Introduction to Algebraic Topology, Part-II

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Lecture - 28
Excision

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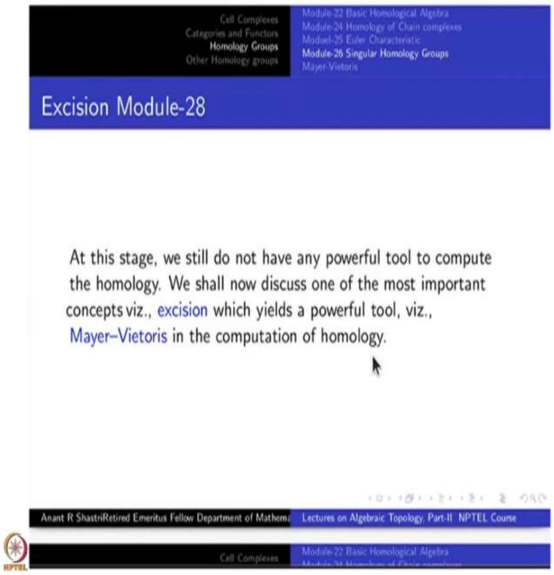

Having introduced the singular homology, having verified a number of its interesting properties and having computed the homology of a single point and proved that the singular homology is a direct sum of the homology of its components etc. yet at this stage, we still do not have any powerful tool to compute the homology.

So now, we shall discuss one of the most important concepts, namely, excision, which will yield a powerful tool for computing homology. This is called Mayer-Vietoris principle in real terms, which will give you a long homology exact sequence, and that will help a much more than whatever we have done so far.

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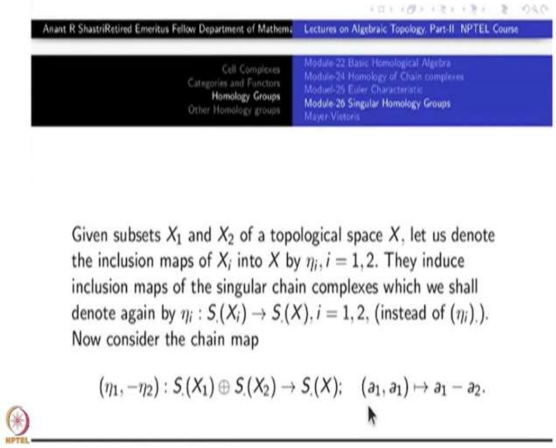

So, let us start with two open subspaces X_i of a topological space X and denote the inclusion maps by η_i. They will induce inclusion maps again from $S.(X_i)$ to $S.(X)$ of the singular chain complexes. Now, consider the chain map from the direct sum to $S.(X)$, namely, (a_1, a_2), (a general element in the direct sum) going to $\eta_1(a_1)$–$\eta_2(a_2)$.

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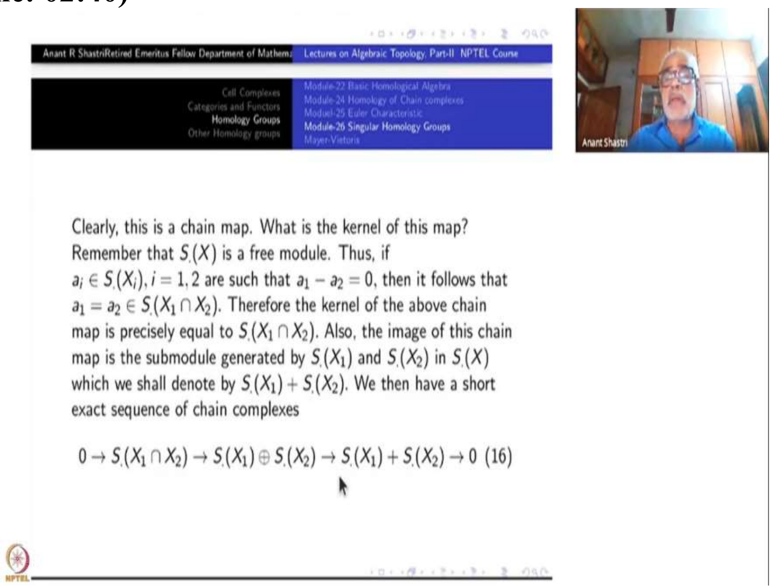

This will be clearly a chain map. The linear combination of chain maps is a chain map and the inclusion maps are chain maps. What is the kernel of this map? Remember $S.(X), S.(X_1)$ and $S.(X_2)$ they are all free abelian groups. So, the direct sum is a free abelian group. So, an element is 0 means all the coefficients of each singular simplex is 0. That is the meaning of certain element is zero in a free abelian group.

