Introduction to Algebraic Topology (Part-II)
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Lecture - 32
The Singular Simplicial Homology

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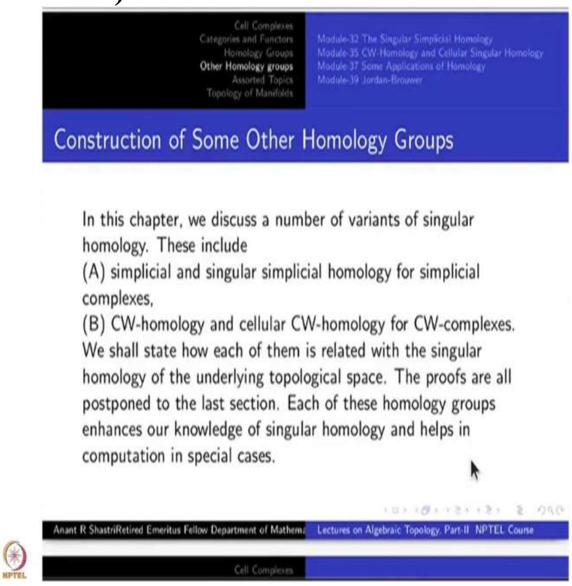

So, in this chapter we shall discuss a number of variants of singular homology. These include the simplicial and singular simplicial homologies for simplicial complexes, CW-homology and cellular CW-homology for CW-complexes and maybe one or more also. So, we shall state how each of them is related with the standard singular homology that we have studied so far. But they are for special types of topological spaces and not for arbitrary topological spaces. That you have to keep in mind.

The proofs are all postponed to the last section. Each of these homology groups enhances our our grip on singular homology and helps in computations in the special cases and it gives you some fantastic results about the singular homology of topological spaces themselves finally.

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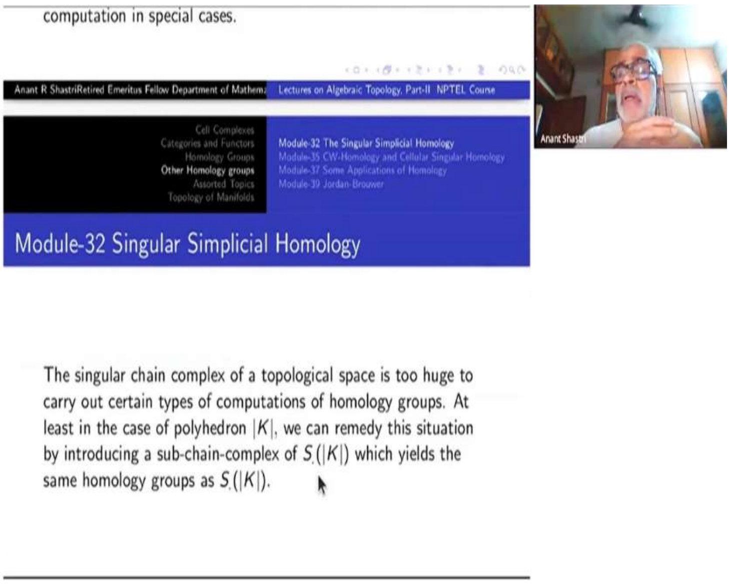

So, the first one is singular simplicial homology. In singular homology, we understand, we are just taking continuous functions from Δ_n to a topological space X at the starting point, what are called the singular n-simplexes, whereas now we are going to replace them by simplicial maps. So, for that we need X to be not an arbitrary space but a simplicial complex, that is the whole idea.

The singular chain complex of a topological space is too huge to carry out certain types of computations. At least in the case of a polyhedron, namely, when the underlying topological space has a simplicial complex structure, we can remedy this situation by introducing a sub chain complex of $S.$ So that new thing that we are going to consider is a sub chain complex of the singular chain complex of $|K|$ which yields the same homology groups as $S.(|K|)$ When you go to the homology, they will be the same. Same means what? canonically isomorphic.

