Introduction to Algebraic Topology, Part - II
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Lecture - 27
Basic Properties of Singular Homology

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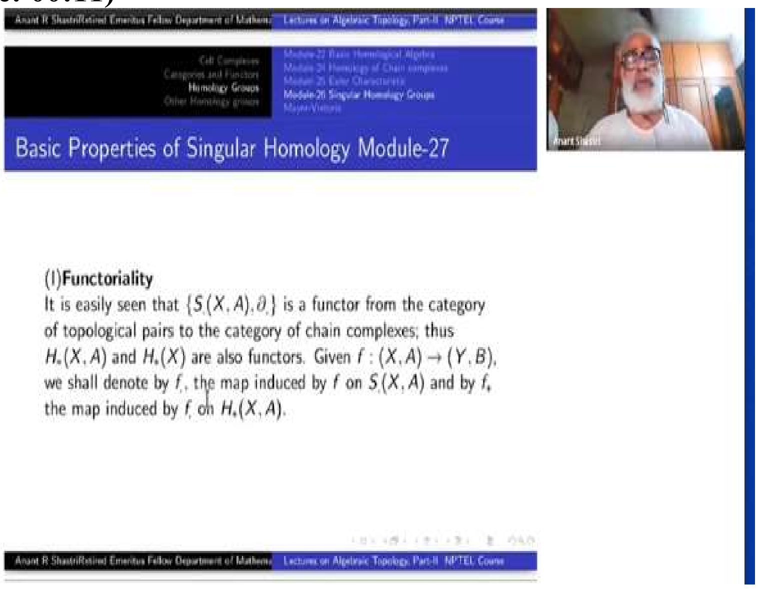

Having defined the singular chain complex and the singular homology of topological pairs, let us examine today a few very very important properties of the singular homology. These are going to be the guidelines for the entire homology theory later on. The properties of singular homology, some of them not all of them will be picked up as the fundamental properties and declared as axioms.

So, the very first one is the functoriality. So first we have seen that from topological pairs to the chain complexes, it is a functor okay. (X, A) leads to $(S.(X, A), \partial)$ is a covariant functor from the category of topological pairs to the category of chain complexes. We have also seen that taking homology is also a functor from the chain complexes to the graded abelian groups, or graded modules. Thus composing these two functors what we get is that the association leads to $H_*(X, A)$ as well as X leads to $H_*(X)$ are covariant functors. Okay?

Given a map f from (X, A) to (Y, B), we shall denote by $f.$ the morphism induced at the chain complex level and by f_* the morphism induced on the homology level okay? Quite often f_* is used for both of them and that makes it somewhat confusing. So, let us try to follow this convention, okay? The star would be for the homology and dot will be for chain complex.

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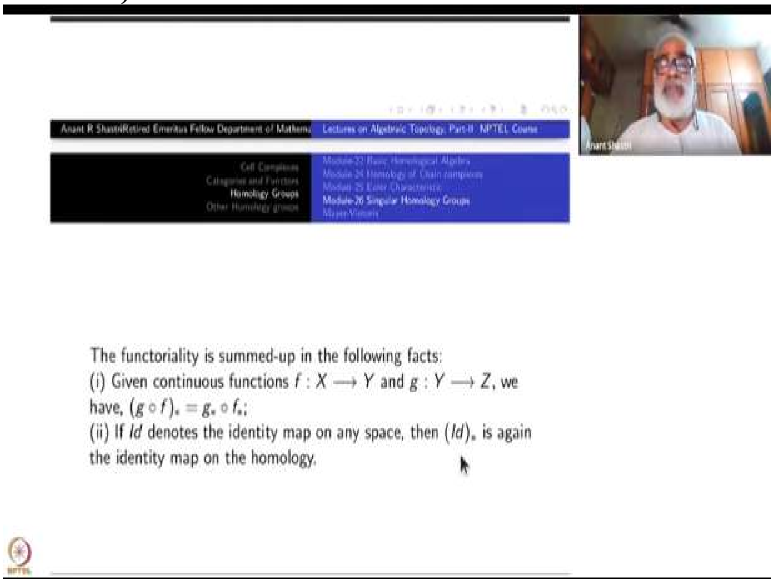

The functorialty is summed up in the following facts. Okay? I am repeating it. We have seen it earlier, there is nothing new, but just to emphasize this one. If f from X to Y and g from Y to Z, then $(g \circ f)_*$ is same thing as $g_* \circ f_*$. Also, if Id denotes the identity map of any space X to X, then Id_* from $H_*(X)$ to $H_*(X)$ is also the identity map on the homology groups, Okay? These are the two factors you have to understand. Okay?

