Introduction to Algebraic Topology (Part-II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 31
Applications - Continued

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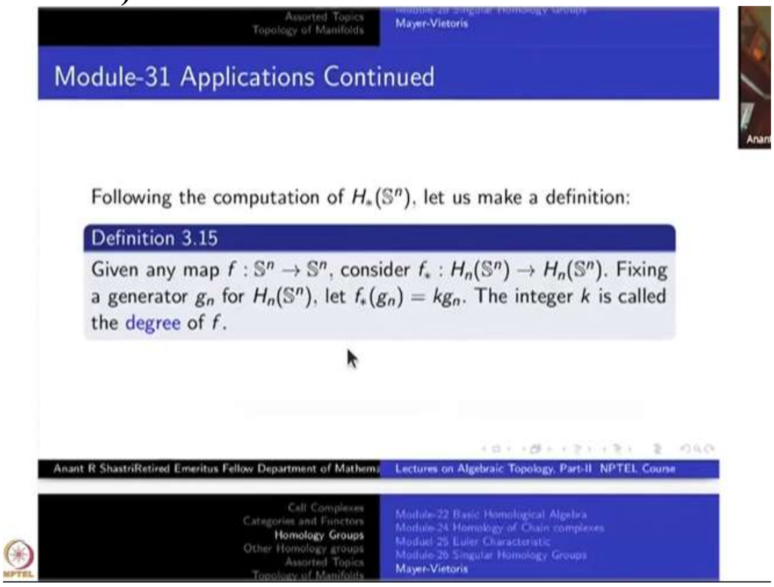

Following the computation of the homology groups of the spheres and some specific generator in $H_n(\mathbb{S}^n)$ for each n, let us do a little more computation here. With one definition first. Take any continuous function from $\mathbb{S}^n$ to $\mathbb{S}^n$. Look at the induced morphism f_* from $H_n(\mathbb{S}^n)$ to $H_n(\mathbb{S}^n)$. We know that both domain and codomain are infinite cyclic groups.

Fixing a generator for this infinite cyclic group denoted by $g_n, f_*(g_n)$ will have to be some integer kg_n. This integer k is called the degree of f. So this is defined whenever you have a map from $\mathbb{S}^n$ to $\mathbb{S}^n$. The degree would be an integer in general. okay?

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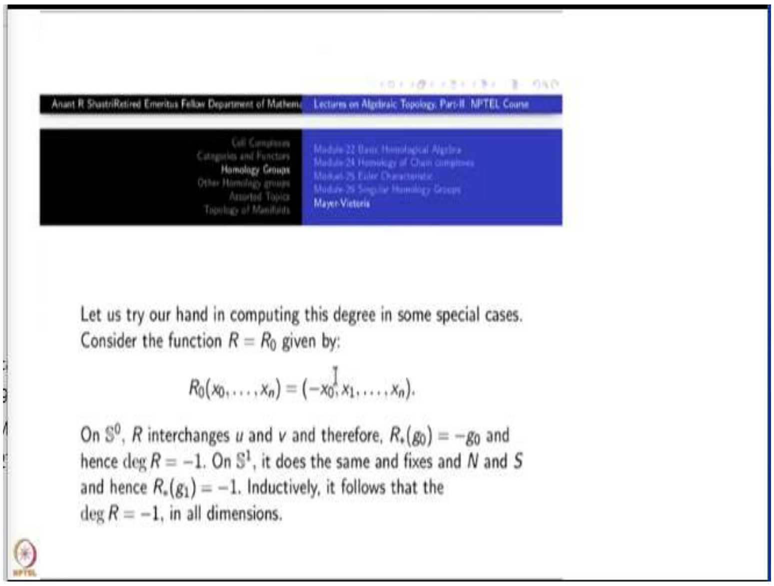

Let us try our hand in computing this degree of some very special maps. Obviously, if you take f to be identity map then f_* will be also identity map and hence g_n will go to g_n so k will be 1. So identity map has degree 1. Another important thing you may notice is that the homomorphism f_* depends only on the homotopy type of f. If you change the map by a homotopy, the degree does not change because f_* does not change okay?

So let us now consider the functions on $\mathbb{S}^n$, which are reflections, reflection in one of the coordinate planes. Let us take first simplest one, namely the first coordinate itself, the 0-th coordinate itself goes to its opposite and all other coordinates are kept fixed; $(x_0, x_1, \ldots, x_n)$ goes to $(-x_0, x_1, \ldots, x_n)$. So that is a reflection. On $\mathbb{S}^0$ itself, (which is a subset of $\mathbb{R}$, okay? Remember it is subspace of unit vectors in $\mathbb{R}$ which means $\mathbb{S}^0 = \{-1, 1\}$. But we have carefully avoided writing -1 or $+1$ because it may create confusion with the real numbers or integers which you are going to use as coefficients from the ring.) We are denoting it by $\mathbb{S}^0 = \{u, v\}$. What is effect of this reflection R_0 on that one? u will go to v, v will go to u okay. So therefore, $(R_0)_*$ of g_0 will be $-g_0$, okay? Just interchange u and v. $u - v$ is a generator g_0, remember that. So $v - u$ is $-g_0$. Therefore, this degree = -1 on $\mathbb{S}^0$.