There is a sub chain there and the inclusion map, when you pass on to homology, the inclusion induces an isomorphism. That is the final statement okay?

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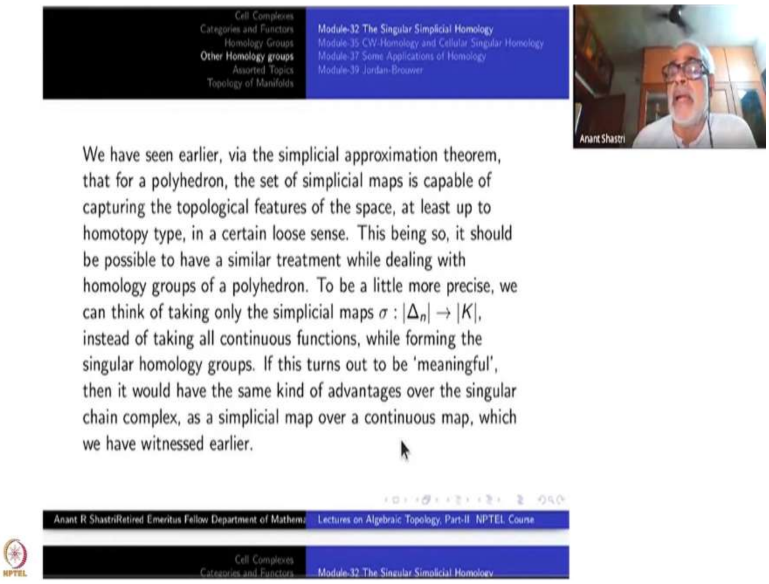

So, we have seen earlier the simplicial approximation theorem. Remember that for a polyhedron, by polyhedron I mean the topological space $|K|$, where K is a simiplicial complex. the set of simiplicial maps is capable of capturing topological features of the space, at least up to homotopy type, okay? The simplicial approximation theorem just says that any continuous map can be replaced by a simplicial map up to homotopy.

After that we have seen a lot of applications of this result. So it should be possible to have a similar treatment while dealing with homology groups also. So, to be a little more precise, we can think of taking only simplicial maps σ from Δ_n to K, In the general case of an arbitrary space X, all contiguous maps are taken. Now use the extra structure on X namely, X is $|K|$. So, instead of taking all continuous functions you can take only those which are given by simplicial maps.

So, without taking bars, you can first take σ to be a simiplicial map from Δ_n into K, and then pass on to the corresponding continuous function $|\sigma|$ from mod Δ_n to $|K|$. That is the meaning of a simplicial maps in short for me. Now we can just take them as the basis elements and take the abelian group (or module) generated by them, that will be a subgroup of $S.(X)$. So, if this turns out to be meaningful, it would have same kind of advantage over the singular chain complex as simplicial maps have over continuous functions right? The set of continuous functions is too huge, whereas simplicial simplicial maps can all be written down and counted. You can analyse them easily. So that is the whole idea okay?

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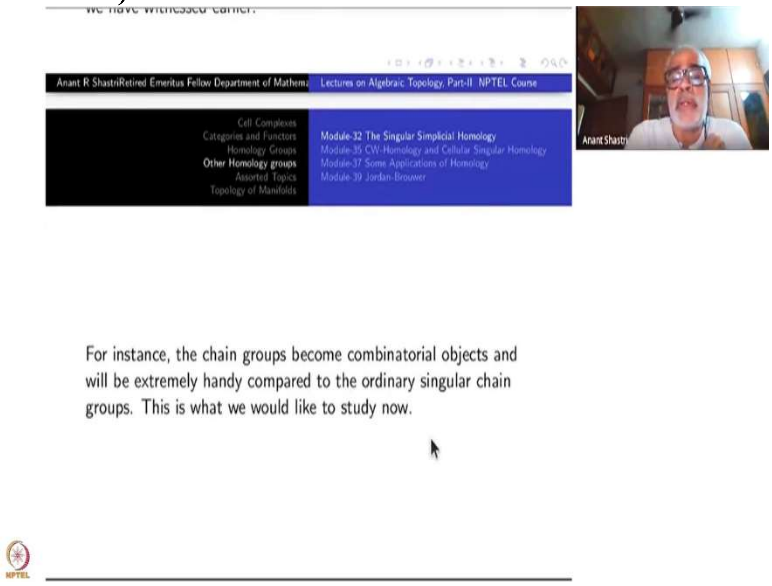