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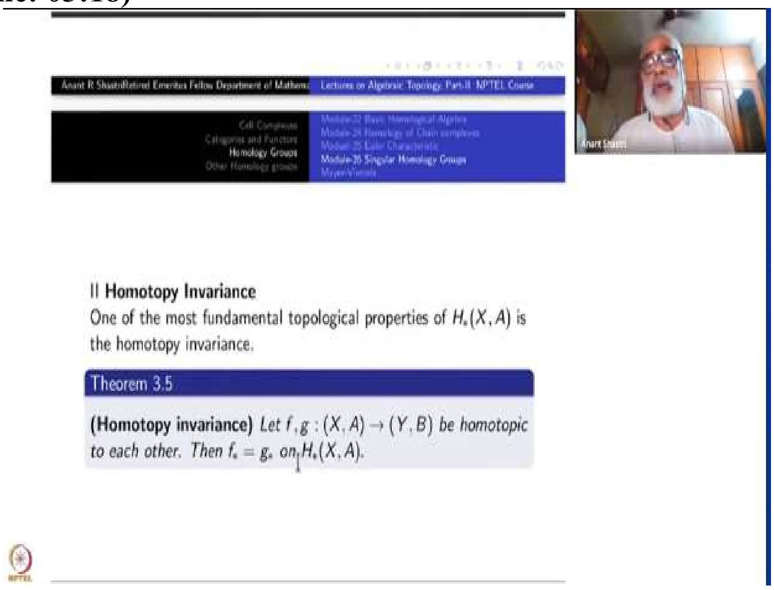

So, this is one of the fundamental properties of the homology. It is not just some group or a module out of nothing, okay? It is a functorial association. That is what one has to understand.

Next important thing is the homotopy invariance of $H_*(X, A)$. For any reasonable space, $S_n(X)$ is a very huge group. Okay? Why we are taking such a huge group we do not know. Because the set of all maps from Δ_n to any reasonable space is going to be very huge. Of course, if the space is a discrete space or a single point and so on, then only we can know what looks like and so on. Otherwise, it is quite a big group. What is important is that each $S_n(X)$ is a topological invariant. But it is too huge topological invariant. However, $H_*(X)$ is defined as some sub quotient of this group. You take some subgroup and then the quotient of that group right? It is somewhat strange that this sub quotient group is going to be homotopy invariant, which is stronger than being homomorphism invariance. For any homotopy invariant property is also a homeomorphism invariant property, Because if f from X to Y is a homeomorphism, then it is also a homotopy equivalence okay?

So, this homotopy invariance we are going to state very clearly, namely, if f, g from (X, A) to (Y, B) are homotopic to each other, then $f_* = g_*$ on $H_*(X, A)$. Okay? So, this is the statement of homotopy invariance. Morphisms induced by two homotopic maps. they are actually the same at the homology level.

Note that we are not defining homotoy invariance in terms of topological spaces but in terms of the maps, which is a stronger okay? That's what you have to pay attention to. From this definition, you will see that if f from (X, A) to (Y, B) is such that g is its homotopy inverse, then $f \circ g$ we will be homotopic to identity, therefore, $f_* \circ g_*$ will be will be equal to identity. Similarly, the other way round. This means f_* is the inverse of g_*. So, the homotopy invariance of the homology for the spaces will follow if you prove the homotopy invariance of the induced morphisms, okay?

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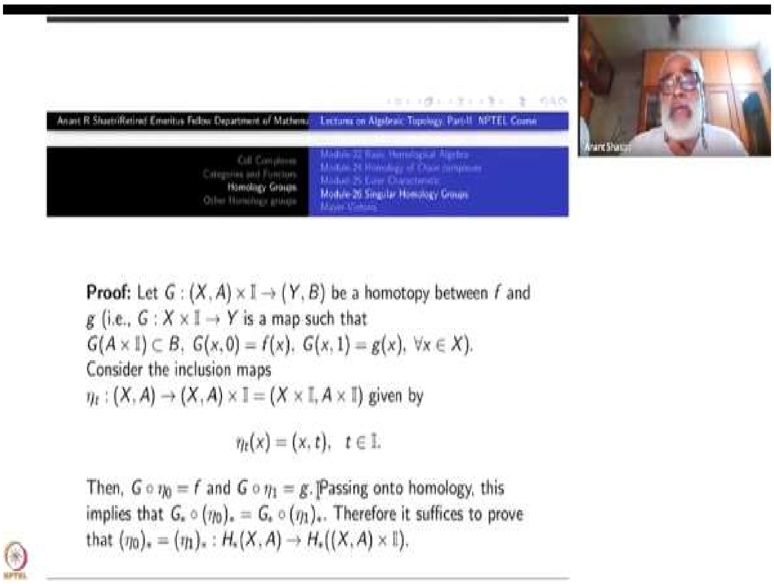

So, we will take a few steps toward proving this one, but we will not complete the proof of this one. We will leave it at a stage wherein things become bit too difficult okay. So, let us see what is the easy part of the proof of this. Start with a G which is a homotopy from $(X, A) \times \mathbb{I}$ to (Y, B) which just means that it is a map from $X \times \mathbb{I}$ to Y, such that $A \times \mathbb{I}$ goes inside B, that is all. The topological pair $(X, A) \times \mathbb{I}$ is nothing but same as the pair $(X \times \mathbb{I}, A \times \mathbb{I})$.

