Introduction to Algebraic Topology, Part – II
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Lecture – 29
Examples of Excision – Mayer Vietoris

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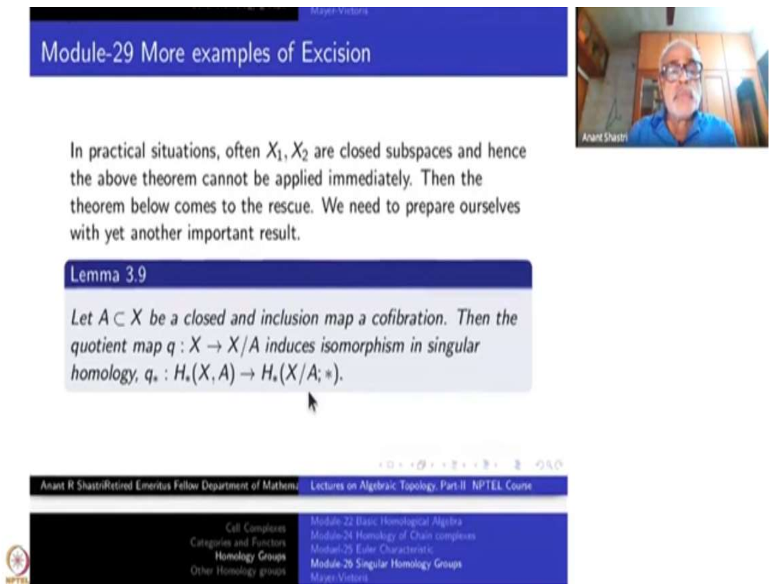

Last time we introduced the notion of excision, excessive couples, gave two easy examples of excessive couples, namely, $\{\mathbb{D}^n, \mathbb{R}^n \setminus 0\}$ that is one such pair; and another pair was $\{\mathbb{S}^n \setminus N, \text{the upper hemisphere}\}$. In practical situations often X_1 and X_2 nice subspaces often closed subspaces covering the entire space but they may not satisfy that their interiors cover the whole space. So, you cannot expect expect theorem to be directly used.

So, we have to do a little bit of circus here. So, there are quite a few situations which can be taken care by the following lemma, which is quite an important result, namely: Suppose A is a closed subspace of X, and the inclusion map A to X is a cofibration. Then the quotient map q from X to X/A induces isomorphism in the singular homology; q_* from $H_*(X, A)$ to $H_*(X/A, \{*\})$ is an isomorphism. Here I have denotes the point $\{A\}$ in the quotient space by $\{*\}$, since all points is A are identified to a single point in the quotient space.

One may anticipate that the relative homology is always the same as the homology of the quotient but there must be a slight modification. The role of taking the relative homology of $(X/A, \{*\})$ as we have anticipated is same thing as taking the reduce homology of X/A, $H_*(X/A, \{*\}) = \tilde{H}_*(X/A)$. But without the cofibration condition such a thing may not be true in general. So, the relative homology is not the same thing as the homology of the corresponding quotient space in general.

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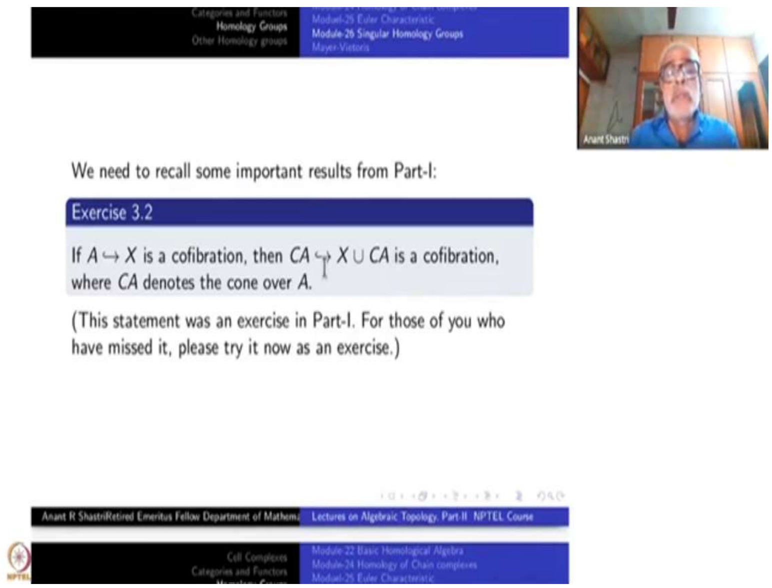

