Introduction to Algebraic Topology (Part – II)
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Lecture –03
Subcomplexes and Examples

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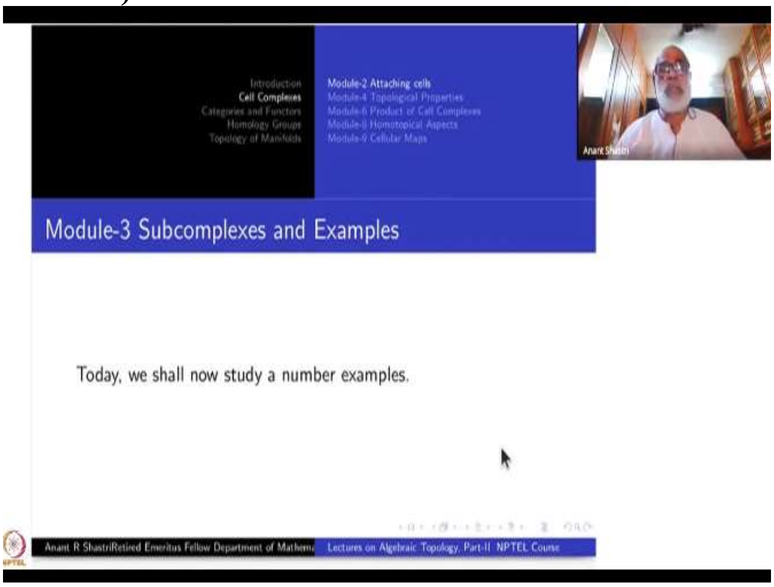

Today, we shall now study a number of examples. Before that I will give you one more definition, namely, subcomplex.

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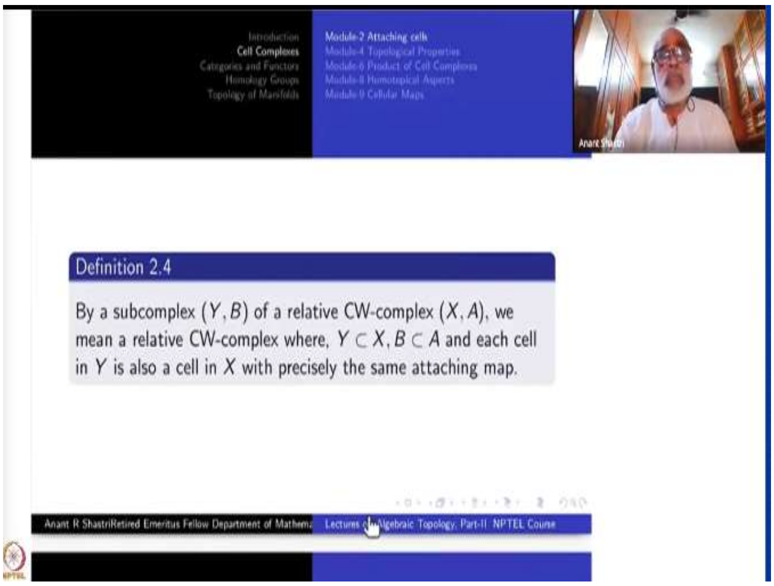

Like in the case of subcomplex of a simplicial complex, there is a definition of subcomplex in the case of CW-complexes also. Let us define the notiton of a subcomplex of a relative CW-complex. It is going to be a relative CW complex (Y, B) and a subcomplex of a relative CW complex (X, A). So, (X, A) is a relative CW complex, and (Y, B) is a subcomplex. What is the meaning? First of all

(i) the topological pair (Y, B) itself should be a CW complex on its own and then

(ii) Y is a subset of X, B is a subset of A;

(iii) each cell that you are attaching in getting Y from B along with the attaching maps should come from the corresponding cells that you are attaching in getting X from A.

Each cell in Y should be also cell in X with precisely the same attaching map. In other words, in the collection of attaching maps of X you may delete some of them to obtain Y.

But if you delete arbitrarily it may not be a subcomplex. Because, whatever attaching maps are remaining their codomain must be appropriate. If you have deleted cells which form part of codomain of the attaching map of a latter cell, then that cell also have to be deleted. So, there is a very strong restriction on being a subcomplex. Recall for a subgroup of a group, there is a group operation, the set must be a subset of the original set, but the group operation should be also the same as the original one. In the case of subcomplexes, it is the attaching maps and attaching cells, they must be the same as original one, and for each k, the collection of attaching maps of k-cells for Y should be also a subset of the collection of the attaching maps for X. That is the meaning of this subcomplex.

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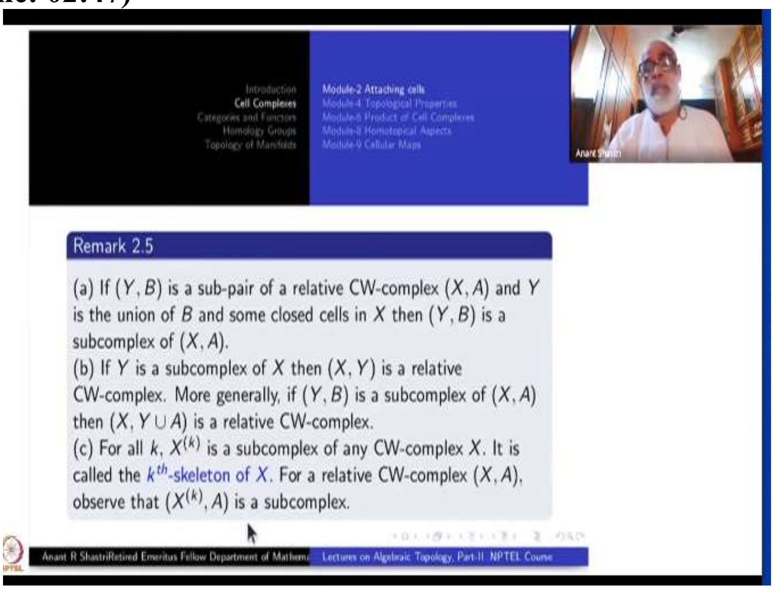

Note that to say that (Y, B) is a topological sub-pair of a relatives CW complex (X, A) means just B subset of A and Y is a subset of $X.Y$ is the union of B and some closed cells in X then there is a good chance that (Y, B) is a subcomplex. All that you need to check is that for each k-cell, the codomain of the attaching map is already in the $k-1$ skeleton of Y. So, if you look at the picture of a CW-complex, then it will be much easier to determine whether something is subcomplex or not. Just by looking at the data given to you, you may not be able to tell whether you have a subcomplex or not. So, drawing a good picture is an easy way of determining if something is a subcomplex or not.