Now, on $\mathbb{S}^1$ what happens? The map is (x_0, x_1) going to $(-x_0, x_1)$ okay? If you think of this as a complex number it is somewhat similar to the conjugation but the conjugation is different, this minus of that. So on $\mathbb{S}^1$, again changes u and v, but the North pole and South pole are kept fixed.

So go back to the definition of $g_1 = \Sigma_*(g_0) = u[S]-v[S]-u[N]+v[N]$. So, u and v are interchanged. So the sign of this one changes again. Therefore on $\mathbb{S}^1$ also degree is -1. Inductively you can see that each time u and v we are the only two things interchanged all the latter the vertices are fixed, okay? It follows that degree of R_0 is always -1 in all dimension.

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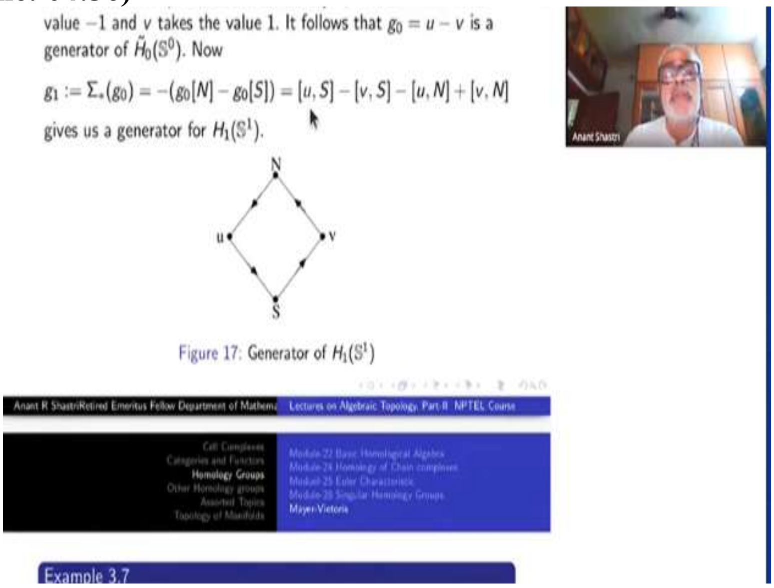

So, next we can take other reflections, instead of changing the 0-th coordinate, e.g., in $\mathbb{S}^1$ if you take (x_0, x_1) to $(x_0, -x_1)$ okay, then as complex number it is actually the conjugation map. Then it keeps u and v fixed and interchanges N and S. Check that it is again of degree –1. So, more generally, if only one coordinate is getting changed to the negative, then the generator changes its sign.

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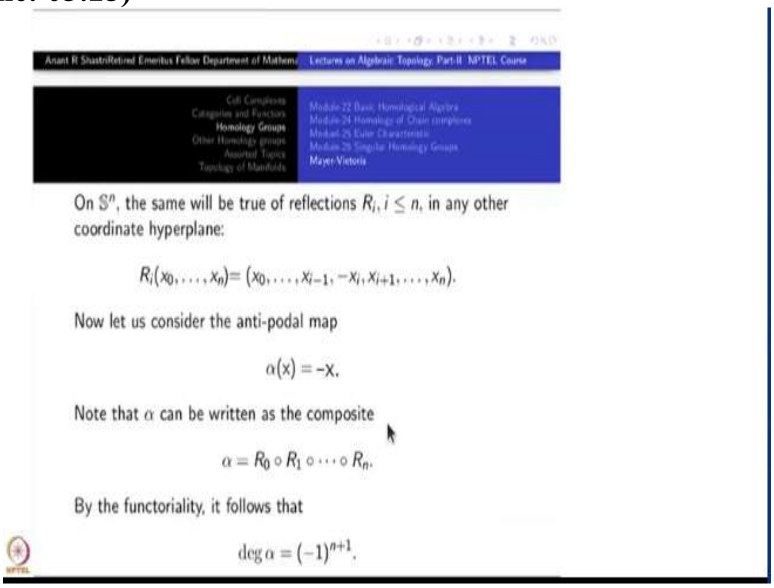

Therefore, for all i, if R_i takes $(x_0, \ldots, x_i, \ldots, x_n)$ to $(x_0, \ldots, -x_i, \ldots, x_n)$, (all x_j, $j \neq i$ are kept fixed and i-th coordinate becomes minus of itself), viz., if R_i is the reflection in the hyperplane perpendicular to i-th coordinate axis, okay, then its degree is -1.