For instance, these sub chain groups become now purely combinatorial objects. You can feed it in a computer. And they will be extremely handy compared to the ordinary singular chain group. This is what we would like to study now okay?

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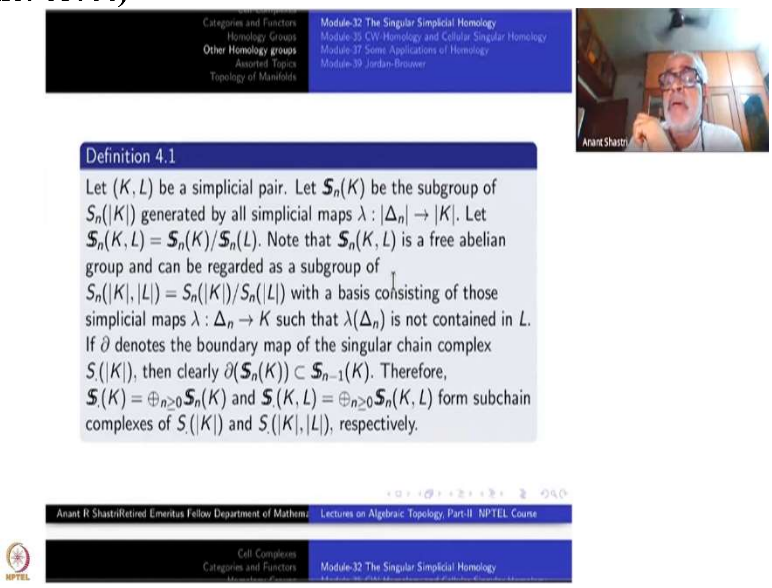

So that is all pep-talk or sales-talk. We will now come to brass tacks. So, just like we started with (X, A) as a topological pair, let (K, L) be a simplicial pair. K is a simplicial complex and L is sub complex okay? Now I am going to write down a new symbol double $S_n(X)$, you know, just to distinguish from ordinary $S_n(X)$, okay?

So, double of $S_n(K)$ is the subgroup of $S_n(|K|)$ generated by all simplicial maps λ from Δ_n to $|K|$ okay? This is similar to taking just polynomial functions or linear functions instead of all continuous functions, yeah? that is what I am doing here. So, let double of $S_n(K, L)$ be the quotient group double $S_n(K)$ modulo double of $S_n(L)$. So, any simplicial map Δ_n to L, would be also a simplicial map into in K and hence double of $S_n(L)$ is a subgroup of double of $S_n(K)$ and the quotient maps sense. Similar to the case of $S.(X, A)$, this is also a free abelian group with the set of simmplical maps into K with their image not contained in $|L|$. Even if one of the vertex goes outside L, such a simplical map will be inside this basis of double $S.(K, L)$. So, though it is obtained as a quotient group, it is still a free abelian group, with the basis consisting of those simplicial maps λ such that $\lambda(\Delta_n)$ is not contained in L.

Similarly, now take the same face maps. Remember the face maps were linear maps. So, when you take the boundary operator which is the sum of these face maps, if you have started with σ as a simplicial map, the boundary will be also a sum of simplical maps. So, if ∂ denotes the boundary map of singular chain complex, then ∂ of double of $S_n(K)$ will be contained in double of $S_{n-1}(K)$.