Suppose G is a homotopy between f and g Okay? Consider the inclusion map η_t from (X, A) to $(X, A) \times \mathbb{I}$, which I have written here. I have written fully from (X, A) to $(X \times \mathbb{I}, A \times \mathbb{I})$. What is η_t? $\eta_t(x) = (x, t)$. So, I am putting X inside $X \times \mathbb{I}$ at t-th level okay? So, all of these inclusion maps. η_0 and η_1 are important specific ones. Then, $G \circ \eta_0(x)$ will be what? $G(x, 0) = f(x)$ and $G \circ \eta_1(x) = G(x, 1) = g(x)$ okay?

When you pass to the homology this will imply that $f_* = G_* \circ (\eta_0)_*$ and g_* is $G_* \circ (\eta_1)_*$, because of the functoriality. Therefore, it suffices to prove that $(\eta_0)_* = (\eta_1)_*$. This is what we want to prove. If $(\eta_0)_* = (\eta_1)_*$ then we get f_* will be equal to g_* Okay?

So instead of worrying about general maps, f and g, we you have to just prove just that $(\eta_0)_* = (\eta_1)_*$ from $H_*(X, A)$ to $H_*((X, A) \times \mathbb{I})$, just for these two inclusion maps, the induced morphisms must be equal. Note that they are homotopic of course. So, the general homotopy has been cut down from arbitrary f and g to the case for inclusion maps η_0 and η_1, the two coordinate inclusions. This is precisely the first step that is done in the proof of the Poincare lemma in differential calculus.

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So, for this purpose we construct a chain homotopy h from $S_*(X, A)$ to $S_*(X \times \mathbb{I}, A \times \mathbb{I})$, between $(\eta_0).$ and $(\eta_1).$, okay? at the chain complex level. If two chain morphisms are chain homotopic, then we know that they induce same morphism at the homology level. So, this chain homotopy is called the prism operator. Okay? Because Δ_2 to $\Delta_2 \times \mathbb{I}$ looks like passing on to the prism.

For future reference we just state this as a separate lemma so that we have to prove this lemma not the whole theorem Okay? What is the lemma?

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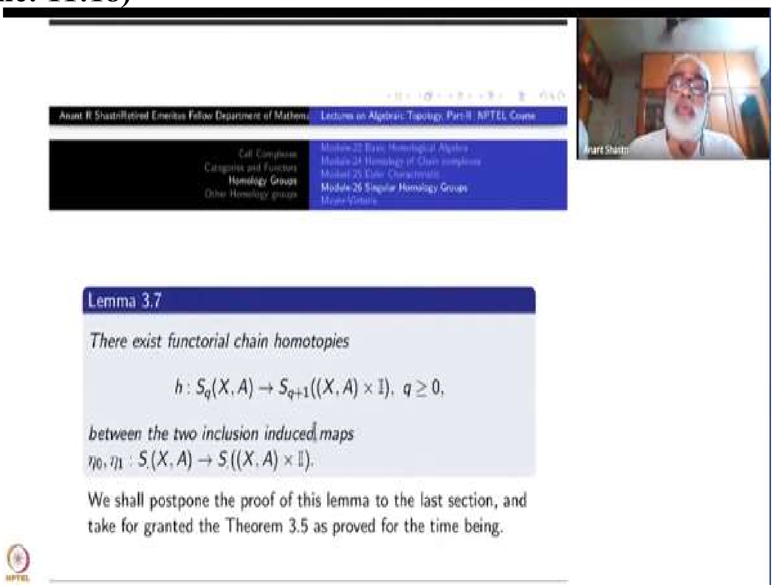

There exist functorial chain homotopies, whole lot of them, h for more elaborately $h_{(X,A)}$. (That is the way to write a functor Okay? This usage of $h_{(X,A)}$ indicating certain functoriality) h from $S_q(X, A)$ to $S_{q+1}((X, A) \times \mathbb{I})$ for all $q \geq 0$, between the two inclusion maps η_0 and η_1. Okay? We shall postpone the proof of this lemma to the last section and take this lemma and hence the homotopy invariance theorem for granted for the time being, okay.

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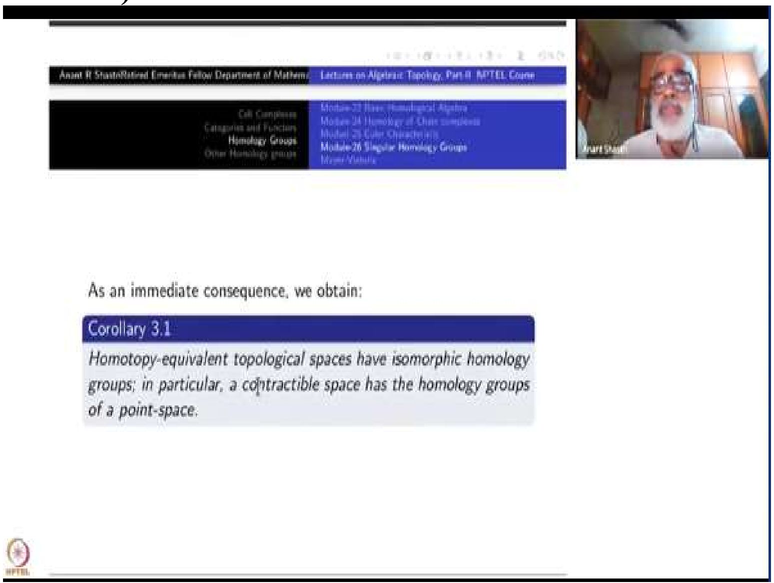