If Y is a subcomplex of X, then (X, Y) itself is a relative CW complex.

Suppose you start with a pure CW complex X. (X is a CW complex on its own that means that A is empty.) In that case, a subcomplex by definition, is also a pure CW complex, because, B is empty.

If Y is a sub complex of X, then what happens is (X, Y) itself is relative CW complex; X can be got out of Y by attaching all those cells which are missed from Y. More generally, if (Y, B) is a subcomplex of (X, A), then $(X, Y \cup A)$ will be relative CW complex. So, you have to throw in A also in the relative part, not just Y, A because Y may not contain the whole of A, Y will contain B. So, take $Y \cup A$ and then you can attach all those cells which are missing to get X.

You also see that for a relative CW complex, and for all k, $(X^{(k)}, A)$ is a subcomplex. If you stop at the k^{th} step, in the attaching process, that itself is a subcomplex, because what you have done, you have not attached the k-cells beyond $(k+1)$-cells, $(k+2)$-cells and so on. That is another example of a subcomplex. So, these subcomplexes have a special name; they are called k^{th} skeleton of (X, A). The subcomplex $(X^{(k)}, A)$ is also written as $(X, A)^{(k)}$.

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So, now let us have a number specific of CW-complexes as well as relative CW complexes. To start with any discrete space is a CW complex, any discrete space is obtained by attaching 0-cells to the empty set. So, any discrete space itself is a CW complex, no relative pair etc. If you take a subset of that, you can then think of the pair as relative CW complex also. And what are the cells 0-cells? What is the dimension? Dimension is also 0. This is the easiest example but it is also important one.

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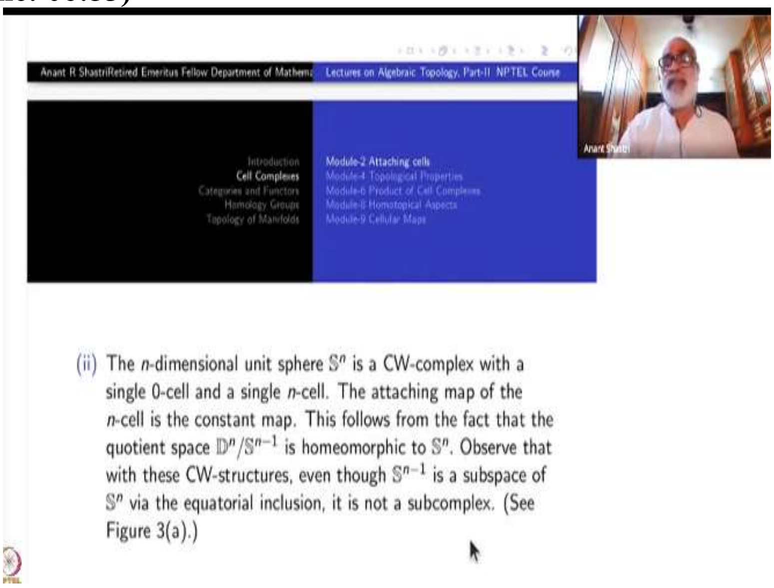

Let us take the standard sphere. (The spheres and the discs are our basic topological objects. If you left them out, then you will be in trouble.) So, an n dimensional sphere is a CW complex. How? Where do you start? Remember, if you a pure CW complex, you have to have a 0-cell there. $X^{(0)}$ cannot be empty. So what is the 0-cell you have to tell. So, you can start with any point in $\mathbb{S}^n$, for example, you can take $p = (0, 0, \ldots, 0, 1)$ or $(1, 0 \ldots, 0)$, any one of them as a single 0-cell.

If you remove one point from a sphere, what do you get? You will get an open cell namely $\mathbb{S}^n \setminus p$ is homeomorphic to $\mathbb{R}^n$, and $\mathbb{R}^n$ is homeomorphic to open unit ball. So, the entire boundary has gone to a single point. Take $\mathbb{D}^n$. On the boundary you take a constant to p. Constant map to a single point p, no matter what it is, that is the $0-$cell. What is the quotient space? Quotient space is precisely homeomorphic to $\mathbb{S}^n$.

The simplest case is when you take $\mathbb{D}^1$, which is the closed interval $[-1, 1]$, both -1 and +1 are mapped to a single point. Then you get your circle. So, this is a generalization, one 0-cell and one n-cell will give you a CW structure on $\mathbb{S}^n$, that is the simplest way you could have got a CW complex other than the trivial example that we have taken earlier of a discrete space. The attaching map is a constant map again. Because if you take the quotient space of $\mathbb{D}^n$, wherein the entire boundary is identified to a single point, that quotient space is a homeomorphic to $\mathbb{S}^{n+1}$.

Observes that even though $\mathbb{S}^{n-1}$ is a subspace of $\mathbb{S}^n$ via the equatorial inclusion it is not a subcomplex, with respect to the CW structure that we have introduced. For example, $\mathbb{S}^1$, the circle is contained as an equator on $\mathbb{S}^2$. Now this is a subspace but it is not a sub complex. Because for $\mathbb{S}^1$ you have to attach 1 cell, for $\mathbb{S}^2$ you have attached only 2-cells directly to single point, the single point can be made common to both of them that is fine.

But the 1-cell which is inside the subcomplex is not present in the bigger one at all. So this is not a subcomplex. So, I am giving you an example of a nice picture which may fail to be a subcomplex picture, equatorial inclusion from $\mathbb{S}^1$ to $\mathbb{S}^2$ or $\mathbb{S}^2$ to $\mathbb{S}^3$, $\mathbb{S}^3$ to $\mathbb{S}^4$ and so on all equatorial inclusion from $\mathbb{S}^k$ to any higher $\mathbb{S}^{k+1}$, $\mathbb{S}^{k+n}$ of where they are not subcomplexes.