So, this is the beauty of this boundary operator here, which is a sum of the face maps, actually linear maps. So, double of $S.(K)$ which is the graded direct sum of double $S_n(K)$ and double of $S.(K, L)$ which is the of graded direct sum of double of $S_n(K, L)$ form sub chain complexes okay? What are boundary operators here? Just the restriction of the same ∂ on $S.(K)$ and $S.(K, L)$ respectively. Okay?

So, the construction of simplicial chain complex was easy. But the difficult task is to show that the inclusion induced homomorphism in the homology is an isomorphism. That is not coming so easily okay you have to have some patience.

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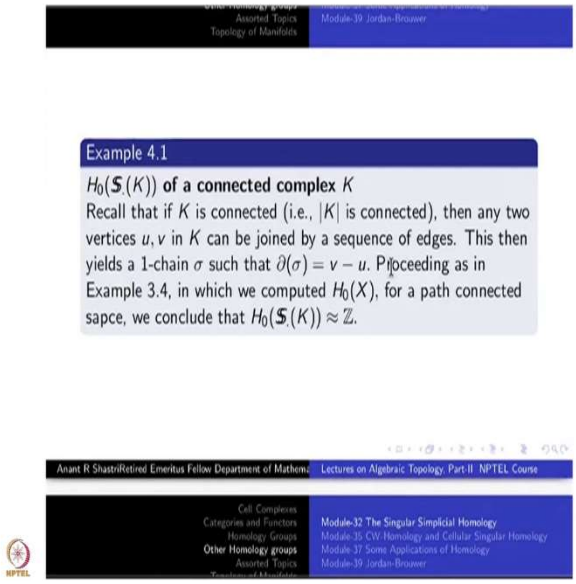

Instead of doing that let us get familiar with this new object first. So, let us compute a few things with this new chain complex. What is H_0 of double of $S.(K)$ for a connected complex K? Remember a simplicial complex K is connected if and only if any two vertices inside K can be joined by any edge-path. An edge path means what? A sequence of directed edges okay, with the next edge starting at the vertex where the present one ends. Like this take a finite sequence from one vertex to another vertex. If you can do that for any two vertices then K is connected. This result we have seen in part I okay. So, if you take such an edge path, then each edge gives you a 1-simplicial simplex in K. You can take the sum of these to get a chain. What is the boundary of this? It is the sum of the boundaries of each edge which is the end vertex minus the initial vertex. So, only the starting vertex and the end vertex will remain, the starting vertex will come with a minus sign and the end vertex comes with a plus sign, the in between vertices cancel out. So, what you get is that the boundary of this chain sigma is $v - u$, where v and u are end points of this edge-path okay?

The rest of the argument is the same as for the singular chain complex of any path connected space. We can use the augmentation map ϵ from double of $S_0(K)$ to $\mathbb{Z}$, right, which sends all the vertices to 1 in $\mathbb{Z}$. In particular $\epsilon(v - u)$. So, all these arguments will be same way as before and what you conclude is that $H_0(\text{double } S_d \cot(K))$ is isomorphic to $\mathbb{Z}$, the isomorphism being induced by the augmentation.

So, the only difference is that earlier we took arbitrary path but in a connected simplicial complex you can choose by an edge path.

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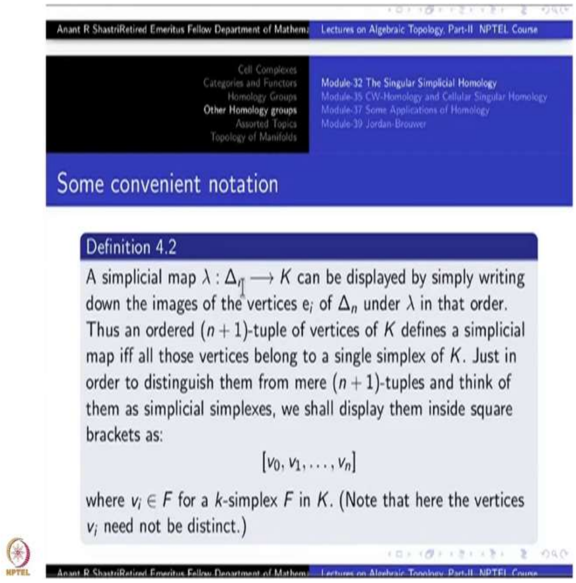