We will now start using it. And the first corollary is homotopy equivalent topological spaces have isomorphic homology groups. In particular contractible spaces have homology groups of a single point okay? So, both of these results will be used again and again. Contractible spaces, as far as homology is concerned, are just like singleton spaces. Okay?

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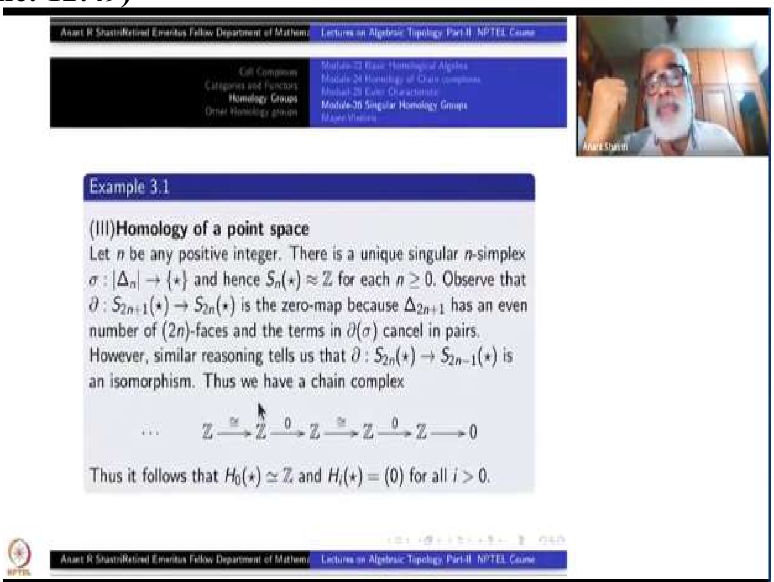

Do we know the homology of the single point? So, that is what we have to do now. So, luckily, we can do that from the scratch. Though we cannot do much of computations here. So, take n to be any non negative integer. What are all maps from Δ_n into a single point space, The $\{*\}$ denotes a single point space, okay? What are all continuous functions here? What are all functions? There is only one function okay? And it is continuous alright?

Therefore, $S_n(X)$ is nothing but a group generated by one element free group generated by 1 element that is infinite cyclic. This is true for all $n \geq 0$. Note that Δ_0 is also a single point space. okay? What are S_n's for n negative? Of course, we have defined them to be 0.

Now comes the crucial thing about the boundary operator ∂. Look at ∂_{2n+1} from S_{2n+1} to S_{2n} . I would say it is the 0 map. Why? Consider the case $n = 1$. It has two vertices and $\partial_0(\sigma) = \sigma \circ F^0 - \sigma \circ F^1$. But both $\sigma \circ F^0$ and $\sigma \circ F^1$ are the same constant map Δ_0 to $\{*\}$. Therefore, they cancel out.

More generally, if you look at Δ_3 or $\Delta_5, \ldots \Delta_{2n+1}$ has an even number $2n + 2$ faces of dimension $2n$ and the constant function restricted to each of them is the same constant function. Since they are taken wit alternate signs the sum total is zero. For the same reason, ∂_{2n} of σ consists of an alternate sum of odd number of constant functions, and hence is equal to plus or minus of the generator.

So, what we have is in this sequence, starting with the zero map from S_1 to S_0, going backward, alternatively, we have an isomorphism and a zero map. Therefore, the $n = 1$ onward, we have the kernel and the image are always the same and hence homology groups are all zero.

At $n = 0$, however, H_0 will be infinite cyclic. Because here the kernel is the whole group $\mathbb{Z}$ whereas the image is zero. So H_0 is infinite cyclic and $H_n = 0$ for $n > 0$.
Of course, $H_n = 0$ for $n < 0$. So only the 0-th homology of a single point survives and rest of the homologies are all 0.