We need to recall some important facts from part I to prove this lemma, some of them not so results. The first thing was an easy one and it was left as an exercise in part 1. So, I will leave this as an exercise here so as to just refresh your memory. You worked it out as an exercise. Namely, if the inclusion map A to X is a cofibration then the inclusion map CA to $X \cup CA$ is also a cofibration, where CA denotes the cone over A. A is already a subset of X. Only on the part of A, you take the cone okay? So, $X \cup CA$ contains CA and the inclusion map is a cofibration.

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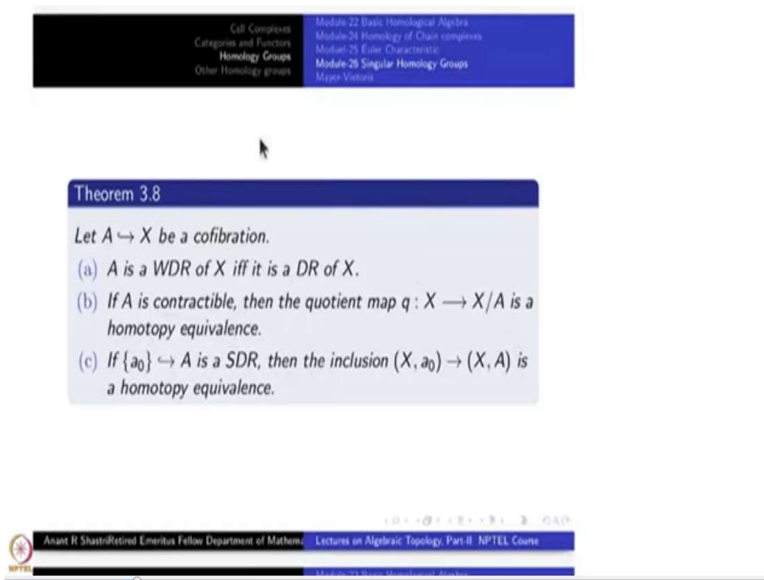

The second thing is if A to X is a cofibration and if A is contractible then the quotient map X to X/A is a homotopy equivalence, okay? So this is theorem 3.8 or some other theorem perhaps from part I. Here I have quoted the entire theorem for your ready reference. We need only the part (b) of it.

So, here in the above lemma, there is no hypothesis that A is contractible. Yet you can get homology isomorphism. The trick is to use this fact that CA which is a cone over A is contractible. Okay? So you can put this exercise and Part (b) of the above theorem, and get something. Okay? So that is what we're going to do now.

So, proof of lemma 3.9, this lemma we're going to prove okay? q_* from $H_*(X, A)$ to $H_*(X/A, \{*\})$ is an isomorphism. So, I recall what is the cone over A. The cone over A is the quotient of $A \times [0, 1]$, where $A \times \{1\}$ is identified to a single point. Note that A is identified with the image of the subspace $A \times \{0\}$ under the quotient map, via a going to $q(a, 0)$. By the theorem quoted above, by part (b) okay, this one combined with exercise 3.4 okay, we get that the quotient map p from $X \cup CA$ to $X \cup CA$(collapsing CA to a single point) is a homotopy equivalence, because CA is contractible and the inclusion map CA to $X \cup CA$ is a cofibration. So, this quotient map p becomes a homotopy equivalence, okay?