Now let us consider the antipodal map, antipodal map changes sign of all the coordinates. Therefore, you can write it as α equal to the $R_0 \circ R_1 \circ \cdots \circ R_n$, the composite of $(n+1)$ reflections. By the functoriality α_* will be the composite of $(R_0)_*, (R_1)_*, \ldots, (R_n)_*$. Therefore it follows that degree of α is $(-1)^{n+1}$ okay.

So we have computed the degree of α very effortlessly okay? This could have been done (and it is done) in different methods. In a differential topology course, you may come across the degree concept in a different way. Indeed, it actually comes from complex analysis of 1-variable. ok? For example, while learning the fundamental group of a circle also you must have studied the degree of a map from $\mathbb{S}^1$ to $\mathbb{S}^1$.

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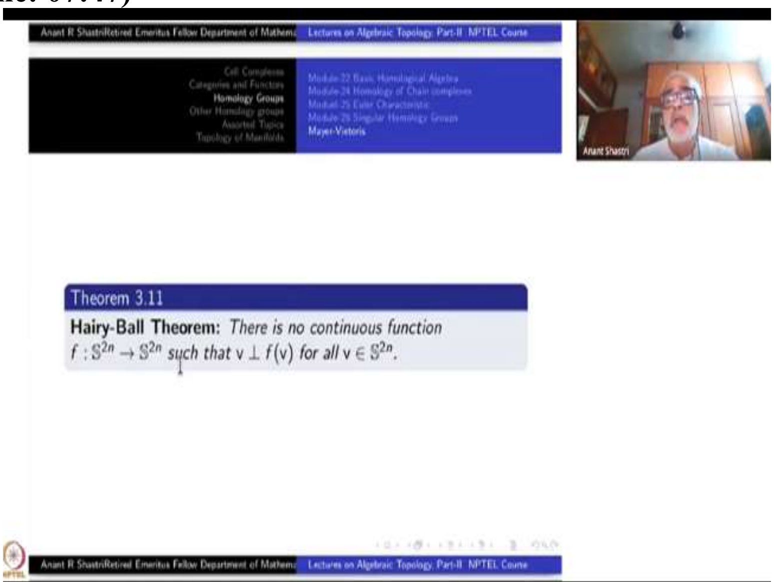

Here is a quick and a very interesting theorem which you must have learnt elsewhere also alright? Once you know the degree of the antipodal map you have this theorem which is called Hairy-Ball theorem in differential topology and which has roots in differential topology. What it says is that there is no continuous function f from $\mathbb{S}^{2n}$ to $\mathbb{S}^{2n}$ such that $f(v)$ is perpendicular to v at every v in $\mathbb{S}^{2n}$.

Such a function is actually called a nowhere-vanishing vector field. Actually, here it is the unit vector field, because I have $f(v)$ also a unit vector. On an even dimensional sphere every vector field has to vanish somewhere, that is the statement in differential topology.

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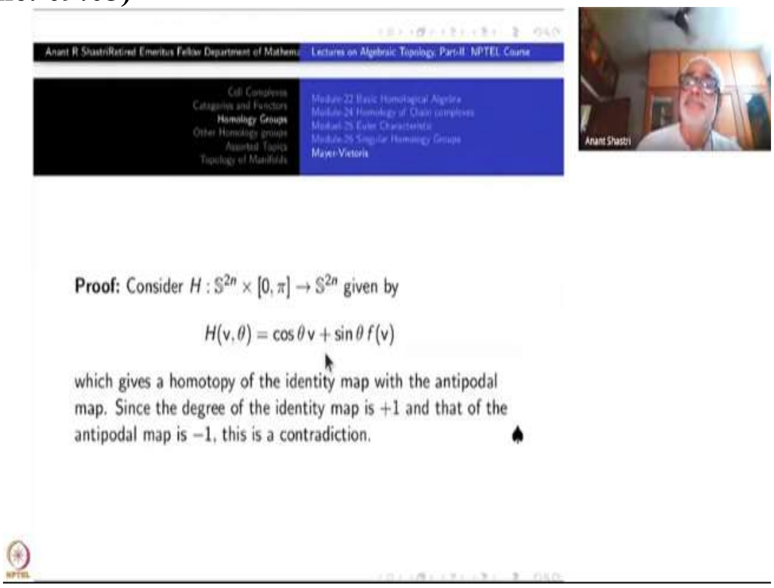

The proof is just one line, namely, if there is such an f, then consider the homotopy H from $\mathbb{S}^{2n} \times [0, \pi]$ to $\mathbb{S}^{2n}$ given by $H(v, \theta) = cos\theta v + sin\theta f(v)$. Remember v and f are perpendicular to each other. Therefore, if you compute the norm of this one, this will be also equal to one because $H(v, \theta) = cos\theta\ v + sin\theta\ f(v)$. okay? The norm of $H(v, \theta)$ will be again 1. Therefore H takes values inside $\mathbb{S}^{2n}$. Because f is continuous, it follows that H is continuous, Okay?