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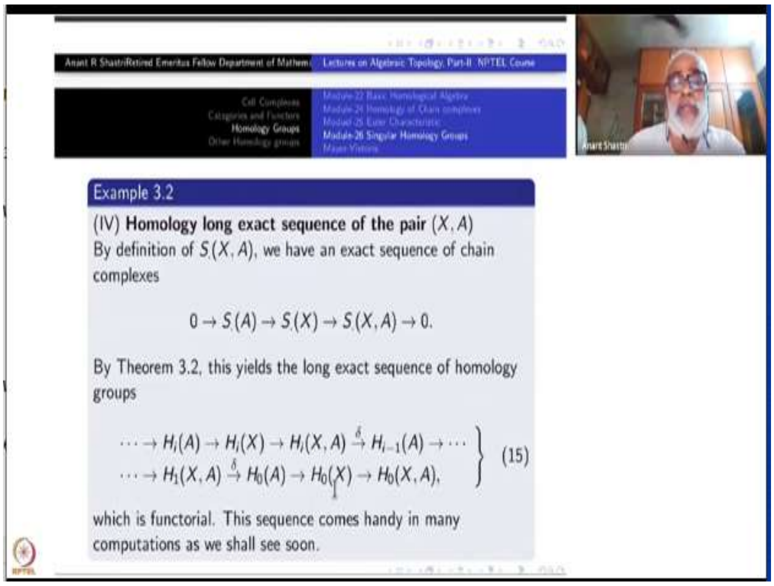

So let us go ahead. Homology long exact sequence of the pair: Just for this one result, we made a lot of algebraic preparation, namely, the snake lemma okay? So we use that one and see how we get this here, as an almost ready result. So by definition, $S.(X, A)$ is the quotient of $S.(X)$ by $S.(A)$. Therefore, we have short exact sequence of chain complexes here: 0 to $S.(A)$ followed by the inclusion map into $S.(X)$ followed by the quotient map to $S.(X, A)$ to 0.

Okay, so, this is a short exact sequence of chain complexes. The snake lemma and then the theorem following that will now yield the long exact sequence of the homologies. What are these homology groups? $H_n(A)$ to $H_n(X)$ to $H_n(X, A)$ and then the connecting homomorphusm δ to $H_{n-1}(A)$... The last 4 terms you can write down: $H_1(X, A)$ to $H_0(A)$ to $H_0(X)$ to $H_0(X, A)$. After that everything will be 0 okay?

We know these terms for a singleton space. In general, we do not know even know the last few terms completely. So, better to start computing these things now. There is a long exact sequence which is functorial. This sequence comes handy in many computations as we will see presently, okay? When you have some additional information here and information here, this group will be trapped between them.

So, you can get a lot of information on this one if not completely okay? Suppose two of these groups are both 0. Then this middle one will be 00 So, such things we will have to keep using. These four properties are so fundamental, okay? they have been raised to the status of axioms for homology theories, There are some more of them, which will take consider a little later, but now, we will come to some other properties of the singular chain complex which are special to the singular chain complex itself but may not be shared by other homology groups. Okay? some of them may be true, but they are not made a part of the axiomatic set up.

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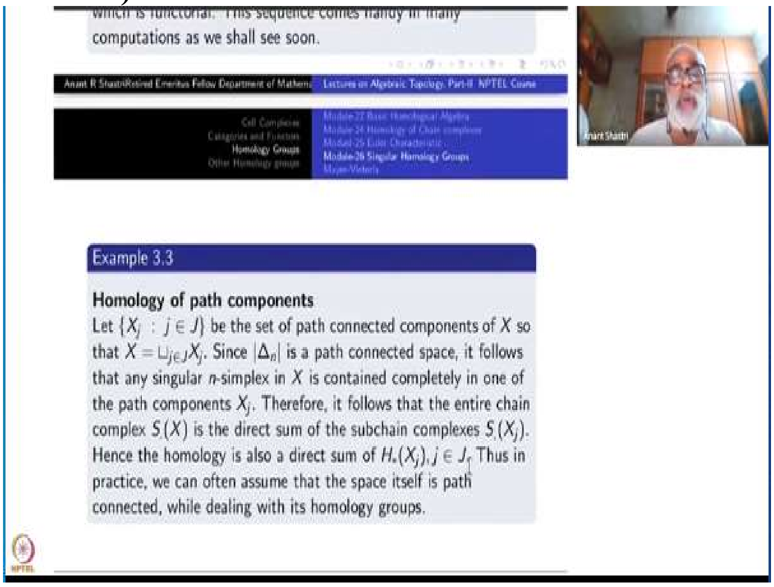

So, here is an extra property of homology. The homology of path components: I have put it as an example, rather than under properties $1, 2, 3, 4$. So, you will have more properties later. Suppose you have a space X which has path components $\{X_j\}$ indexd by $j \in J$. Clearly, X is a disjoint union of X_j's. Okay? Since Δ_n is path connected, if you take any singular n-simplex, namely a continuous function σ from Δ_n into X, it will be completely inside one of the X_j's, okay?

Therefore, the set of all singular simplexes of X is just the union of the sets of all singular simplexes of X_j, the union taken over $j \in J$. Thus the basis itself is partitioned like this, therefore, the free abelian group $S_n(X)$ over that will be the direct sum of the free abelian group over each of sets of singular simplexes of X_j. So, $S_\cdot(X)$ is a direct sum of $S_\cdot(X_j)$'s. Using one of the first things that we proved for the homology of a direct sum of chain complexes, we conclude that the homology $H_*(X)$ is the direct sum of $\oplus H_*(X_j)$ taken over $j \in J$, okay?

So, here we have used the construction of the singular homology not just some functorial properties and so on. Okay? It comes out of the property of continuous functions on path connected components. Alright?