So kernel means what now? If a_1 and a_2 are chains in X_1 and X_2, and (a_1, a_2) is in the kernel of ($\eta_1 - \eta_2$), then corresponding coefficients of the singular chains must be identical for a_1

and a_2. That just means that a_1 is identically equal to a_2 in $S.(X)$. But one is in X_1 and another inside X_2. Therefore $a_1 = a_2$ must be inside $S.(X_1 \cap X_2)$. Further you can easily check that $S.(X_1 \cap X_2)$ is actually the entire kernel.

Next, what is the image of this? That is easy to check. It is just like sum of two elements one element from here and another element from here. The sum or the difference are the same sum of elements here. So that will be a submodule here and that submodule is usually written as just the without the without the direct sum notation, just the ordinary sum, $S.(X_1) + S.(X_2)$ is the image.

Therefore we have a short exact sequence, 0 to $S.(X_1 \cap X_2)$ to $S.(X_1) \oplus S.(X_2)$ to $S.(X_1) + S.(X_2)$ to 0. The first morphism here is a maps to (a, a) and the second one is $(\eta_1, -\eta_2)$. So, you know exactly what the morphisms are, though it is not written in the slide.

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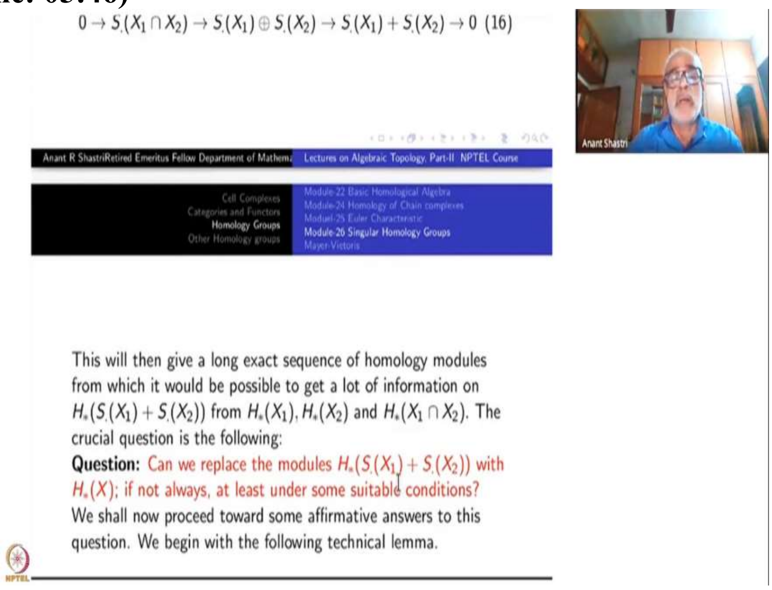

So this will then give you a long exact sequence of homology modules from which we hope, by experience that there will be some information on H_* of this chain complex here. Since H_* of the direct sum is the direct sum of H_*'s, this we know if you also know $H_*(X_1 \cap X_2)$, the intersection of the two space subspaces. We should know what is happening in X_1, X_2 and $X_1 \cap X_2$. Then we may be able to say something about the H_* of the sum. So, this theme which is known as Mayer-Vietoris, was there in the study of fundamental group under the name Van-Kampen's theorem. So, this is a general principle here which goes under the name Mayer-Vietoris, after two Austrian mathematicians of early last century.