Next to apply the excision theorem, what do you want to do is: X_1 equal to $(X \cup A) \times [0, 1/2]$, and X_2 equal to CA. Then it follows that $X \cup CA$= interior of X_1 union interior of X_2, the interiors cover the whole thing. If you take the interior of CA in the whole space it is equal to $CA \setminus (A \times \{0\})$ and hence is open. And interior of X_1 is $X \cup [0, 1/2)$. So they cover the whole space.

Moreover, there is actually a strong deformation retraction of the pairs $(X_1, X_1 \cap X_2)$ to (X, A). Yeah, X_1 contains only the bottom half of the cone, $A \times [0, 1/2]$. So, you can push the whole thing back to X, via $(a, t) \mapsto a$. So, that gives a deformation retraction which is identity on X. $X_1 \cap X_2$ goes inside $A \times 0$. Therefore, the inclusion map here induces an isomorphism from $H_*(X, A)$ to $H_*(X_1, X_1 \cap X_2)$ which I have denoted by α. Okay, so, step by step we have to build up this final isomorphism namely q_*.

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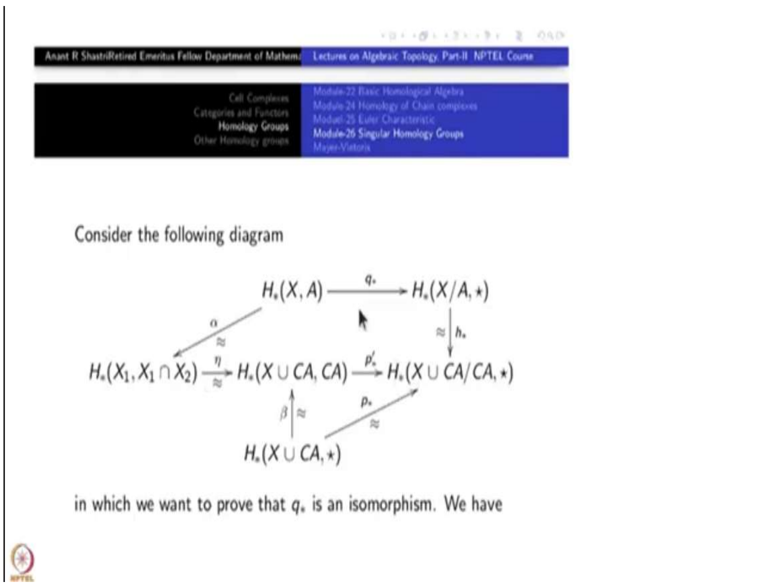

So now, with all this background, let us consider this somewhat complicated looking commutative diagram. The top arrow here is what we want to prove to be an isomorphism. To monitor Left most arrow is the isomorphism alpha that we saw. The right vertical arrow is actually indiced by a homeomorphism. A is a closed subset of X and you are collapsing it into a single point which is the same thing as first putting your cone over A and then collapsing the whole cone to a single point. Just the inclusion map X to $X \cup CA$ itself induces the homeomorphism h of the corresponsing quotients and h maps the special point to the special point as well. So, h_* is an isomorphism in homology.

Now, you can come to the second row here. The first arrow η is the isomorphism given by excision theorem. Okay? Now, what is this upward vertical arrow β? This is the inclusion induced morphism. Since CA is a contractible, if you use the homology exact sequence of the two pairs, you will see that every third map will be a zero morphism and the next ones will be identity ismomorphisms. Therefore, β is an isomorphism. Indeed both domain and codomain of beta are isomorphic to the reduce homology $\tilde{H}_*(X \cup CA)$. Finally, we have already seen that p_* is an ismorphism since p is a homotopy equivalence.

It follows that p'_* whatever this map p' is, again a quotient map where CA has been collapsed to a single point. Thus the morphism q_* is an isomorphism, since the entire diagram is commutative.