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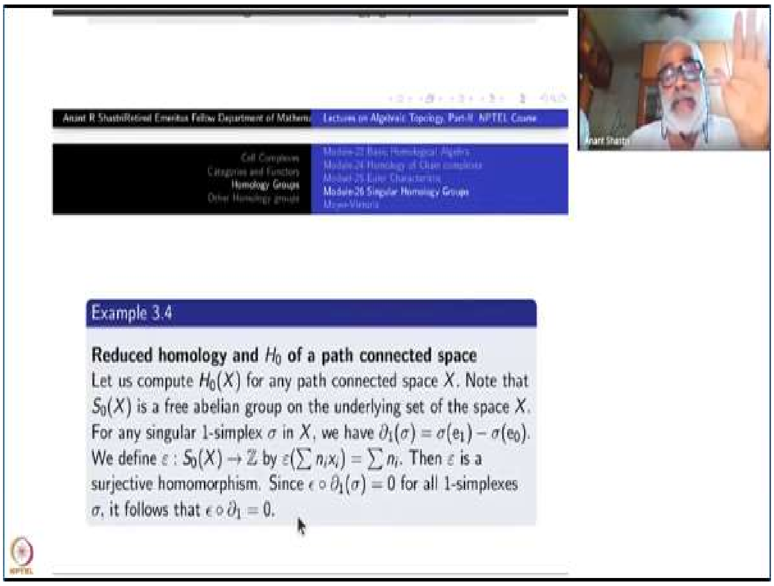

Now, we will do some more thing. These things are concerned in the general construction also. But this one is motivated by combinatorial aspects. The reduce homology and H_0 of a path connected space. Now, why suddenly I want to take path connected space? Because to compute the homology of any space X, by the above discussion, you have to do the homology computation of each path component and then you may simply take the direct sum.

Therefore, you are going to concentrate on computations of path connected spaces, okay? Presetnly, we shall do this at least for H_0, okay? So, let us compute $H_0(X)$ for any path connected space X, Okay? What is H_0? The boundary operator on S_0 is 0 and therefore the kernal is the whole group S_0, therefore $H_0(X)$ is the quotient of $S_0(X)$ by the image of ∂_1 from $S_1(X)$ to $S_0(X)$.

What is S_0? By the very definition you take the set of all functions from Δ_0 to X, Δ_0 is just a single point, functions from a single point is nothing but a point of X, and therefore, this set is X itself. That means $S.(X)$ is the free abelian group over X. For any one singular 1-simplex sigma in X, the boundary of σ is nothing but $\sigma(e_1) - \sigma(e_0)$, where e_0, e_1 are the vertices of Δ_1.

So, we define an augmentation map, usually denoted by ϵ from $S_0(X)$ to the integers. (This the map that I was mentioning that is going to be generalized. Right now it is motivated by our computational need.) So, ϵ is defined by this formula $\epsilon(\sum n_i xi)$ is equal to $\sum ni.$ S_0 is a free abelian group over X. $\mathbb{Z}$ is just the infinite cyclic group. So, take $\epsilon(x) = 1$ for all $x \in X$ and extend it linearly.

That is just add all the coefficients and that is the function ϵ, okay? Then ϵ is surjective homomorphism because X is non empty. I have to start with a non-empty space okay? Now $\epsilon \circ \partial_1(\sigma)$ is 0 for all σ, because $\partial_1(\sigma)$ consists of two terms one is positive sign and another with negative sign, the sum total of the coefficients is zero. So, $\epsilon \circ (\sigma)$ is 0 for all 1-simplexes. Therefore, $\epsilon \circ \partial$ itself is 0, because the singular 1-simplexes form a basis for $S_1(X)$.

The surjectivity property of this ϵ will be taken as a definition later on in the definition of augmentations for arbitrary chain complexes. Right now, what we are interested in is how does this help us to compute H_0, okay?

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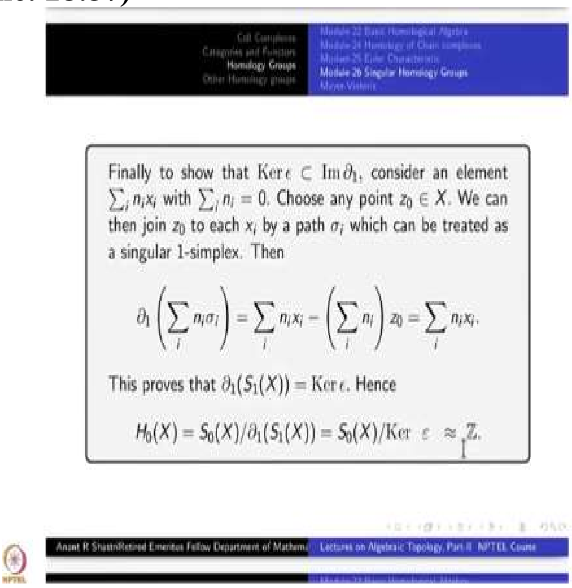

Finally, what we want to show is that kernel of ϵ is equal to the image of ∂_1. One way containment is already seen. Now, I want to show that kernel of ϵ is contained in the image of ∂_1 okay? So, for this consider an $\sum n_i x_i$ with $\sum n_i = 0$, an element in the $ker(\epsilon)$. Like $\sigma(e_0) - \sigma(e_1)$, you know the sum total of the coefficients must be 0. Take such an element. Take any point $z_0 \in X$. Okay?