So, the question is can we replace $H_*(S.(X_1) + S.(X_2))$ with $H_*(X)$? Why do we hope such a thing. To begin with we must assume that the space X is the union of X_1 and X_2. Without that topological hypothesis as the starting point, we should not proceed. But even after that can you do this algebra, viz., can one replace $H_*(S.(X_1) + S.(X_2))$ by $H_*(X)$? Does the inclusion induced from the submodule to the whole at the chain level induce isomorphism at the homology level? So this is the question, we shall now proceed toward some affirmative answers to this question.

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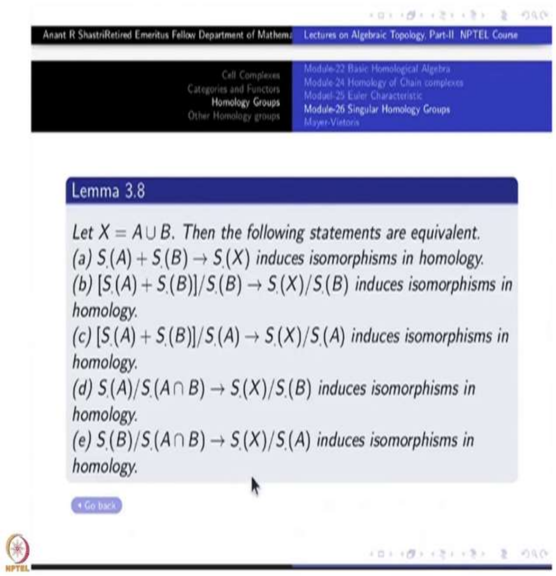

The first technical step is this mega lemma here before we go to the positive answer finally, the beautiful answer given by Mayer Vietoris sequence which will be our aim. So, this is lemma algebraic preliminary for that, about how to deal with this question, namely, four other equivalent statements to the statement that the inclusion induced morphism is an isomorphism.

I am changing notation here instead of X_1 and X_2, because this is applicable in a larger context. Start with $X = A \cup B$. The following statements are equivalent.

(a) the inclusion morphism of $S.(A) + S.(B)$ to $S.(X)$ induces an isomorphism in homology. So, this is a very clear statement namely the inclusion induced the homomorphism this homomorphism must be itself isomorphism that is what is demanded in this.

(b) the second statement is that $(S.(A) + S.(B))/S.(B)$ on both sides induces an isomorphism on the homology. Note that these quotient complexes are chain complexes, and

the chain maps are again induced by inclusion maps. When you pass to homology, it must be an isomorphism.
(c) The third condition is similar to (b) and is obtained by interchanging A and B. After going modulo $S.(A)$ instead of $S.(B)$. Obviously, (b) and (c) are These two are obviously equivalent to each other by symmetry.
(d) The fourth condition is slightly different now. Just take $S.(A)$ and go modulo $S.(A \cap B)$ on the left and take $S.(X)/S.(B)$ on the right. Obviously, there is a homomorphism at the chain complex level, again induced by inclusion maps. The statement is that this inturn induces isomorphism at the homology level.
(e) The fifth statement is similar to fourth one, again by interchanging A and B.

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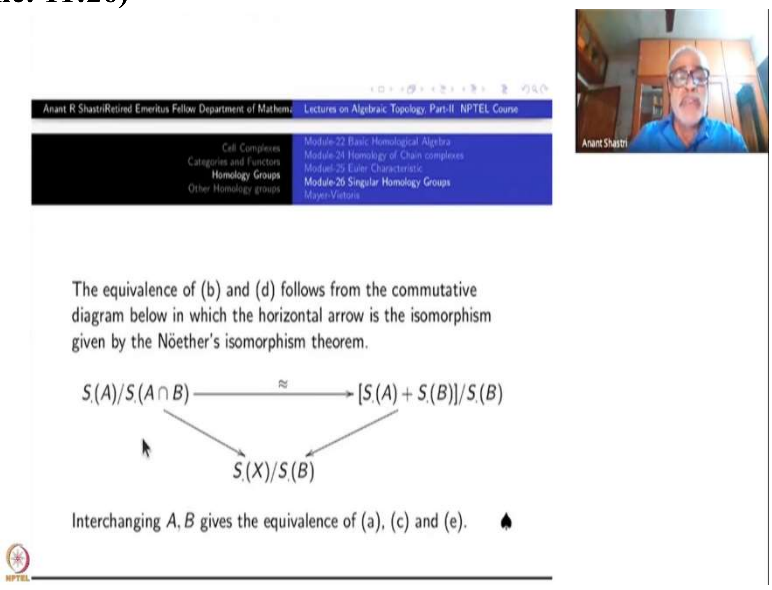

So, you know, these statements are similar to what you may call Noether's isomorphism theorem, first isomorphism and so on. It is of that nature. So let us quickly go through these equivalences. Equivalence of (a) and (b) is an easy consequence of five-lemma.