For $\theta = 0$, it is v, so it the identity map and for $\theta = \pi$, it is $-v$ which is the antipodal map. That means that identity map and antipodal maps are homotopic to each other. That is a contradiction because identity map is always degree $+1$ and whereas the degree of α is equal to $(-1)^{2n+1} = -1$, okay?

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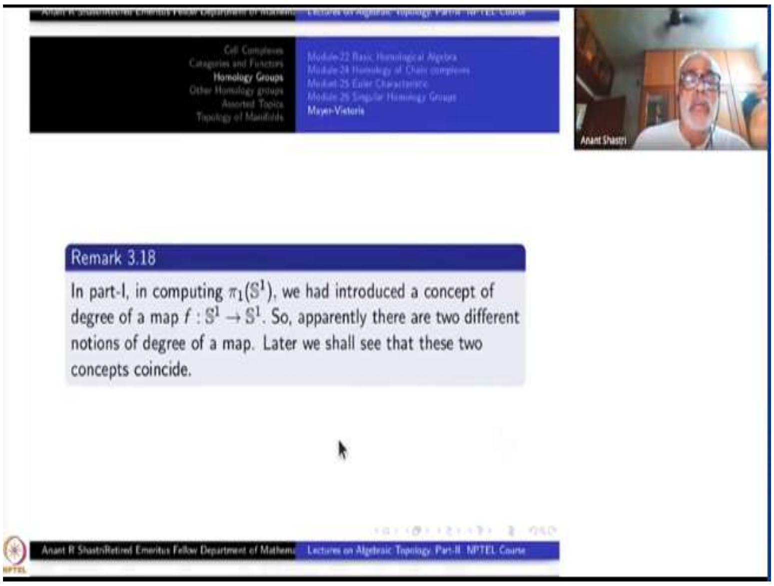

So let us carry on with one more comment here. In part one, we have introduced the degree of a map from $\mathbb{S}^1$ to $\mathbb{S}^1$, using which we computed the fundamental group $\pi_1(\mathbb{S}^1)$ and showed that it is isomorphic to the group of integer. The two definitions of the degree are quite different ones but they represent the same thing.

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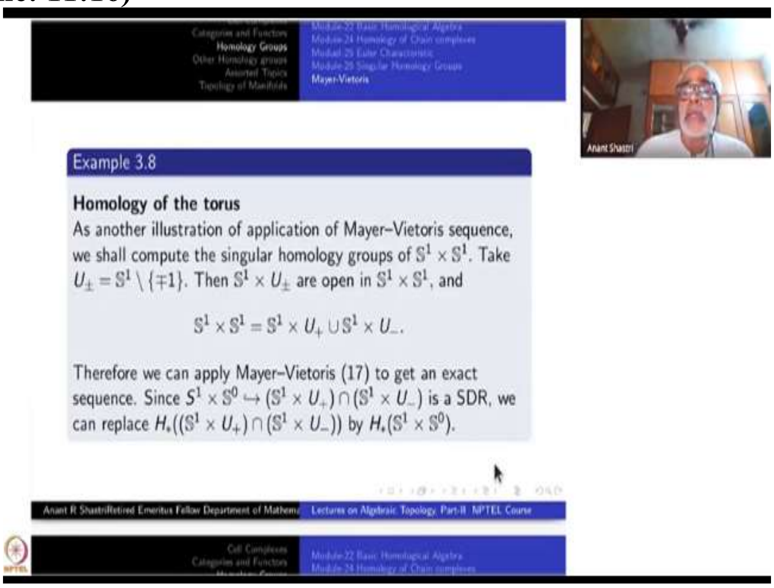

So now let us do some more computation this time with a torus. By a tours I mean the product of $\mathbb{S}^1$ with yourself, $\mathbb{S}^1 \times \mathbb{S}^1$, okay? So to apply Mayer-Vietoris sequence, just like in the case of $\mathbb{S}^1$, but this time we will break it up along the second factor, okay? Keep the first factor $\mathbb{S}^1$ as it is. Write the second factor $\mathbb{S}^1$ as the union of U_+ and U_- equal to $\mathbb{S}^1 \setminus \{-1\}$ and $\mathbb{S}^1 \setminus \{1\}$. So when you delete -1 then you call it as U_+ and when you delete $+1$, you call it U_- Okay?