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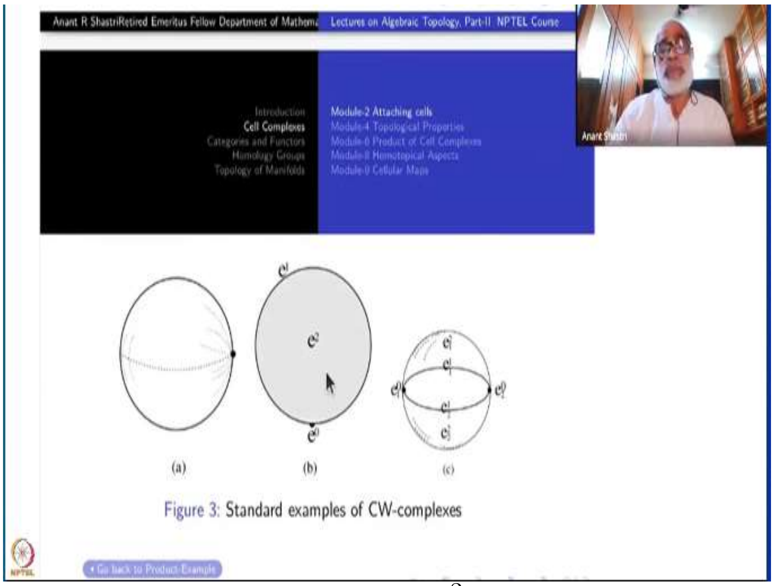

So, here are the pictures that are given, so this is $\mathbb{S}^2$ and ...... is representing the equatorial $\mathbb{S}^1$ there is not as a subcomplex, but each of them is a CW complex with a single n-cell for $n = 1, 2$ etc.

Now, we need to consider more examples. Here also, I will refer to those pictures again.

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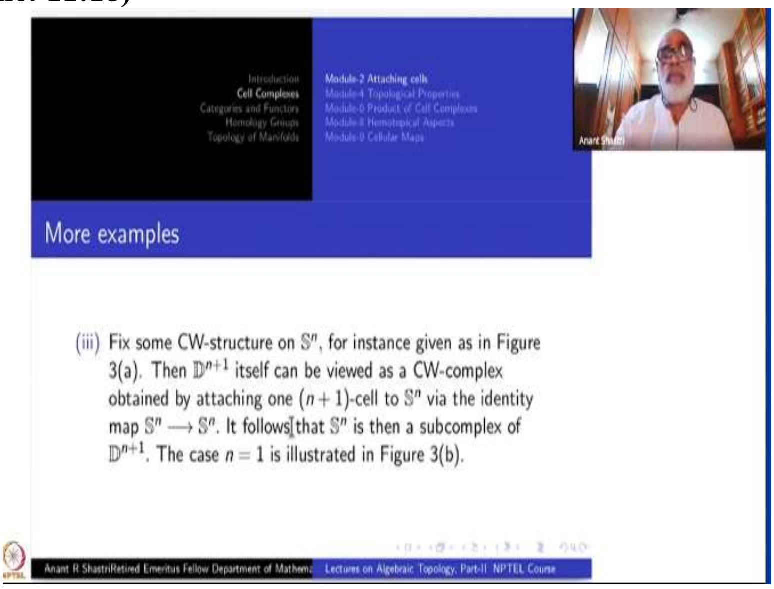

Fix a CW structure on $\mathbb{S}^n$, namely a single point and n cell. Then for $\mathbb{D}^{n+1}$ you can give a CW structure wherein $\mathbb{S}^n$ will be subcomplex. All that you have to do is to fill in the $\mathbb{D}^{n+1}$ n cell with identity map from $\mathbb{S}^n$ to $\mathbb{S}^n$ as the attaching map. Thus you can view $(\mathbb{D}^{n+1}, \mathbb{S}^n)$ as relative CW-complex also, because $\mathbb{S}^n$ is then a subcomplex of $\mathbb{D}^{n+1}$. Which is a nice example of a sub complex. The boundary is a subcomplex of the disc $\mathbb{D}^{n+1}$.

So, how many cells are there in this CW structure of $\mathbb{D}^{n+1}$? $\mathbb{D}^{n+1}$ itself is the $(n+1)$-cell. Before that there is an n-cell $\mathbb{S}^n$ and even before that there is a 0-cell. So, there are three of them.

So, for instance, $\mathbb{D}^2$ is a CW complex with 3-cells: one 0-cell, one 1-cell and one 2-cell. Same picture you can get for any n: one 0-cell, one $(n-1)$-cell and then one n-cell. You do not go through $1, 2, 3, \ldots$ That is not possible by the way. For that, there will be more complications. That is what is shown in the next picture, but let us come to that one later on.

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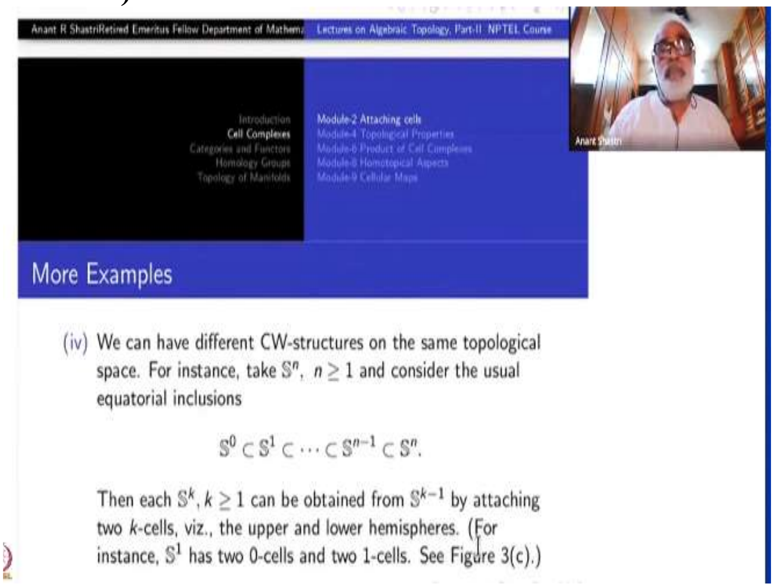

We can have different CW-structure on the same topological space just like we can have different triangulation of a space. Your topological space, may not have any CW structure or that there can be more than one CW structure.

For instance, we have given $\mathbb{S}^n$, a CW structure with a 0-cell and one n-cell. But now I will give you another so that equatorial inclusions become subcomplexes. So, consider the usual equatorial inclusions. $\mathbb{S}^0$ which is $\{-1, 1\}$ has not just one but 2 points. So, I start with 2 points as my vertices, then how do I get $\mathbb{S}^1$? I attach two 1-cells, to $\mathbb{S}^0$ and to get $\mathbb{S}^1$. Having got $\mathbb{S}^1$, how do I get $\mathbb{S}^2$? I will get two 2-cells, upper hemisphere and a lower hemisphere.