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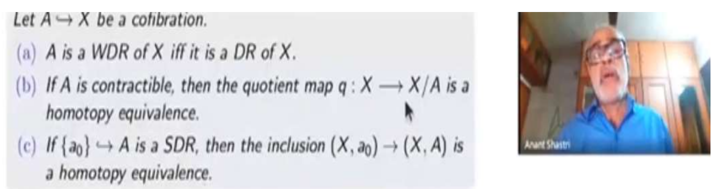

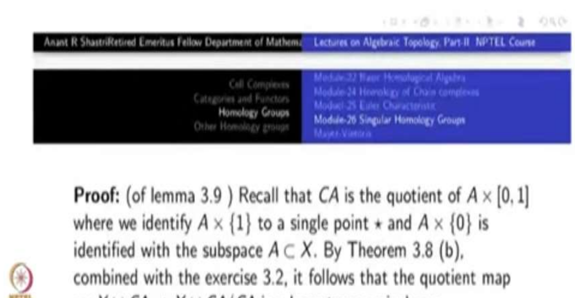

Thus you see that it takes so much effort to show that the q_* from $H_*(X, A)$ to $H_*(X/A, \{*\})$ is an isomorphism.

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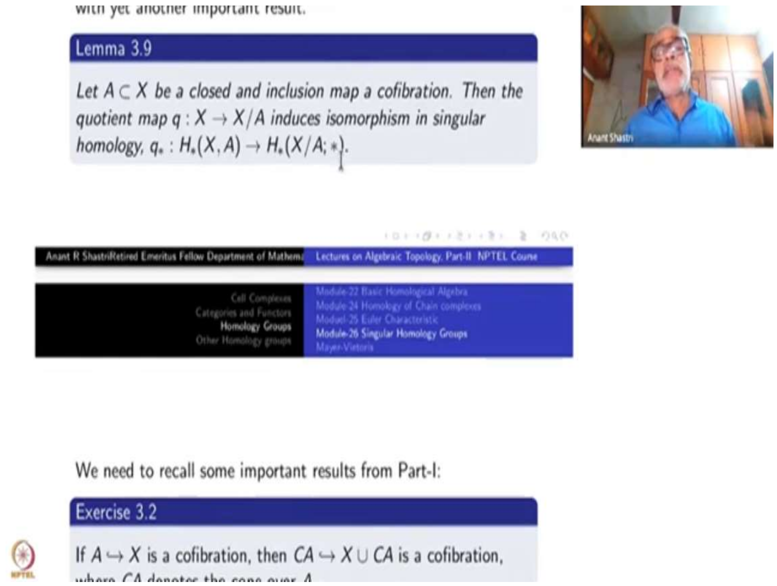

To sum up what we have is: whenever A is closed subspace which is contractible and the inclusion A to X is a cofibration, then the quotient map induces isomorphisms of homology modules q_* from $H_*(X, A)$ to $H_*(X/A, \{*\})$.

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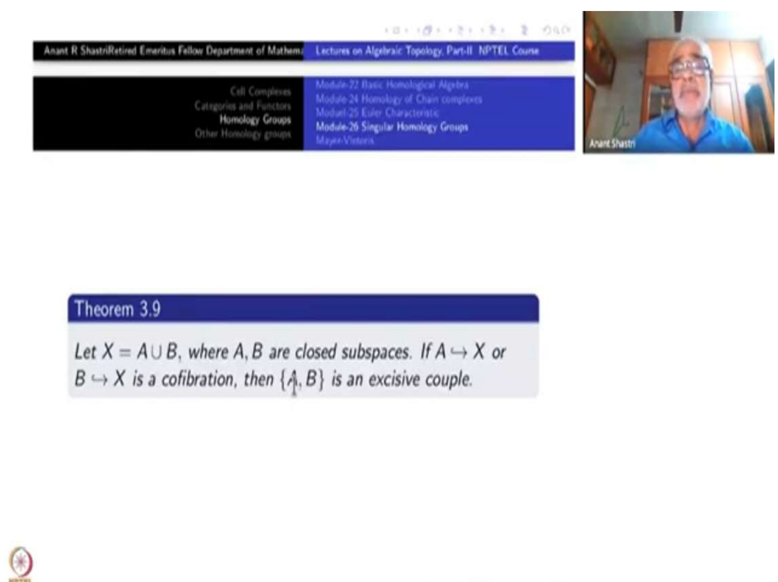

Now, let us see how we can proceed further. Now, come back to the situation of Excision. Suppose X is $A \cup B$, where A and B are closed subspaces. If the inclusion map one of them A to X or B to X is a cofibration, then $\{A, B\}$ is an excisive couple.

