplicial homologies are the same. No problem.

Thus for a triangulable topological space, the simplicial homology does not depend upon what triangulation you choose. This freedom of choice is very important. Historically, this was a very very deep and strong result which had no proof until singular homology was introduced.

This analogous, you know, to a phenomenon in real analysis. A problem in Riemann integration theory was solved only after the invention of a more general theory viz., Lebesgue integration theory. The characterization of functions which are Riemann integrable in terms of the size of the set of points of discontinuities of the function. Indeed, Poincare had tried to prove that the simplicial homology is independent of the choice of the simplicial structure on a given topological space. He then just assumed the result.

So, we will stop here and continue the study next time.