So we have to use this result which you have been introduced to, just a couple of days back. From $S.(A)+S.(B)$ to $S.(X)$, there is the inclusion map and this is the quotient morphism onto $S.(A)+S.(B)$ by $S.(B)$ and that is the kernel $S.(B)$. So, this is an exact sequence. Similarly, we have another exact sequence in the second row.

Let us now look at (b) and (d). $(S(A)+S(B))/S(B)$ here and I am taking modulo S. So, this is basically like the isomorphism theorem, (b) and (d) follow from the commutative diagram below in which the horizontal arrow is an isomorphism theorem given by Noether's isomorphism theorem. So, all the 5 statements are equivalent equal to each other.

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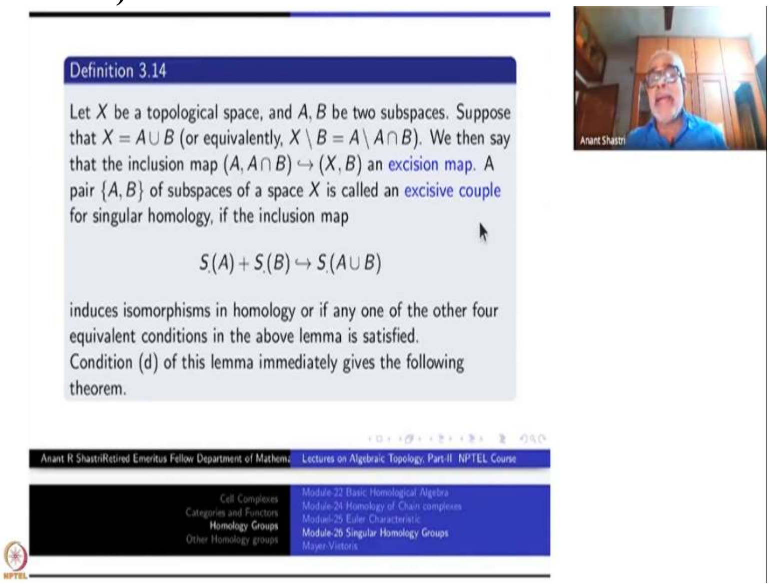

So, now we shall make a definition. Take a topological space X and A and B any two subspaces. (Just to our sake of definiteness, assume that X is equal to $A \cup B$, which is not necessary it will follow automatically from what we say next.) Suppose $X \setminus B = A \setminus (A \cap B)$. We then say that the inclusion map of the pairs $(A, A \cap B)$ to (X, B) is an excision map. So, I could have also said that $(B, A \cap B)$ to (X, A) is as excision map because both are equivalent to the condition that $X = A \cup B$. So, this is purely set theoretic condition.

Next, a pair (A, B) of topological subspaces is called an excisive couple (note that this definition is something different, and not really a set theoretic one) for the singular homology (this definition is with resect to a particular homology theory) excisive couple for singular homology excisive couple for some other homology and hence depends upon what homology you choose) if the inclusion map $S.(A) + S.(B)$ to $S.(A \cup B)$ induces isomorphism of the singular homology, that is, the first condition of this lemma is satisfied, once you assume $X = A \cup B$.

So, we are taking conveniently the first statement of the lemma, that is what our aim is after all, and converting it into a definition.