Like this, you can keep on going... each time upper hemisphere and lower hemisphere of one higher dimension. I can go on getting $\mathbb{S}^1, \mathbb{S}^2, \mathbb{S}^3, \mathbb{S}^4, \ldots, \mathbb{S}^n \ldots$ Each $\mathbb{S}^k$ can be obtained by $\mathbb{S}^{k-1}$ by attaching two k-cells, one above and one below, namely, the upper and lower hemispheres. This is the picture here e_0^1, e_0^2 are the two points to -1 and $+1$, which are two 0-cells, then e_1^1 and e_1^2 make up the circle, then in the top there is one 2-cell; e_2^1 and in the bottom there is other 2-cell e_2^2. This is the picture of $\mathbb{S}^2$. How many cells it has in all? $2+2+2$; in each dimension it has two of them.

So, keep going on like this...each time you attach two cells to get the sphere of the next dimension, all the way to $\mathbb{S}^\infty$. So, this is a nice picture wherein each n-dimensional sphere will be a subcomplex.

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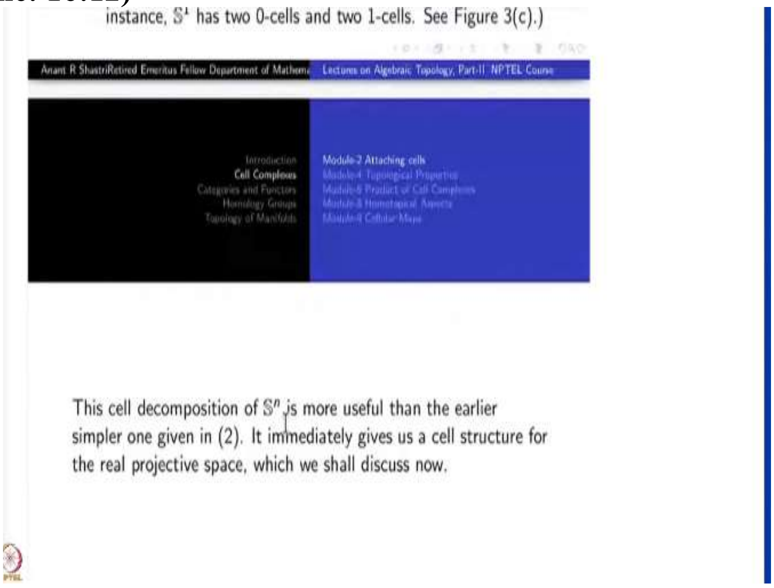

This cell decomposition is more useful than the earlier one. It immediately give us a cell structure for the real projective spaces because this cell structure is invariant under the antipodal action: x equivalent to $-x$. 0-cells will be interchanged, the 1-cells will be interchanged, the 2-cells will be interchanged etc, that is the invariance. So, therefore, you know if you identify $+1$ and -1 all that you have to do is to identify the corresponding cells that will give you a cell structure on the projective space.

How many 0-cells will be there? Originally, -1 and $+1$, there were two of them. So, in the projective spaces $\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2, \ldots$ there will be only one 0-cell, one 1-cell, only one 2-cell, and so on. So, $\mathbb{P}^n$ has 1-cell for each dimension $0, 1, 2, 3, \ldots, n$. So, that is the structure coming out of the equatorial structure for $\mathbb{S}^n$, because it is invariant under the antipodal action.

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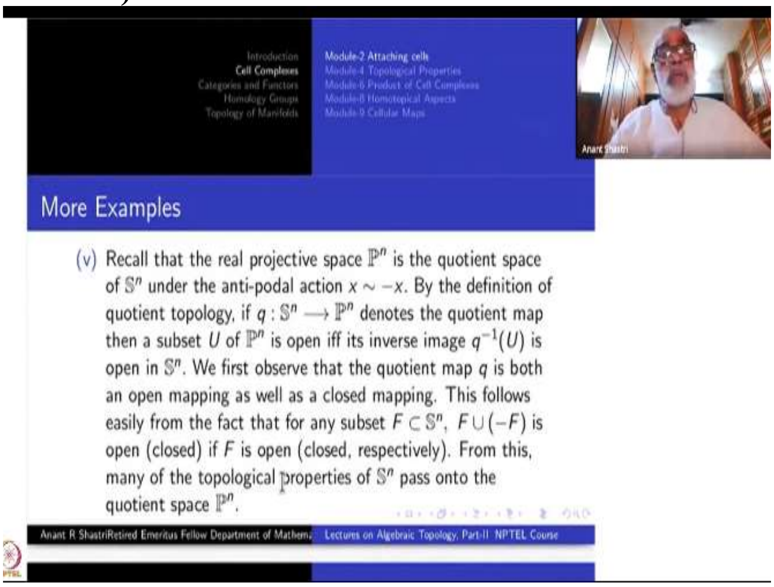

So, that is the picture of $\mathbb{P}^n$. Here I am repeating this one. Recall that $\mathbb{P}^n$ is a quotient space of $\mathbb{S}^n$ under the antipodal action x equivalent to $-x$. By the definition of quotient topology suppose q from $\mathbb{S}^n$ to $\mathbb{P}^n$ is quotient map then a subset U of $\mathbb{P}^n$ is open if and only if $q^{-1}(U)$ is open $\mathbb{S}^n$. So, when you first observe that the quotient map q is both open as well as a closed mapping. This follows easily from the fact that for any subset F of $\mathbb{S}^n$, $F \cup (-F)$ is open (or closed) if F is open (or closed) respectively, because $-F$ is a homeomorphic copy of F. From this many of the topological properties of $\mathbb{S}^n$ pass onto the quotient space. You can use this to prove that $\mathbb{P}^n$ is a Hausdorff space. This is a wonderful thing to happen, because quotient spaces are quite often not Hausdorff. So, you have to be careful here. Very easy to get counterexamples.