X is path connected. Therefore, you can join z_0 to all these $x_i \in X$, okay, by some paths. What do you do? Select a path from z_0 to x_i and call it σ_i, Okay? Each σ_i can be thought of as a continuous function from the closed interval $[e_0, e_1]$ into X and hence as a singular 1-simplex. Now, you look at $\sum n_i\sigma_i$ and its boundary ∂_1. That is nothing but $(\sum n_i)z_0 - \sum n_i x_i$.

But this summation is 0 so, therefore, ∂_1 of this 1-chain is $\sum n_i x_i$. This proves that the kernel of ϵ is contained in the image of ∂_1. Therefore, the image of ∂_1 is actually equal to kernel of ϵ. It follows that $H_0(X)$ which is $S_0(X)$ divided by the image of ∂_1 is the same as $S_0(X)$ divided by the kernel of ϵ. By the first isomorphism theorem this is isomorphic to the image of ϵ which is the group of integers.

So, we have computed that $H_0(X)$ is isomorphism to $\mathbb{Z}$, okay? whenever X is path connected. Note that just connectivity is not enough to conclude this, by the way. You can give easy counter examples. okay?

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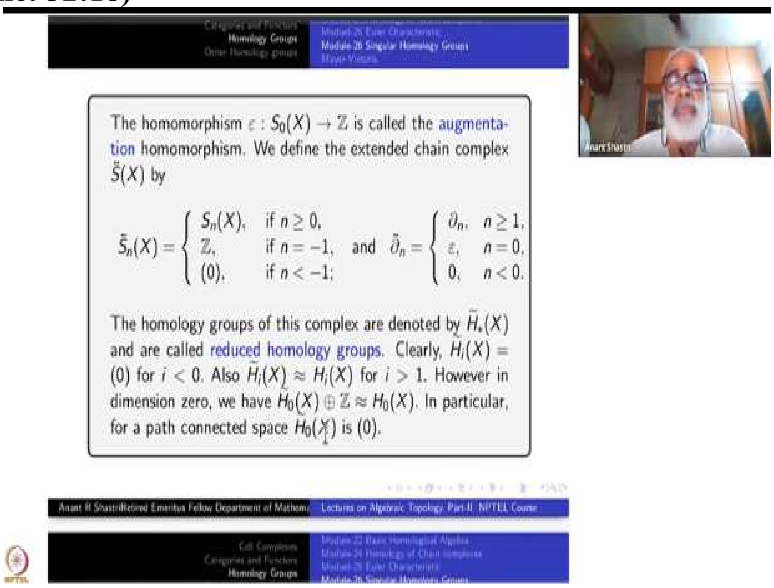

The homomorphism $S_0(X)$ to $\mathbb{Z}$ is called the augmentation, okay. Using this we extend the chain complex by altering it only at -1 level, okay. So, $\tilde{S}.(X)$ is defined as follows:

$\tilde{S}_n(X) = S_n(X)$ for n non negative, equal to $\mathbb{Z}$ for $n = -1$ and equal to 0 for $n < -1$. Accordingly the operator $\tilde{\partial}_n$ is taken to be the same as ∂_n for n positive, $\tilde{\partial}_0 = \epsilon$ and $\tilde{\partial}_n = 0$ for $n < 0$.

$\tilde{S}.$ is called the extended or the augmented singular chain complex. If you look at the homology of this one okay, that will differ only in 0-level okay? Everywhere else it is same as the homology of $S.$ We call the homology of $\tilde{S}.(X)$ the reduced homology groups of X and denote it by $\widetilde{H}_*(X)$.

Clearly, $\widetilde{H}_i(X) = H_i(X)$ for all $i \neq 0$. But in dimension 0, $\widetilde{H}_0(X)$ direct some with $\mathbb{Z}$ will be equal to $H_0(X)$. So, the reduced homology is reduced by one factor $\mathbb{Z}$ exactly at the 0-level. (That is why it is called reduced homology.) In particular, for a path connected space, $\widetilde{H}_0(X)$ is will be also 0. Without a twiddle, namely, the unreduced 0-th homology is $\mathbb{Z}$; that is what we have proved, but this $\mathbb{Z}$ factor goes away here. So, $\widetilde{H}_0(X)$ will be 0 okay? So, that is what you have to remember. Okay?