Then $\mathbb{S}^1 \times U_+$ and $\mathbb{S}^1 \times U_-$ are two open subsets; they cover the whole of $\mathbb{S}^1 \times \mathbb{S}^1$, so we can apply Mayer-Vietoris sequence here because they form an excessisve couple okay? So Mayer-Vietoris sequence (22) can be applied to get a long exact sequence okay? But before writing down this, we will make some simplifications, namely the subspace $\mathbb{S}^1 \times \mathbb{S}^1$ is sitting inside the intersection of $\mathbb{S}^1 \times U_+$ and $\mathbb{S}^1 \times U_-$ is strong deformation retract, right?

This we have seen in the computation of homology of $\mathbb{S}^1$ itself. So therefore, we can replace the H_* of the intersection by $H_*(\mathbb{S}^1 \times \mathbb{S}^0)$. Since $\mathbb{S}^1 \times \mathbb{S}^0$ is just the disjoint union of two copies of $\mathbb{S}^1$, we know its homology groups completely.

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So we get the following exact sequence, 0 to H_2 of the total space then you have the connecting homomorphism to H_1 of the intersection which has been replaced by $H_1(\mathbb{S}^1 \times \mathbb{S}^0)$, then you have i_* here which is defined to be a maps to $(a, -a)$, right? To $H_1(\mathbb{S}^1 \times U_+) \oplus H_1(\mathbb{S}^1 \times U_-)$ okay? then you have j_* which sends (a_1, a_2) into $a_1 + a_2$, into the H_1 of entire space, $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$. Once again there is the connecting morphisn δ to $H_0(\mathbb{S}^0 \times \mathbb{S}^0)$ to the direct sum and then to H_0 of the total space. That is where the exact sequence stops.

What are these terms? The term before this, what is it? This is H_2 of the direct sum? What are the spaces? $H_2(\mathbb{S}^1 \times U_+)$ is same as $H_2(\mathbb{S}^1)$ because U_+ is contractible and hence $\mathbb{S}^1 \times 0$ is a deformation retract of $\mathbb{S}^1 \times U_+$. And we have computed $H_i(\mathbb{S}^1)$ to be 0 for all $i > 1$. It follows that $H_i(\mathbb{S}^1 \times \mathbb{S}^1)$ is also 0 for all $i > 2$. So, we we need to concentrate on on the above end tail-end of the exact sequence.

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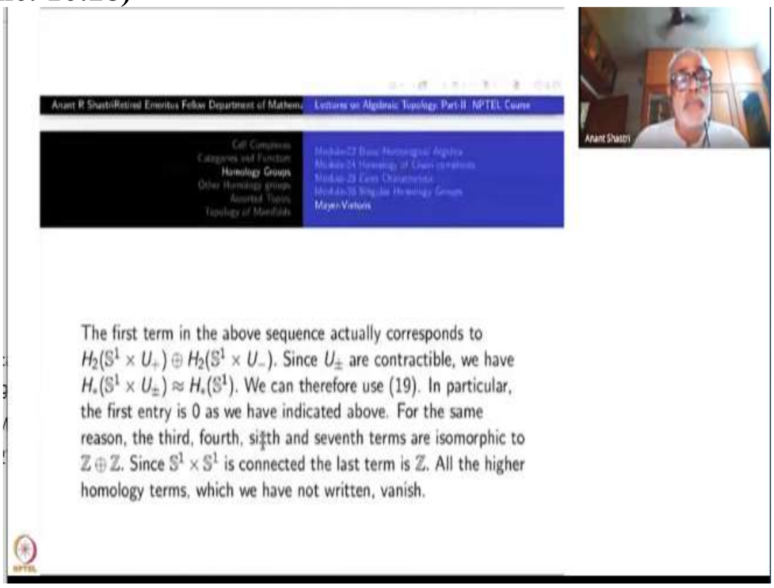

The first term in the sequence, as we have indicated above is 0. For the same reason the third, fourth, sixth and seventh terms are isomorphic to the direct sum of two copies of $\mathbb{Z}$. Okay? Since $\mathbb{S}^1 \times \mathbb{S}^1$ is path connected, the last term $H_0(\mathbb{S}^1 \times \mathbb{S}^1)$ is infinite cyclic. For any continuous function f into a path connected space, it is easily checked that f_* at the H_0 level is surjective. Therefore the last morphism j_* is surjective. Hence its kernel is isomorphic to $\mathbb{Z}$, which is equal to the image of i_* from $H_0(\mathbb{S}^1 \times \mathbb{S}^1)$. Therefore, the kernel of i_* is also infinite cyclic which is equal to image of δ from $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$.