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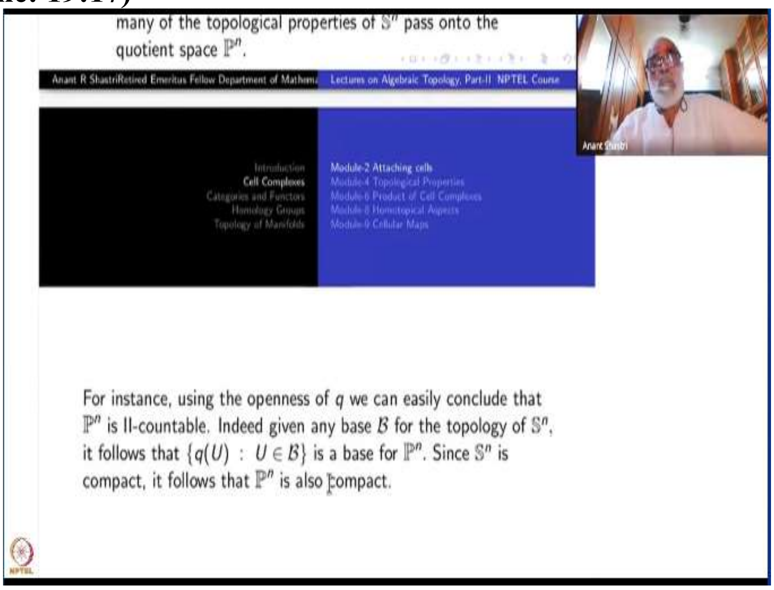

So, for example you prove that $\mathbb{P}^\infty$ is second countable. $\mathbb{P}^n$ is compact of course.

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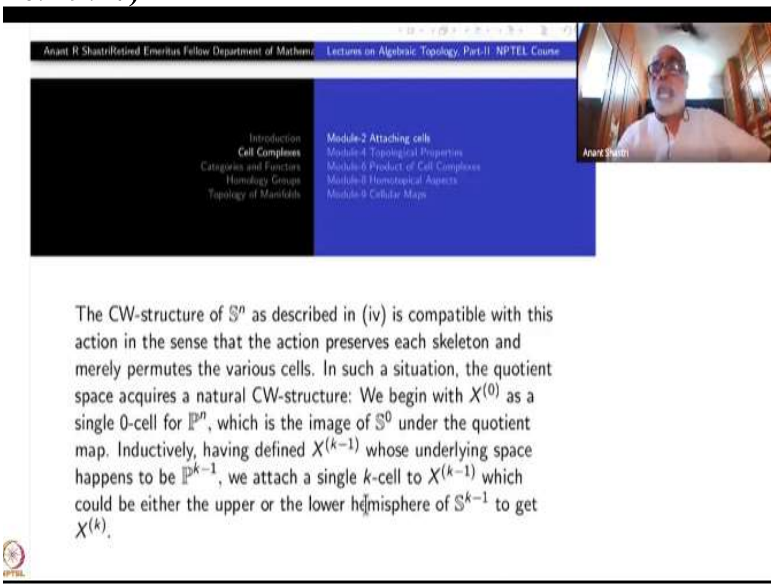

A CW structure $\mathbb{S}^n$ as described in (iv) is behaved well with $\mathbb{Z}_2$-action, in the sense that the action preserves each skeleton and actually merely permutes the various cells by homeomorphisms. In such situations, the quotient space acquires a natural CW structure which you can call it quotient structure. This is a general remark.

So, coming back to the special case what is happening in $\mathbb{P}^n$? We begin with x_0 as a single 0-cell in $\mathbb{P}^n$. This is the image of $\mathbb{S}^0$ under the quotient map.

Inductively having defined $X^{(k-1)}$ whose underlying space happens to be $\mathbb{P}^{k-1}$, we attach a k -cell: there are two of them in $\mathbb{S}^k$, you have to choose only one of them. But what is the attaching map? Attaching map is now the quotient map from $\mathbb{S}^{k-1}$ to where $\mathbb{P}^{k-1}$, the cover double cover so, one single sphere $\mathbb{S}^{k-1}$ wraps around twice, in some sense, around $\mathbb{P}^{k-1}$, that is not a sphere of course so, you have to understand what is the attaching map carefully, it is no longer identity map. In the case of sphere $\mathbb{S}^{k-1}$, the boundary of upper (or lower) hemisphere is precisely $\mathbb{S}^{k-1}$.

So, attaching map was identity there. But below in the quotient space structure, it is simply the quotient map.

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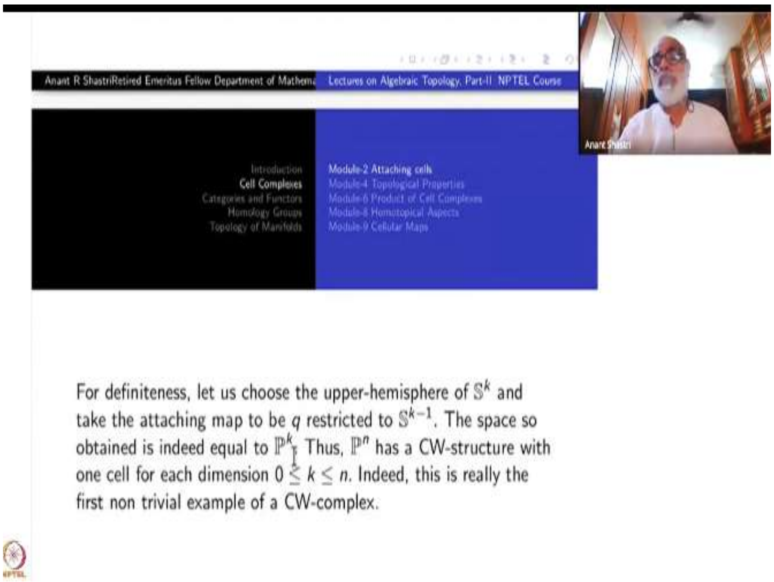

So, for definiteness, let us choose the upper hemisphere and take the attaching map to be the q restricted to $\mathbb{S}^{k-1}$. The space obtained is equal to $\mathbb{P}^k$. So, $\mathbb{P}^k$ is obtained from $\mathbb{P}^{k-1}$ by attaching a k-cell, the boundary of the k-cell maps $\mathbb{S}^{k-1}$ to $\mathbb{P}^{k-1}$ and that is the quotient map.