How are we going to prove this? We don't know whether interiors of A and B cover X. If that is the case then we can directly appeal to the excision theorem. I have already illustrated this situation namely, when you have written sphere as the union of upper hemisphere and lower hemisphere. But here, we cannot do that. So, there are many other situations like this. So, an alternative approach is that they are closed subspaces and the inclusion map of one of them is a cofibration.

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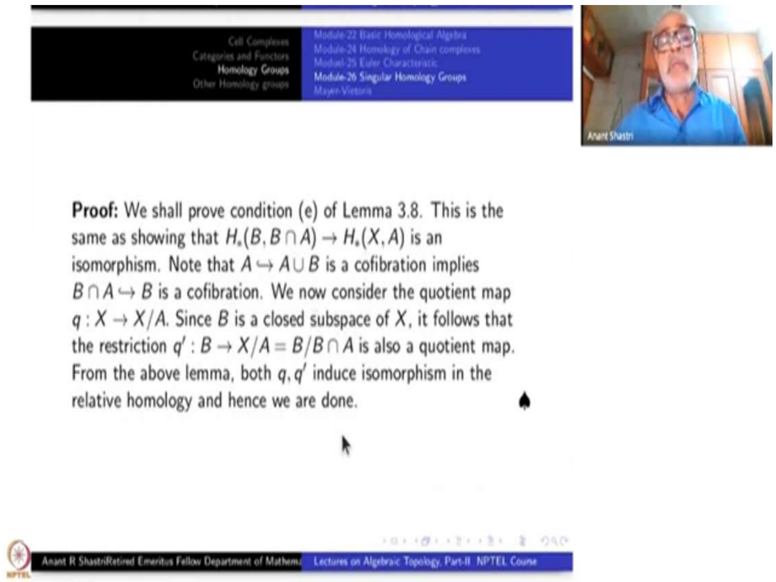

We shall prove condition (e) of lemma 3.8. Remember there are 5 different conditions for Excision okay? So, you can use one of them any one of them. So, condition (e) seems to be suitable now. Assume for definiteness that A to $A \cup B$ is a cofibration. So we shall show that the inclusion induce morphism $H_*(B, B \cap A)$ to $H_*(X, A)$ is an isomorphism. Note that A to $A \cup B$ is a cofibration implies $B \cap A$ to B is a cofibration. Okay? Take this as an exercise here.)

So, we have A to $A \cup B$ is cofibration, and $B \cap A$ to B is a cofibration. We now consider the quotient map X to X/A. We want to use this lemma. Since B is a closed space of X it follows that the restriction q' of q from B to X/A is also a quotient map.

From the above lemma, q and q' induce isomorphism of the relative homology. But we have already seen that the two quotient spaces are the same.

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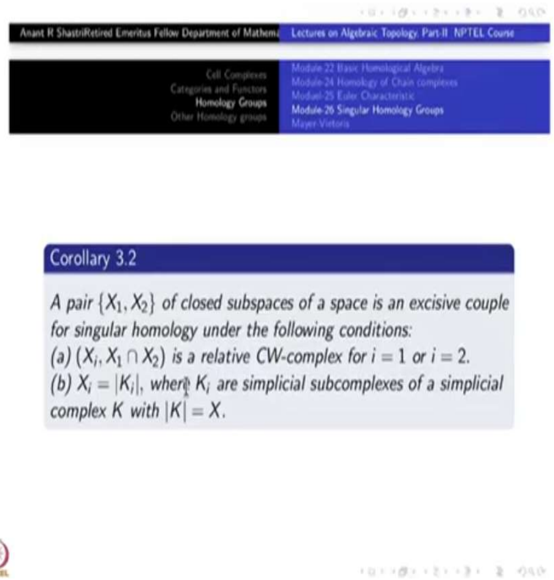

So, here is a corollary. Take a pair $\{X_1, X_2\}$ of two closed subspaces of a space X, okay? This is an excisive couple for singular homology under the following conditions, (a) or (b) whatever.