We introduce some convenient notation now for going further. A simplicial map Δ_n into X can be displayed completely, unlike an arbitrary continuous function, which may be very complicated. You do not have any formula in general, but the case of some polynomial functions or trigonometric functions and so on right? But here, we are in a better situation of linear functions. A linear function can be displayed by simply writing down the image of the generators. Similarly, a simplicial singular n-simplex is completely determined by it values on the vertices $e_i(\Delta_n)$. So we can store it in a sequence $[a_0, a_1, \ldots, a_n]$ and enclosed it in square bracket, where $a_i = \lambda(e_i)$ which are some vertices of K. Thus a singular simplicial n-simplex is encoded by an ordered $(n+1)$-tuple of vertices in K. Since the entire λ is a simplicial, it follows that an ordered n-tuple of vertices of K form unique singular n-simplex in K if and only if all those vertices belong to a single simplex in K.

The only thing is that it may not be an n-simplex. It may be a 0-simplex also when all vertices of Δ_n are mapped to the same vertex of K. It may be 1-simplex or it may be a k-simplex for any $k \leq n$. So, what happens is that the map λ may not be injective, so $v_0, v_1, \ldots, v_n$ may not be distinct okay? This is the difference between a singular simplical n-simplex in K and a n-simplex in K. The ordered n-tuple $[v_0, v_1, \ldots, v_n]$ does not coincide with the set of vertices $\{v_0, v_1, \ldots, v_n\}$.But if you cut down all the repetitions then we get the image set $\lambda(\Delta_n)$. The ordered n-tuple will tell you exactly what the map λ is. That is very important for us okay?

So, let us understand notation correctly, where v_i's are in some simplex $F(K)$ okay? And v_i is the image of λe_i under λ, okay? So, take any simplex F in K. It may have a number of vertices. Take a sequence in that which is a not necessarily consisting of distinct elements okay? Just a sequence $[v_0, \ldots, v_n]$. That will give you a singular simplicial n-simplex. The word `singular' is used functions which may not injective in this sense.

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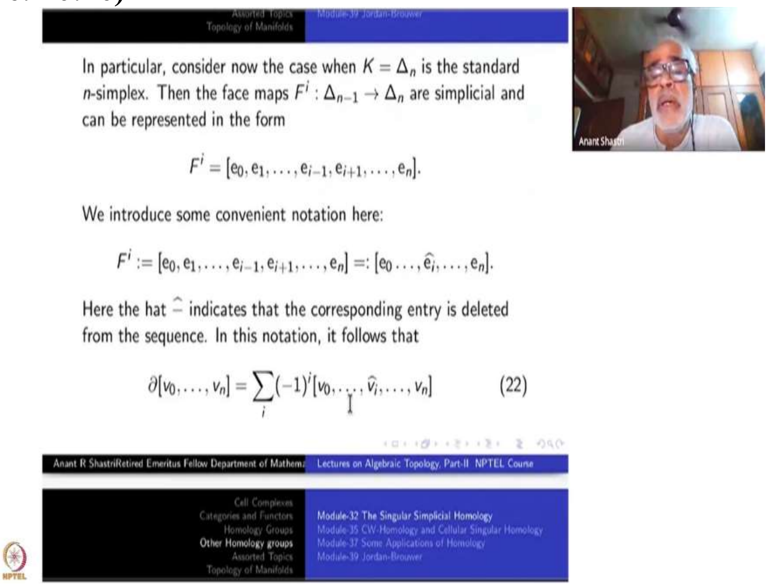