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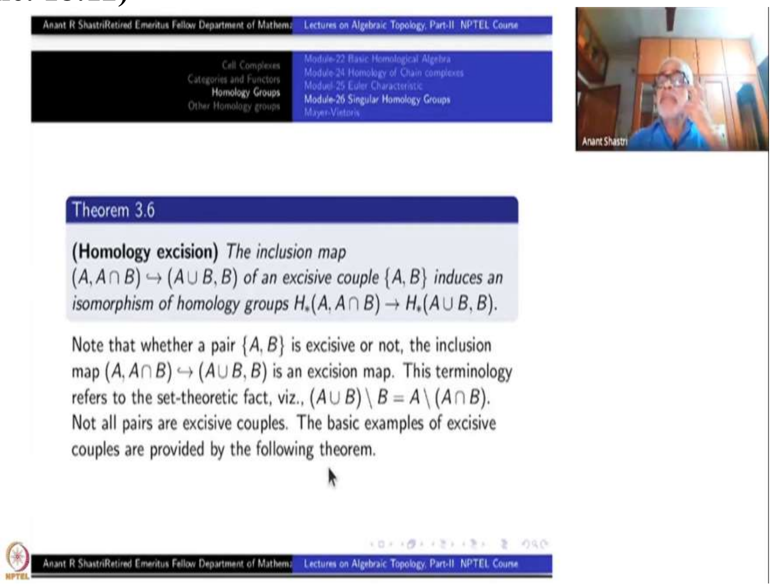

Clearly, we could have taken any one of the four other equivalent statements but now each of them becomes a theorem. Here we take statement (d) and restate it as a theorem.

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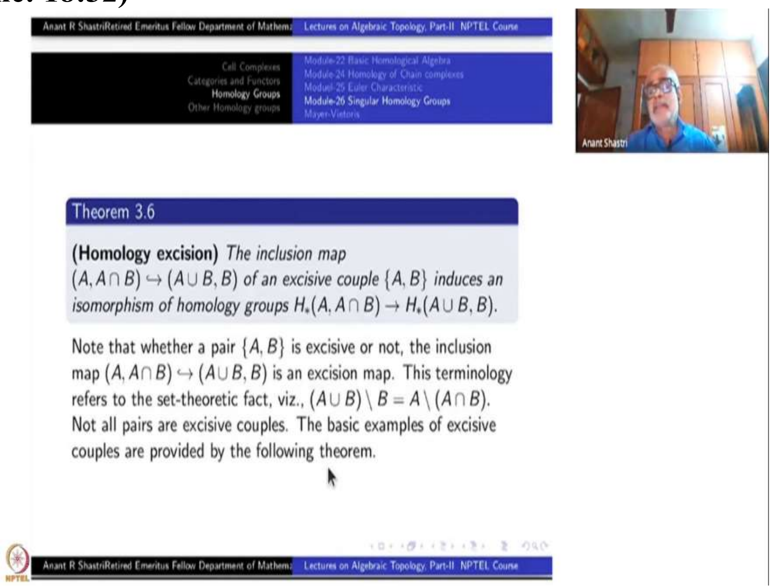

If $\{A, B\}$ is an excisive couple, then the inclusion map induces an isomorphism of $H_*(A, A\cap B)$ to $H_*(A\cup B, B)$. That is the statement (d) here. The first one is $H_*(S.(A)\S.(A\cap B))$ and the second one is $H_*(S.(A\cup B)/S.(B))$.

Note that a pair $\{A, B\}$ of subspaces of a space may or may not be an excessive couple, but the inclusion map $(A, A\cap B)$ to $(A\cup B, B)$ is always an excision map, which is purely to tell you that every thing is inside $A\cup B$.

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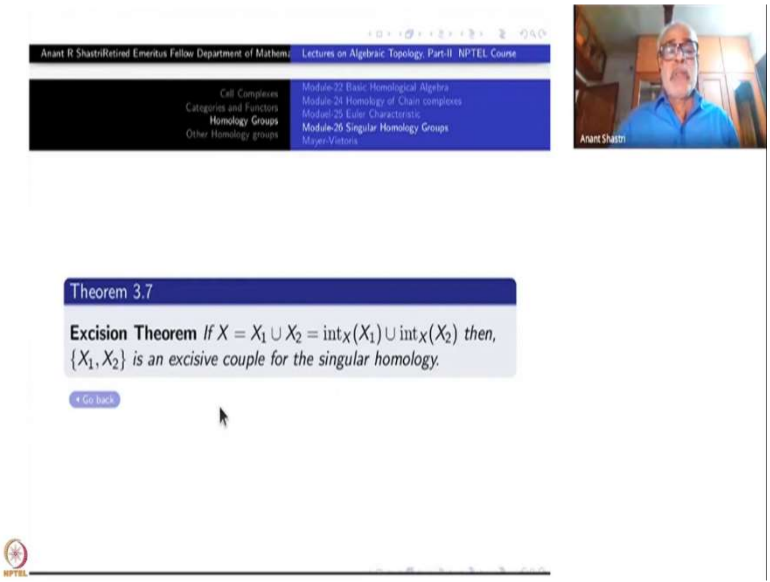