(a) $(X_i, X_1 \cap X_2)$ is a relative CW -complex for $i = 1, 2$.

(b) The other one is simplicial complexes X_i is $mod(K_i)$, where K_i are subcomplexes of a simplicial complex K with $mod(K_i) = X_i, i = 1, 2$.

You have proved that inclusion map of any sub complex whether it is a simplicial sub complex or a CW sub complex into a larger complex is always a cofibration. So this Corollary is a direct consequence of the above this theorem. This is a very useful result.

Thus CW compelxes and simplicial complexes are a rich source of cofibration and many fundamental results were invented about them in the beginning. For instance historically, I think it was Van Kampen who proved his version famous theory about fundamental groups of simplicial complexes.

It was Seifert who gave a much general version. Of course, they didn't collaborate with each other, it is not a joint work, though the theorem is usually called Seifert-Van Kampen theorem. In Europe they will use the name Seifert, whereas in America they call it Van Kampen's theorem. That is just a small historical digression.

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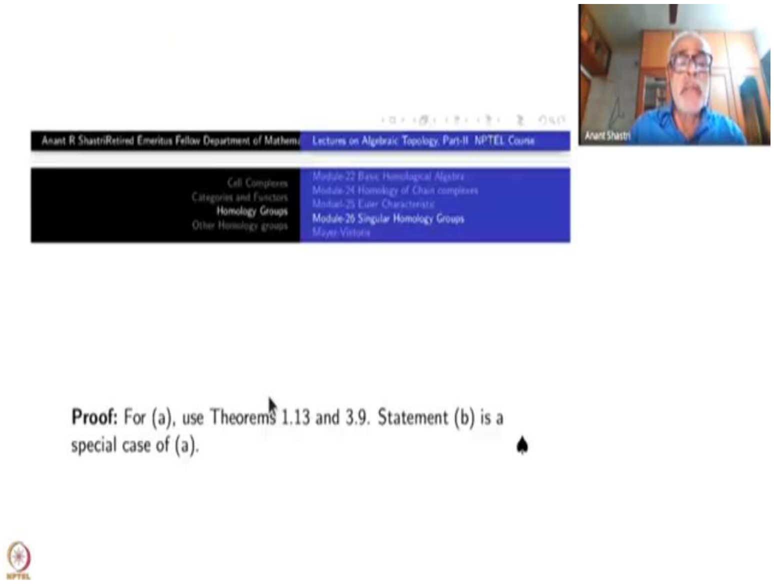

So, finally, let us produce this ready made long exact sequence due to Mayer-Vietoris. But this is just another step in just putting thing in an orderly fashion so that it helps to compute homology. So, we should concentrate on putting the Excision theorem to good use by first deriving an important ready to use result.

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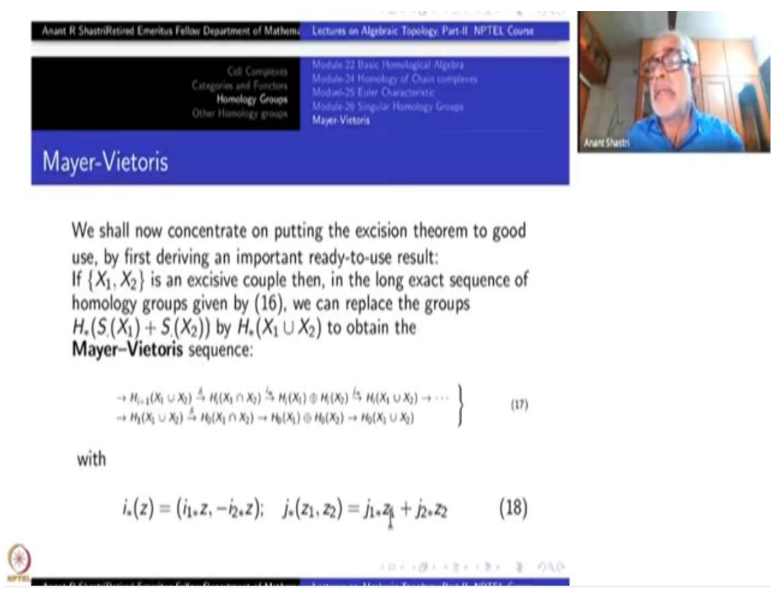