In particular take $K = \Delta_n$ itself, okay? Remember that Δ_n is homeomorphic to the topological space $\mathbb{D}^n$. We have studied $S.(\mathbb{D}^n)$ and $S.$ of a sphere etc. to begin with. So I would like to study them again when K is Δ_n, the standard n-simplex. Then all the face maps F^i from Δ_{n-1} to Δ_n, are simplicial and can be represented in the above form. What happens to e_j under F^i? e_0 etc. upto e_{i-1} will remain the same, e_j will be shifted to e_{j+1} from $j = i$ onwards, right? We can view F^i as a $(n-1)$ simplicial simplex in Δ_n,, okay? It follows that we can just use the notation $[e_0, \ldots, e_{i-1}, \hat{e}_i, \ldots, e_n]$. Put a hat on e_i to indicate that the vertex e_i is missing in the list. It is like hiding e_i, or dropping e_i, the rest of them are there. This is a notation where hat indicates that the corresponding entry is deleted from the sequence. That is all.

This notation is very convenient. Accordingly, boundary of $[v_0, v_1, \ldots, v_n]$ will be nothing but summation from 0 to n of $(-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. This is just the new interpretation of the old formula.

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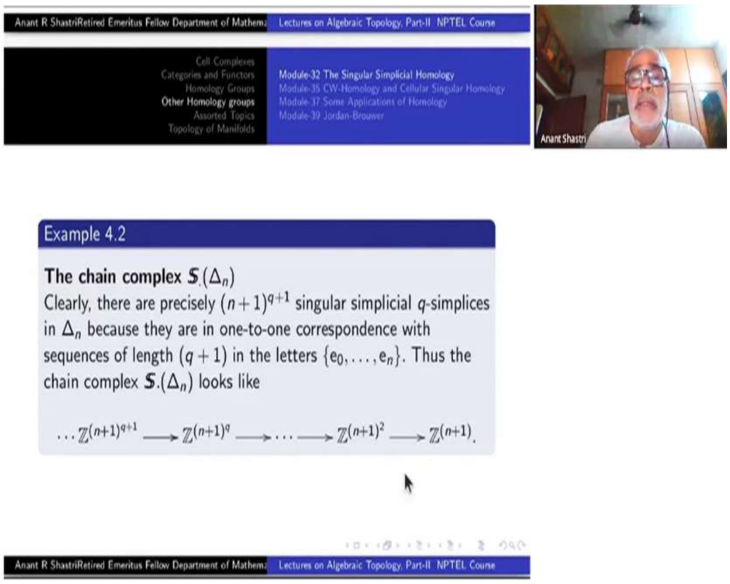

Once you have this notation I can now write down the chain complex of Δ_n itself okay? How many singular simplicial q-simplices are there in Δ_n? Tell me. A singular simplicial q-simplex means any function from the vertex set $\{e_0, e1, \ldots, e_q\}$ to the vertex set of Δ_n, right? which has $n+1$ elements. Δ_q has $(q+1)$ elements, and there is no restriction at all these functions, you have to take all of them.

So, you get $(n+1)^{q+1}$. This may look huge but compared to all continuous function this is very small. All continuous functions from mod Δ_q to $|\Delta_n|$ are uncountable. So, here we have to only take all sequences of length $(q+1)$ taking values in the vertex set of Δ_n. Therefore, double of $S_q(\Delta_n)$ is a free abelian group of rank $(n+1)^{q+1}$.