We now need specific information on j_* from $H_1(\mathbb{S}^1 \times U_+)$ direct sum $H_1(S^1 \times U_-)$ to $H_1(\mathbb{S}^1 \times S^1)$. If p from $\mathbb{S}^1 \times \mathbb{S}^1$ to $\mathbb{S}^1$ is the projection to the first coordinate then we know that $p \circ j(z_1, 0)$ is z_1. Hence it follows that j_* (1, 0) is non zero in $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$. If follows that $j_*(1, -1) = 0$. Therefore both kernel and image of j_* are isomorphic to $\mathbb{Z}$. It follows that there is an exact sequence 0 to $\mathbb{Z}$ to $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ to $\mathbb{Z}$ to 0. Hence $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

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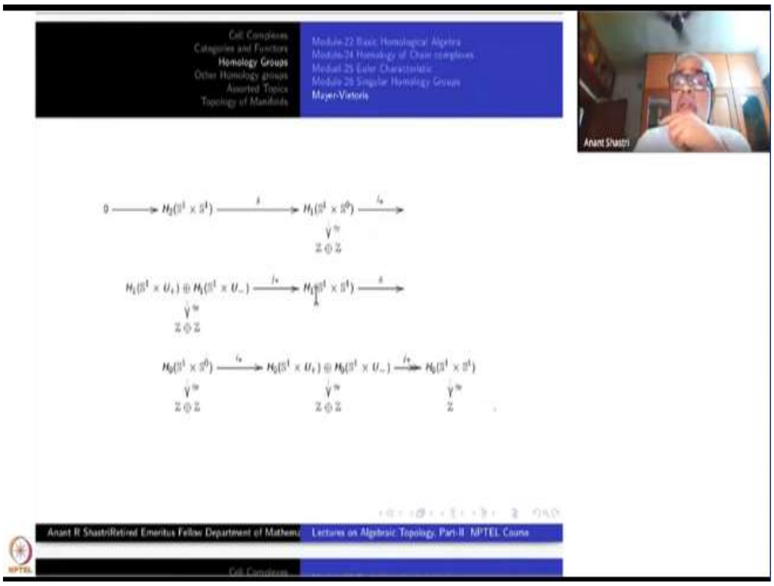

Finally, I have come some information on the image of δ_* Being equal to kernel of j_*, it is infinite cyclic group. But δ is also injective. Therefore, $H_2(\mathbb{S}^1 \times \mathbb{S}^1)$ is infinite cyclic. Once again the same kind of thing happens to that infinite cyclic group will come to direct someone of this one why this I do not know no okay it is kernel will be equal image of j_*.

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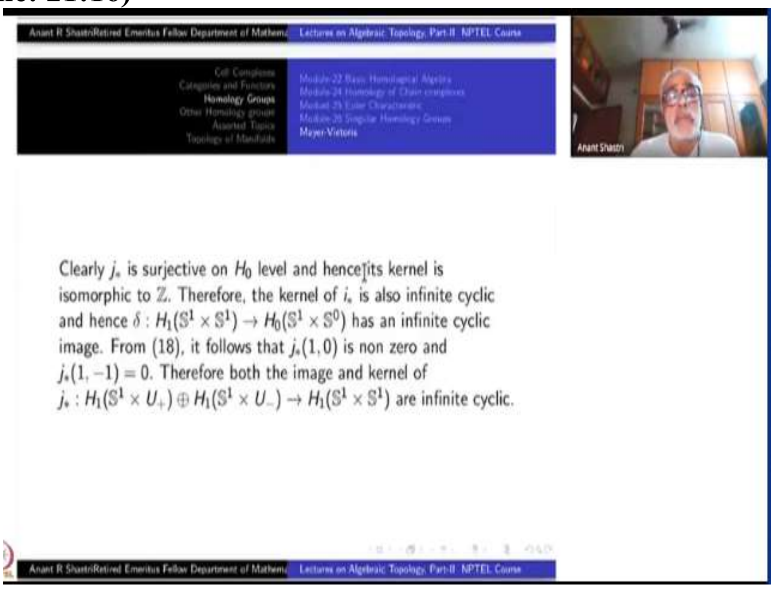

So, let me repeat whatever we have done here. j_* is surjective on $H\_$ level and hence its kernel is isomorphic to $\mathbb{Z}$. Therefore the image and kernel of i_* at the H_0 level are isomorphic to $\mathbb{Z}$. Therefore delta from H_1 to H_0 also has kernel equal to $\mathbb{Z}$. okay? So I have come up to the conclusion that δ from $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ to $H_0(\mathbb{S}^1 \times \mathbb{S}^0)$ has infinite cyclic image. Okay? Now let us look at what this morphism j_*. The domain of j_* is $H(\mathbb{S}^1 \times U_+) \oplus H_1(\mathbb{S}^1 \times U_-)$. Since $U_\pm$ is contrctible, we can take the restriction of j to the subspace $\mathbb{S}^1 \times 0$ on both factors. Now if we write 1 for the generator of $H_1(\mathbb{S}^1)$, then we can use the notation $(1,0)$ and $(0,1)$ for the generators in the two summands respectively. It follows that $j_*(1,0)$ is non zero element in $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ and also $j_*(1,-1) = 0$. Thus at this stage, we have used a very specific property of j_* on the H_1 level.