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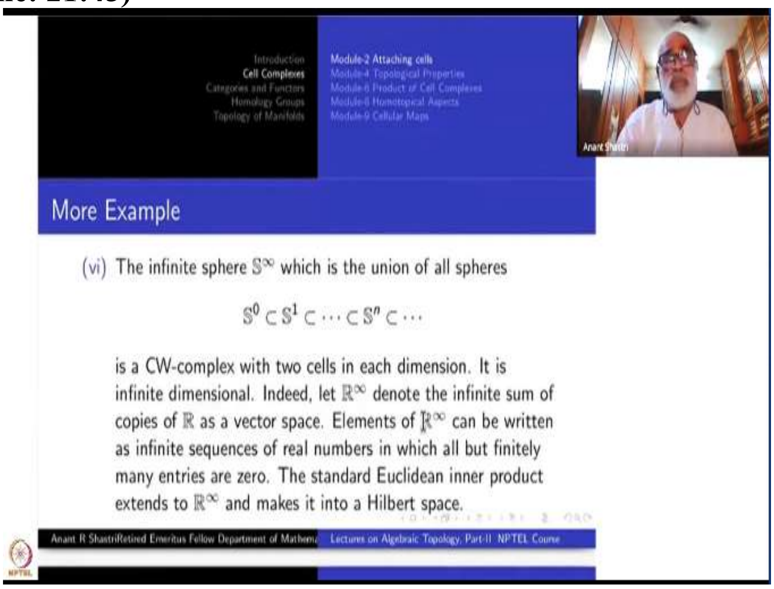

So, what I want to tell you here is that this process can be carried on all the way to $\mathbb{S}^\infty$, $\mathbb{S}^\infty$ is what? Union of all these increasing sequence of spheres. The topology has to be defined by taking a set to be closed if and only if its intersection with each $\mathbb{S}^n$ is closed in $\mathbb{S}^n$. $\mathbb{S}^\infty$ is a subspace of $\mathbb{R}^\infty$ which is an infinite direct sum of copies of $\mathbb{R}$ (not a direct product, do not make that mistake). The standard Euclidean inner product can be taken but the topology is not the metric topology. So, you have to be careful about that. It is the weak topology.

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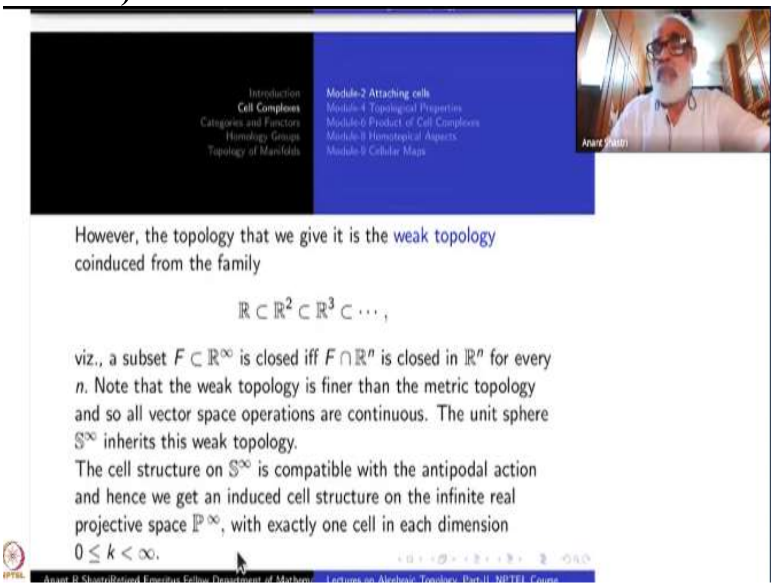

The cell structure on $\mathbb{S}^\infty$ is compatible with antipodal action. The vector space structure is there on $\mathbb{R}^\infty$, and in particular x mapsto $-x$ is an isomorphism and this cell structure is compatible with the antipodal action. Hence we get a cell structure on the infinite projective space. $\mathbb{P}^\infty$. What is k-th skeleton? it is just $\mathbb{P}^k$, the projective space of dimension k. In homotopy theory, it is a very important space. Its fundamental group is $\mathbb{Z}_2$ and all the higher homotopic groups, (whatever you do not know or whatever you know) they are all 0.

So, such a thing is called Eilenberg Maclane space of type $(\mathbb{Z}_2, 1)$. $\pi_1(\mathbb{Z}_2)$ and all π_i for $i > 0$ are 0. This is a very very important space.

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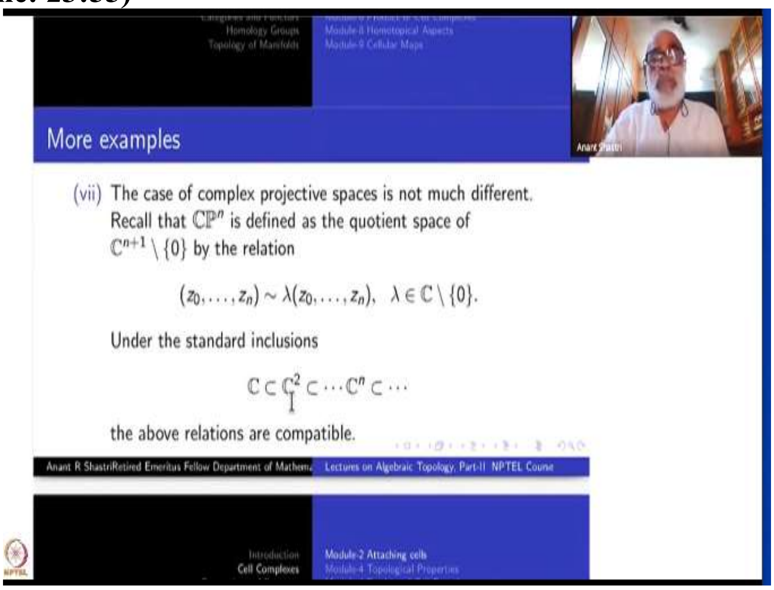

Exactly same way you can define the infinite complex projective space, and before that all the complex projective spaces $\mathbb{CP}^n$ also. Remember $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ under the scalar multiplication, $z = (z_0, \dots, z_n)$ is equivalent to $\lambda z, \lambda$ must be nonzero scalar, scalars are now complex numbers that is all. However, the cell structure is quite a different story now. It is very interesting story here. So, we will this study this one carefully? And perhaps that is the last example for today,

So, we start with $\mathbb{C}$ contained inside $\mathbb{C}^2$ etc just like $\mathbb{R}$ contained in $\mathbb{R}^2$ etc, coordinate inclusions. These are complex vector subspaces and so multiplication by a complex scalar is compatible with the inclusions.