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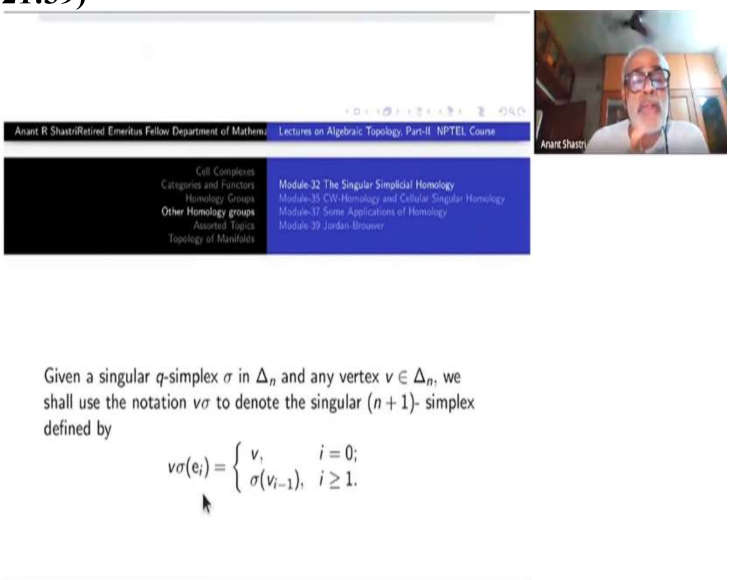

Let us have one more notation. Given a singular simplicial q-simplex sigma in Δ_n and any vertex $v \in \Delta_n$, consider the cone construction which we have done before, viz., the $(q+1)$-singular simpicial simplex $v\sigma$, the cone over σ with this vertex v, defined by $(v\sigma)_(e_0) = v$ and $(v\sigma)(e_i) = \sigma(v_{i-1})$. We have to shift σ because the first slot has been taken by v, afterwards you just apply σ okay so this must be $\sigma(e_{i-1})$, okay?

We can then extend this notation linearly tom all chains, $\sum n_i\sigma_i$ as well. v of that is nothing but $\sum n_i(v\sigma)_i$ okay. So, this is what we mean by cone construction. It follows that boundary of $v\sigma$ is given by exactly the way we have done before viz., $\sigma - v(\partial(\sigma))$, first v will drop out and then minus plus etc v followed by terms in $\partial\sigma$ will appear.

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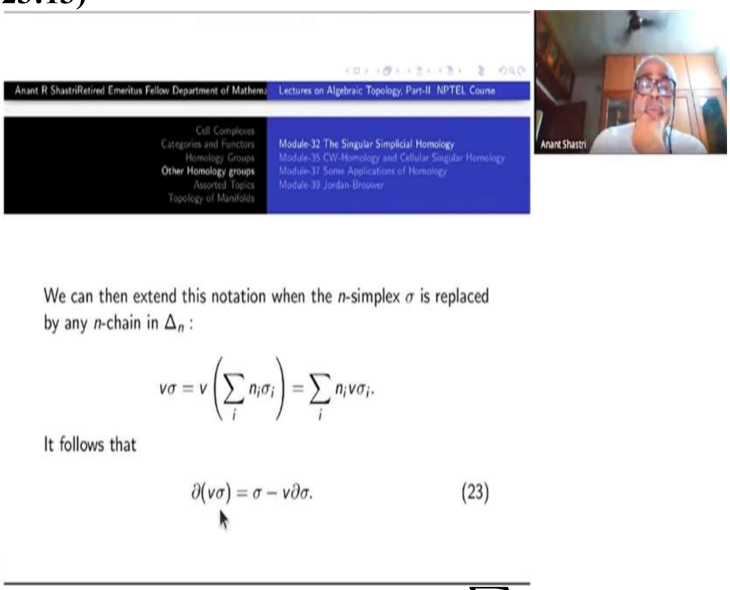