It follows that the image if j_* is infinite cyclic because only one component survives okay? So image of j_* infinite cyclic subgroup of $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$. We have to observe that it is equal to kernel of δ. It is also important that the image of δ is also infinite cyclic. Since a surjective homomorphisnm from an abelian group to an infinite cyclic groups splits, we can now conclude that the domain is the direct sum of the kernel and the image of δ. That is important, otherwise, you will not be able to say it is a direct sum. Just surjective morphism does not mean that you have a splitting, that the image is a free abelian group, an infinite cyclic group, that helps. So $H^1(\mathbb{S}^1 \times \mathbb{S}^1)$ is the direct sum of $\mathbb{Z}$ with itself.

Now here is again $H^1(\mathbb{S}^1 \times \mathbb{S}^1)$ is free abelian group of rank 2, image of i_* is infinite cyclic and hence kernel is also infinite cyclic. Therefore the image of δ is infinite cyclic. But δ is injective. Therefore, δ defines an isomorphism of $H^2(\mathbb{S}^1 \times \mathbb{S}^1)$ with its image which is infinite cyclic group.

So this completes the computation of $H_*(\mathbb{S}^1 \times \mathbb{S}^1)$. The second homology is an infinite cyclic group. The first homology is an abelian group of rank 2, the 0-th homology is also infinite cyclic because the space is path connected. All groups beyond the level 2 are 0.

So you go through the proof again and again because this is a typical way exact homology sequences are used. Though not all the time you will get all these information. You were lucky enough to have many things here. But this is the kind of argument you have to go through whenever you use Mayer-Vietoris sequence to extract some information okay?

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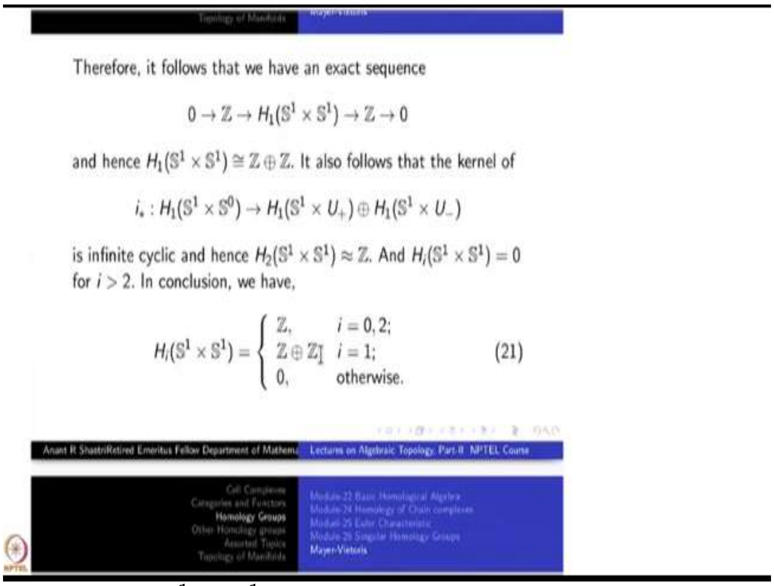

So let us sum it up here: $H_i(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}$, $i = 0$ and 2, H_1 is $\mathbb{Z} \oplus \mathbb{Z}$ and 0 otherwise.

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So the example is a typical way Mayer-Vietoris sequence can be employed in specific situations. Of course, there are more elegant proofs of this this result they are called connect formula.

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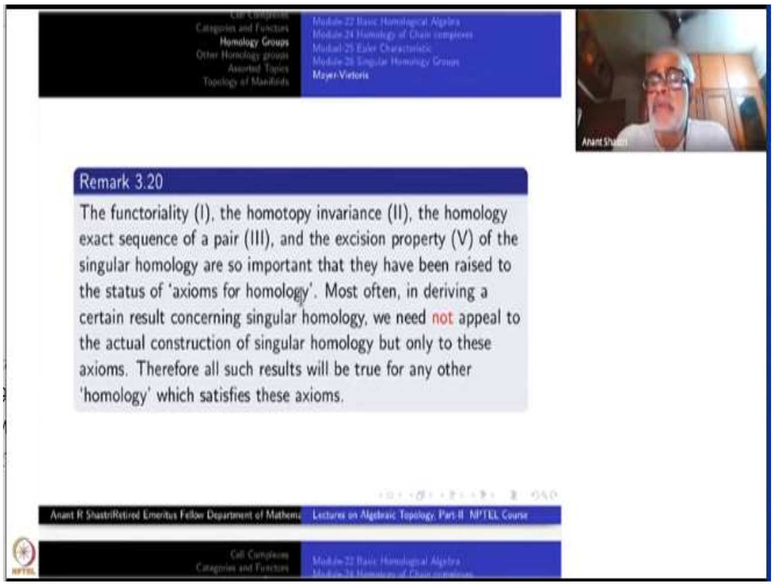