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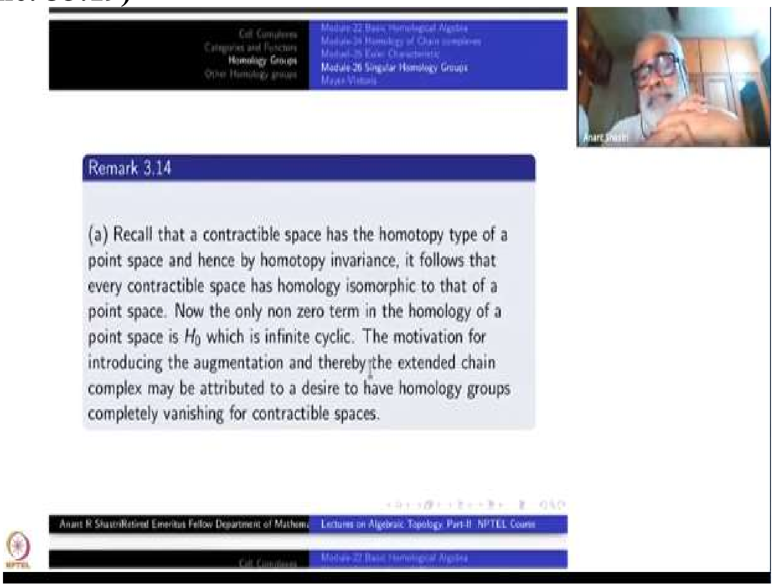

Now, I will tell you something about why this kind of reduced homology is important. We have seen that for a contractible space, the homology is the same thing as homology of a point. And for a point space, we have seen that H_0 is $\mathbb{Z}$. $H_0(*)$ is the infinite cyclic group $\mathbb{Z}$. and everywhere else it is 0. Whereas if you take the reduced homology what happens now? the entire homology will be 0, even at the 0-level. You know psychologically, what mathematicians would like to have is that for a contractible space all the homologies are 0.

That would be a neater conclusion to have. That was not the case with the usual homology that we have defined. H_0 of a nonempty space always survived, with infinite cyclic factor. So, you make a slight modification like this augmentation, which looks somewhat unnatural okay? Later on, we can make it functorial also, okay? So that the homology of a contractible space completely vanishes. Okay? So this was perhaps the motivation for considering is augmentation and reduced homology. Okay?

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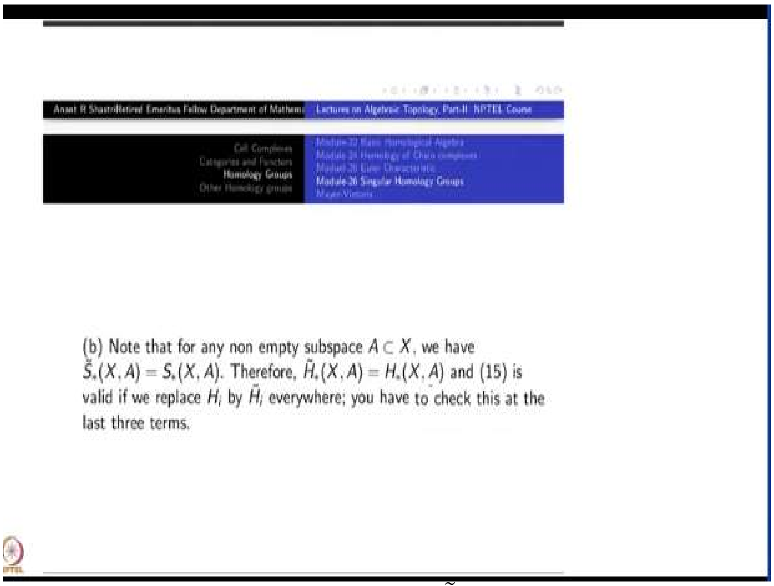

But note that for any non-empty subspace A of X, $\tilde{S}(X, A)$ remember, by definition, is given by a short exact sequence 0 to $\tilde{S}(A)$ to $\tilde{S}(X)$ to $\tilde{S}(X, A)$ to 0, Right? So the extra $\mathbb{Z}$ factor appears in both the first and second term and hence disappear in the third one. So what you are left with is that $\tilde{S}_{-1}(X, A)$ is also equal to 0. Thus there is no change in the S and $\tilde{S}$ for a pair (X, A) where A is non empty.

Therefore, the whole reduce homology for a relative pair when A is non empty is the same thing as homology for the ordinary thing, without reduced, there is no change at all. Okay? Now, what you can go back to the long homology exact sequence here, you can put a tilde everywhere here, no problem, but when you come to the index 0, you have to be careful.

All that you must do is directly apply the proposition to the short exact sequence of augumented chain complexes to obtain the long exact sequence which will be identical with the long exact sequence of the unreduced homology except for the tail end: $\widetilde{H}_1(X, A)$ to $\tilde{H}_1(A)$ to $\tilde{H}_1(X)$ to $\widetilde{H}_1(X, A)$. Even here, the first and the fourth terms coincide with the corresponding terms without tilde. Only in the second one and third terms there is a change.

So, that brings us to the another very important property which is much more topological than whatever you have discussed so far. Of course one of them was homotopy invariance. And the other one was path connectivity. Okay. The next one is much more topological and that is called excision. We will study it separately next time. Thank you.