So, start with $\{X_1, X_2\}$, an excisive couple. Whether they are open subsets or satisfy cofibration etc is immaterial now. In the long exact sequence of homology groups numbered (21), namely associated to the short exact sequence 0 to $S.(X_1 \cap X_2)$ to $S.(X_1) \oplus S.(X_2)$ to $S.(X_1) + S.(X_2)$ to 0, we can replace $H_*(S.(X_1) + S.(X_2))/H_*(X_1 \cup X_2)$ to obtain a long exact sequence called the Mayer-Vietoris sequence associated to the excisive couple $\{X_1, X_2\}$. X_1 and X_2 that is the gist of the excisive couple expression the statement excisive couple actually took up. So, what we obtain is H_{i+1} of the union instead of a $H_{i+1}(S(X_1) + S(X_2))$, then there is this connecting homomorphism to $H_i(X_1 \cap X_2)$ okay. So, one index lower then it will come continue H_i have this one, this i_* is the inclusion remember, X_1, X_2. What is this one, they say? Inclusion, i_* is two start, then j_* is inclusion and $a_1 - a_2$.

So, I have given you the formulae for all the morphisms. $i_*(z) = (i_1)_*(Z) - (i_2)_*(z)$ and $j_*(z) = (j_1)_*(z) + (j_2)_*(z)$, (Earlier I had taken plus in the first and minus sign in the second, Here, I have deliberately interchanged them just to point out that both are valid, no problem at all okay?)

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So, where what are these i_1, i_2 and j_1, j_2? They are all inclusion induced maps of the respective spaces involved. Okay? So Mayer-Vietoris sequence is one of the major ready to use tool used in computing homology.

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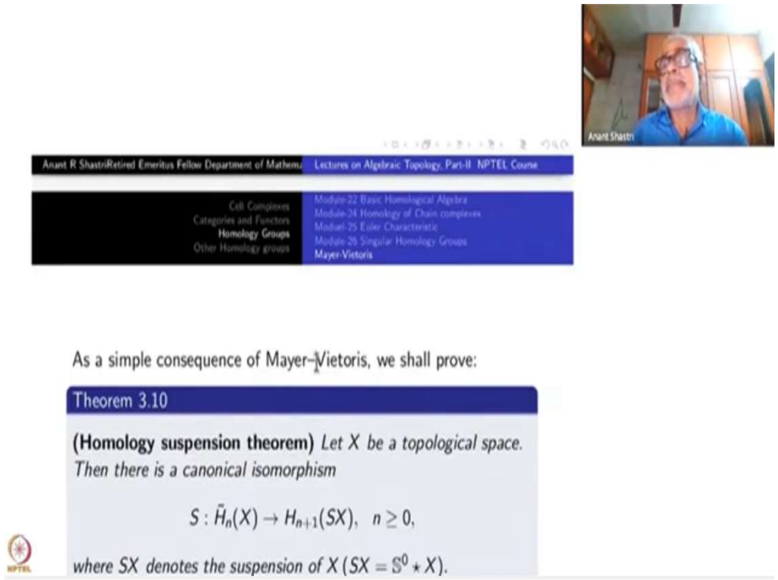

As a simple consequence of Mayer-Vietoris sequence, we are going to prove the result below. But that way we'll do next time. We'll stop here.