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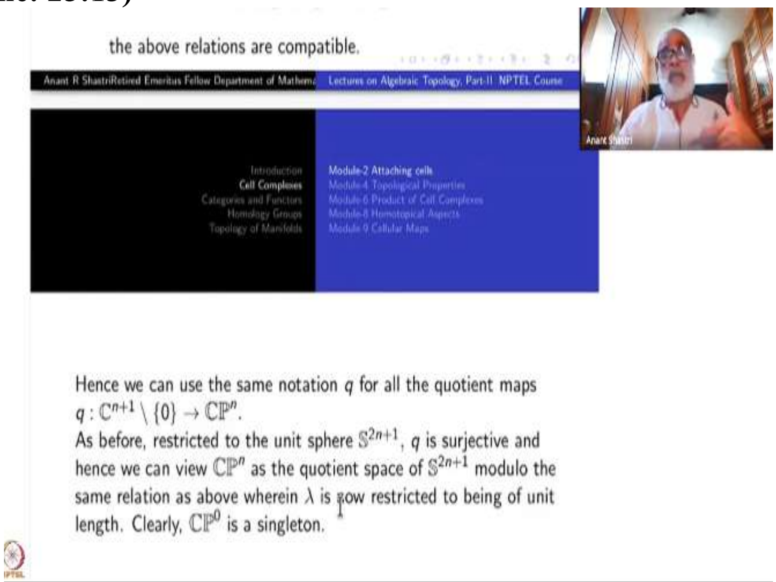

Hence, we can use the same notation q for all the quotient maps, get q from $\mathbb{C}^{n+1} \setminus 0$ to $\mathbb{CP}^n$ and then we can restrict it to the unit spheres here, the unit sphere in $\mathbb{C}^{n+1}$ is $\mathbb{S}^{2n+1}$ because it is of real $2n+2$ dimension. So, the unit sphere there will be $\mathbb{S}^{2n+1}$. You can restrict q to the sphere here that will be surjective because after all every non zero vector is equivalent to a unit vector, there may be many of them namely if you multiply by unit complex number, then you get the whole set which is a circle of those elements representing the same element in the projective space. In the case of real projective spaces, you had only two of them, two unit vectors namely x and $-x$. Here you take any vector and multiply by a unit complex number, it will be still a unit vector. So, $\mathbb{CP}^n$ is the quotient of $\mathbb{S}^{2n+1}$ modulo the scalar multiplication, namely, λ is now restricted to being a unit vector unit length.

In any case what is $\mathbb{CP}^0$? $\mathbb{CP}^0$ is a complex line, one single line, namely one unit vector in $\mathbb{C}$ up to equivalence. Any two unit vectors are related by a complex number. So, $\mathbb{CP}^0$ is a single point, just like $\mathbb{RP}^0$ is single point. What is $\mathbb{CP}^1$? It will be a quotient of $\mathbb{S}^3$ by the $\mathbb{S}^1$ action there.

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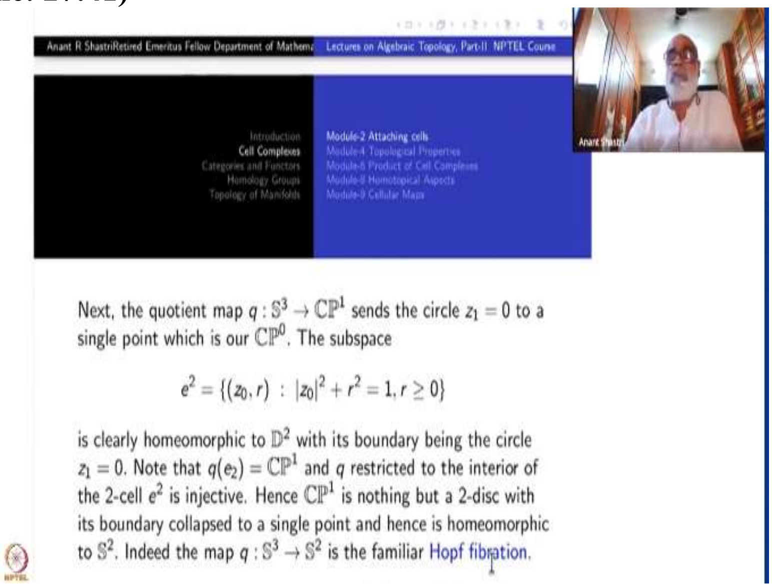

That is an interesting object. First you have to study what is $\mathbb{CP}^1$? The quotient map q from $\mathbb{S}^3$ to $\mathbb{CP}^1$ sends the entire circles to single points. For example, $z_1 = 0$ defines a circle in $\mathbb{S}^3$, it is the intersection of the plane $z_1 = 0$ with $\mathbb{S}^3$. The single point which is our $\mathbb{CP}^0$ given by $z_1 = 0$ which is a plane in $\mathbb{C}^2$. There are two independent planes $z_0 = 0$ and $z_1 = 0$. In terms of complex vector spaces, they are lines.

To get the 2-cell, I am taking some a subspace of $\mathbb{C}^2$ denoted by e_2: points (z_0, z_1) such that z_0 is a complex number but second coordinate is a real numbers $z_1 = r$ greater that or equal to 0 such that $|z_0|^2 + r^2 = 1$. So, that we will a point of $\mathbb{S}^3$. The first coordinate is any complex number, second coordinate is non negative real number such that $|z_0|^2 + r^2 = 1$.

This subspace e_2 is clearly homeomorphic to $\mathbb{D}^2$, why? Take any z_0, since $|z_0|^2$ plus something is 1, $|z_0|$ must be less than or equal to 1. The second coordinate has to be equal to $|z_0|$. Therefore this subspaces is the graph of the function z mapsto $|z|$ restricted to the unit disc $|z_0| \leq 1$. And it is a boundary is given by $z_1 = 0$, which is the same as saying $|z_0| = 1$. So, that is the meaning of the boundary so the boundary is given by $z_1 = 0$. Note that q of this set will cover the whole of $\mathbb{CP}^1$; every point in $\mathbb{S}^3$ is in the equivalence class of some point of the form $(z_0, |z_0|)$ upto scalars. One point we have taken namely, the second coordinate being 0.

That single point is the 0-cell. All other points second coordinate will be nonzero. Given (z_0, z_1) in $\mathbb{S}^3$, once the second coordinate is nonzero, you can divide out $z_1/|z_1|$ to get a point of e_2. If $z_1 = re^{i\theta}$, we are dividing by $e^{i\theta}$. You are left with just r in the second coordinate. You are dividing the first coordinate also by $e^{i\theta}$ so that we get an element in the same equivalence class. Equation $|z_0|^2 + |z_1|^2 = 1$ is not affected. So, we have proved that this $q(e_2)$ is equal to $\mathbb{CP}^1$.