So I want to make a general remark here. Functoriality of the homology is the first thing. Then the homotopy invariance properties of the singular homology, the third one is the homology exact sequence of a pair and the fourth one is the excision property. The fifth one is the dimension axiom. Okay? These things are so important that they have been raised to the status of axioms for homology.

This is not done by one single author, I mean, not by one two authors, though Eilenberg and Steenrod consolidated it around 50, when they established general homology theory. Most often in deriving a certain result concerning singular homology, we need not appeal to the actual construction, but only use these four or five properties in a clever way and get many many results.

Therefore, all such results will be true for any other homology, which also satisfies these axioms. Remember, we are assigned homology modules to a pair of topological spaces, via a chain complex. That is not the case always. One could get many other homology theories which arise, in different ways some times using special structures on topological spaces. Then you can create your own homology in a very different way.

So, what do we mean by homology theory here? Any functor which satisfies some of these very first things, functoriality, homotopy invariance etc., Nobody cares about anything which is not functorial. Similarly, homotopy invariance is another thing, the long homology exact sequence yet another and so on. In case of singular homology, of a chian complex, it was a free gift from algebra. But if your functor is not coming via a chain complex, this free gift may not be there, and so you will have to verify it directly. There are some homology theories which do not satisfy the excision property so strongly. This is again a very very crucial one in, you know what, in computing homology from small open subsets to larger spaces.

One is not so much bothered about the property 4 which was the computation of homology of path connected components, namely, the singular homology of a topological space is a direct sum of the singular homology of its path connected components. Also there is a slightly more general result, namely, if you have a disjoint union of a family of topological spaces then the homology is a direct sum of the homology of these member spaces. These properties are not considered to be fundamental.

Again it is very obvious that we can artificially introduce a kind of shift in the indexing of the singular chain complex and cause the dimension axiom to go wrong. There are many homology theories which do not satisfy the dimension axiom. They are usually called extraordinary homology homology theories. So, K-theory and bordism theory do and so on which we cannot discuss here are examples of extra ordinary homology theories. Okay?

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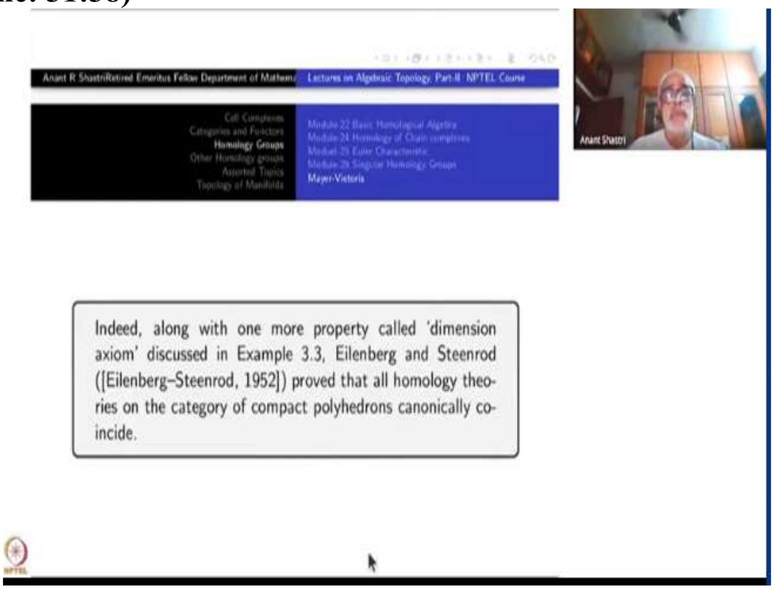

What Eilenberg and Steenrod did is that suppose that any two homology theories which satisfy this set of fundamental axioms on the category of compact polydrons, will coincide. That is a fantastic result, we will not be able to prove that one here. However, we have developed quite a different kind of tools here. I will show you several of them are equivalent, indeed, whatever theory is interesting to us, we will see by hand that it coincides with the singular homology. There are certain techniques there, if you master them, then it will be easy for you to go back and read Eilenberg-Steenrod's results. So that is the idea. So let us stop here. Next time we will try to do different kinds of homologies. Thank you.