So excision theorem below is due to Mayer and Vietoris. The present form is due to Eilenberg and Steenrod. So, between Mayer-Vietoris invention of this idea to the present modern formulation of the statement, it took almost 20-25 years. Mayer-Vietoris did not state it in this way.

If X is the union of subspaces X_1 and X_2 in such a way that if we just take the interior of X_1 and interior of X_2 that must cover the whole of X, then $\{X_1, X_2\}$ is an excisive couple for singular homology. This is a statement if the theorem now. This just means that I can replace $S.(X_1) + S.(X_2)$ by $S.(X)$ itself. Replacing means what? After passing to the homology, the inclusion induced map itself is an isomorphism.

This condition may look a bit strange. But suppose X_1 and X_2 are open subsets. Then interior of X_i is X_i itself. So, this is the easiest situation when the excision theorem can be applied whenever the two subsets are open and they cover the whole space X. Then you are in good shape. You can compute the homology of the whole space in some sense by knowing the homology of X_1, homology of X_2 and the homology of the intersection. How you can do that? Via the long homology sequence. So that part we shall now explicitly state and that is what is called Mayer-Vietoris sequence.

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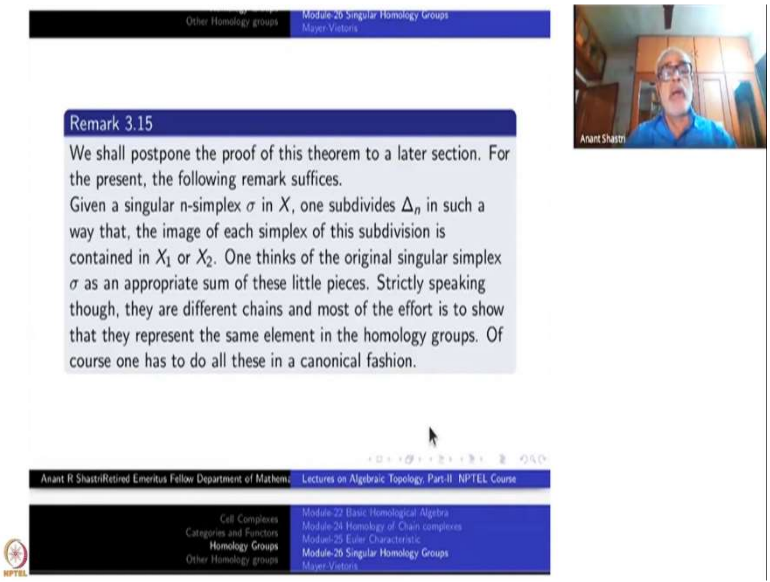

We shall postpone the proof of this theorem just like we have postponed the proof of homotopy invariance of the homology. For the present, we shall only make a remark about the proof. Assume for the time being that X_1 and X_2 are open.

Given a singular n-simplex σ in X, which may or may not be inside X_1 or X_2. Then the typical thing to do is, just like in all work in analysis, to cut down the singular simplex into finer pieces such that each part is inside X_1 or X_2. So, one subdivides Δ_n in such a way that the image of each sub-simplex of this subdivision under sigma is contained inside X_1 or X_2. One thinks of the original singular simplex σ as an appropriate sum of these little pieces.

It is just like in Riemann integration theory. If you have an interval of definition of a continuous function, you cut down the interval into two subintervals, and the integral on the first one plus integral on the second interval is equal to the integral on the entire original interval. Same is true for area integrals etc. The important thing is that the smaller parts would not `overlap' and still cover the whole. This is the motivation for homology, motivation and guidance from what happens in the integration theory in analysis.