And q restricted to the interior of the 2-cell namely when $|z_0| < 1$, there is a unique solution with r positive. When r is 0, there are more solutions but q maps all of them to only one point. Thus, in the interior of e_2, q is an injective map. So, this is precisely what we wanted for a characteristic function to have injectivity on the interior on the boundary some continuous function that continuous function you can take it as the attaching map and the interior gives you the characteristic map. Therefore, $\mathbb{CP}^1$ is nothing but the 2-disc e_2 with its boundary collapsed to a single point and hence is homeomorphic to $\mathbb{S}^2$.

You begin with we did not know what $\mathbb{CP}^1$ is. In the process of getting the CW structure on it we actually showed that it is homeomorphic to $\mathbb{S}^2$. It has a better structure, you can think of this as the so called of extended complex plane because the map z_0 mapsto $[z_0, 1]$ defines a homeomorphism of $\mathbb{CP}^1 \setminus \mathbb{CP}^0$.

The quotient map q from $\mathbb{S}^3$ to $\mathbb{S}^2$ is a very familiar and very important map. It is called the Hopf fibration. This was used by Hopf to get a big landmark result in topology, that $\pi_3(\mathbb{S}^2)$ is non zero. At the time of Hopf, that was a very big invention. It is a landmark inventions, a milestone invention in algebraic topology.

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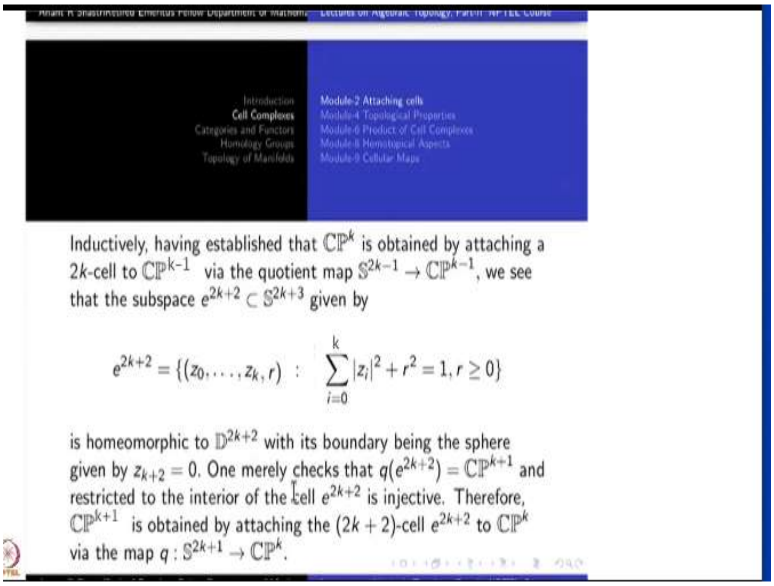

Inductively having established that $\mathbb{CP}^k$ is obtained by attaching a $2k$-cell to $\mathbb{CP}^{k-1}$ via the quotient map $\mathbb{S}^{2k-1}$ to $\mathbb{CP}^{k-1}$; you see that the subspace e_{2k+2} contained in $\mathbb{S}^{2k+3}$ given by e_{2k+2} equal to the set of all $(z_0, z1, \ldots, z_k, r)$, such that $\sum_j |z_j|^2 + r^2 = 1$. This subspace e_{2k+2} is homeomorphic $\mathbb{D}^{2k+2}$, the proof is exactly the same as in the case $k = 1$. there are $k+1$, there are $k+1$ coordinates which are free here, r is completely determined by the values of these by this equation.

So, checking again that $q(e_{2k+2})$ is the whole of $\mathbb{CP}^{k+1}$ is also exactly the same. Look at the last coordinate if it is 0 already, you are in $\mathbb{CP}^{k-1}$, if it is non zero you can solve for this equation that is it. So, therefore $\mathbb{CP}^{k+1}$ is obtained by attaching a $(2k+2)$-cell e_{2k+2} to $\mathbb{CP}^k$ via the map q from $\mathbb{S}^{2k+1}$ to $\mathbb{CP}^k$.

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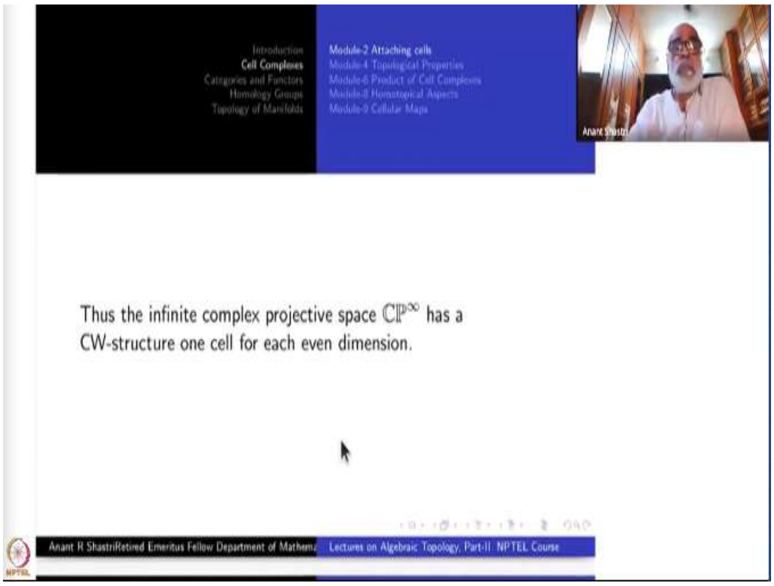

Keep doing this for all k, all the way $\mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^3$, upto $\mathbb{CP}^\infty$. You are getting only attaching one the dimensional cells each time: there is 0-cell then there is e_2, then there is e_4 and so on e_{2k+2}... all the way to $\mathbb{CP}^\infty$.

This is another important space, $\mathbb{CP}^\infty$. This is again an Eilenberg Maclane space. This time it is simply connected, i.e., π_1 is trivial. π_2 is the infinite cyclic group and all other homotopy groups are trivial. So, this is called an Eilenberg Maclane space of type $(\mathbb{Z}, 2)$.

So, one cell in each even dimension $0, 2, 4, 6$ and so on. So, we will study more examples next time. Thank you.