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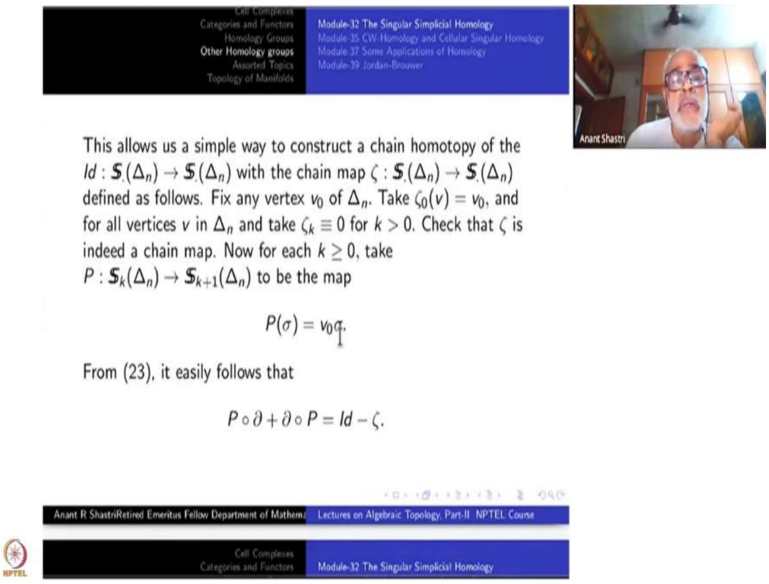

This allows us a simple way to construct a chain homotopy of the identity map of double $S(\Delta_n)$ into itself with the chain map ζ from $S.(\Delta_n)$ to itself defined as follows. So, what is ζ? Fix any vertex v_0 belonging to Δ_n. Then $\zeta = \oplus\zeta_i$, where we have to define ζ_i double $S_i(\Delta_n)$ to itself. Take $\zeta_0(v) = v_0$ for all vertices v of Δ_n, and extend linearly. Take ζ_k to be identically 0 for all $k \neq 0$.

So, this ζ collapses double S_i to 0 in dimensions other than 0 and at the 0th level it collapses double S_0 to an infinite cyclic subgroup generated by v_0. There is nothing more to check than to see that $\zeta \circ \partial = 0$ on double of S_1, to see that this ζ is a chain map.

To verify this, viz., $(P \circ \partial + \partial \circ P)(\sigma) = \sigma - \zeta(\sigma)$, you consider the two cases separately. In positive dimensions, both sides are equal to σ. In dimension zero, both sides operating on any vertex v yield $v - v_0$.

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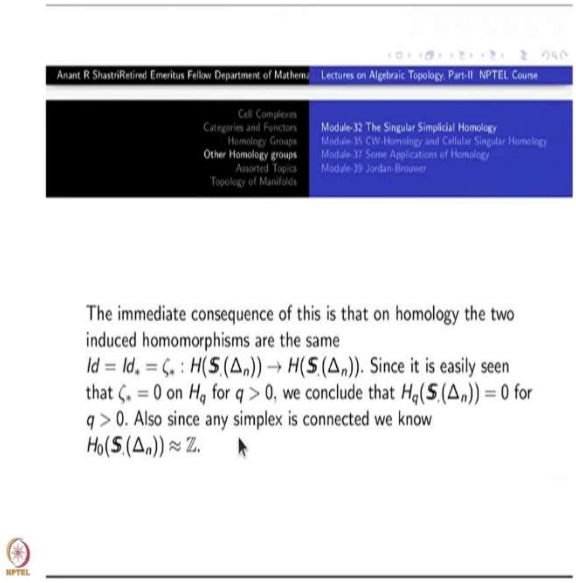

So, an immediate consequence of this is that on the homology, being chain homotopic, the two maps should induce the same homomorphisms. The identity map induces identity homomorphism in the homology. I have to look at what happens to ζ_*, okay?

So, it is easily seen that ζ_* is 0 in positive dimensions, since the morphism itself is 0. It follows that $H_i(\text{double } S.(\Delta_n))$ are all zero for $i > 0$, since the identity morphism of a group is zero means the group itself is 0. It remains to compute H_0. For this we can argue just like in the case of H_0 of the singular homology of a path connected space, via the augmentation map. Notice that the ζ_0 plays the role of augmentation map here and induces an isomorphism of $H_0(\text{double } S.(\Delta_n))$ with an infinite cyclic group.