You cut it down into little pieces and take an appropriate sum. Though they are different chains, when you pass onto homology they will represent the same stuff. Most of the effort goes into proving this part. Somehow, in our definition of singular homology there is no integration, nothing.

Of course, all this thing has to be done in a canonical fashion, not depending upon the actual nature of X_1 and X_2. Once you have the whole thing, the thing should work if you replace X_1, X_2 by some other Y_1, Y_2 subject to the only condition that they are open subspaces. So, let us stop here. For more explanations, actual proof, etc. we will have to wait.

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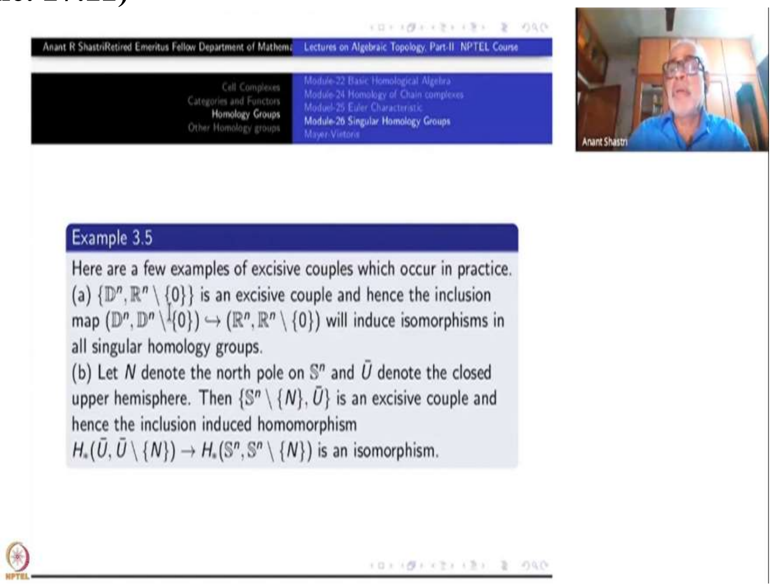

But now, let me give a few examples of excisive couples, which occur in practice. The first one is: take the disk $\mathbb{D}^n$, the closed unit disk or open disk, let us say open disk $\mathbb{D}^n$, and $\mathbb{R}^n \setminus \{0\}$. These are open sets, the union is $\mathbb{R}^n$, I want to say that this is an excessive couple because $\mathbb{D}^n$ is open subset, and $\mathbb{R}^n \setminus \{0\}$ is always an open subset, over. But now the conclusion is that the inclusion map $(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\})$ to $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ induces isomorphisms in the singular homology groups. This is one of the very useful things. This is the starting point of our discussion in manifold theory and so on, we will see that.

The second example: Take the sphere $\mathbb{S}^n$. Let N and S denote the north pole and the south pole respectively. I am just looking at the North pole now. And let $\bar{U}$ denote the closed upper hemisphere. Then $\mathbb{S}^n \setminus N$ is what? The top point, the north pole is deleted. The pair $\{\mathbb{S}^n \setminus N, \bar{U}\}$ is an excessive couple and hence the inclusion induced map should be an isomorphism in the homology $H_*(\bar{U}, \bar{U} \setminus N)$ to $H_*(\mathbb{S}^n, \mathbb{S}^n \setminus N)$.

I have deliberately take $\bar{U}$ here. $\bar{U}$ is not an open set but interior of $\bar{U}$ and $\mathbb{S}^n \setminus N$, they are open subsets and they cover the whole space. So, $\bar{U}$ is not open here. Similarly, in the first example also, see there also I told you $\mathbb{D}^n$ would have been closed disc. The interior of $\mathbb{D}^n$ and $\mathbb{R}^n \setminus 0$, they are open and then they cover whole of $\mathbb{R}^n$. That is enough. That was this kind of statement here in this theorem interior of X_1 and interior of X_2 should cover the whole space.

So these two examples will be used again. that they give excision of isomorphisms. For more examples, you will have to wait. So today let us stop here.