Introduction to Algebraic Topology, Part-II
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Lecture - 26
Singular Homology Groups

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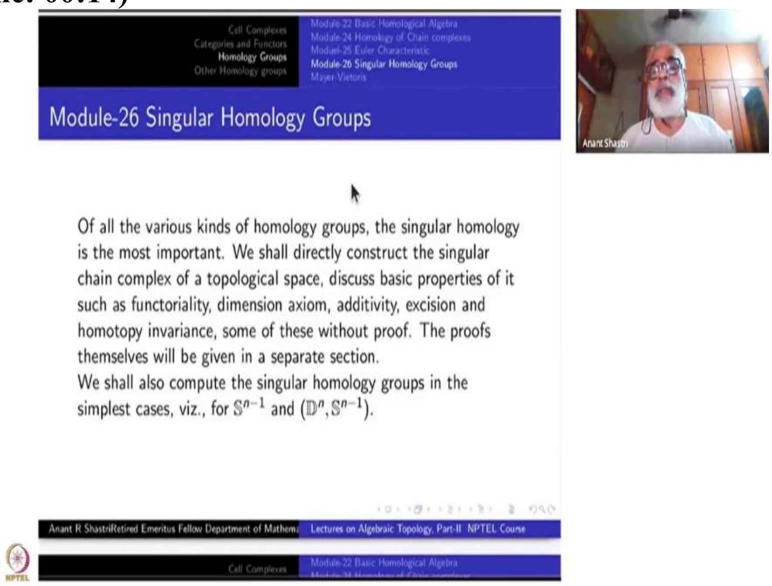

Today we shall start the study of singular homology, which is one of the most important homology theory. There are different approaches. Our approach is to directly construct a single chain complex associated to topological space, discuss its basic properties such as some functoriality, dimension axiom, additivity, excision and homotopy invariance etc. some of them without proof to begin with. Missing proofs will themselves be given in a separate section.

We shall also study some properties which are special to singular homology. It may not be shared by other homologies and then we shall compute the singular homology of the spheres and the relative homology of the disk modulo the boundary of the disk. So, this is the plan. So, let us begin with the construction directly.

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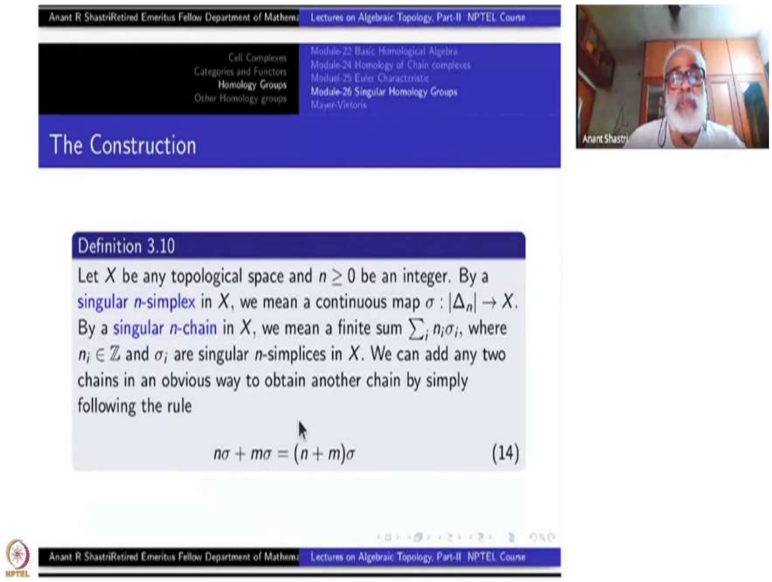

Take a topological space, fix an integer $n \geq 0$. We are going to define the chain groups. So, for the first time the chain complex is going to be a non-negative one. For negative integers we are going to take them all to be 0. So, for $n \geq 0$, a singular n-simplex in X is nothing but a map from Δ_n into X, where Δ_n denotes the standard n-simplex in $\mathbb{R}^{n+1}$ which is the convex hull of the standard basis elements $e_0, e_1, e_2, \ldots, e_n$.

A singular n-chain in X, we mean a formal finite sum of these singular simplexes, i.e. an integer combinations $\sum n_i\sigma_i$, where σ_i is a singular simplex and n_i is an integer. The sum is a finite sum. You can add any two chains in an obvious way, namely, adding the corresponding coefficients. So, the rule is that if you locate a singular simplex σ with coefficients m and n in the two terms to be added, then in the addition you put $m + n$ as the coefficient of σ.

There is an empty sum and that is 0, the identity element for this addition, wherein all the coefficients of each singular simplex is taken as 0.

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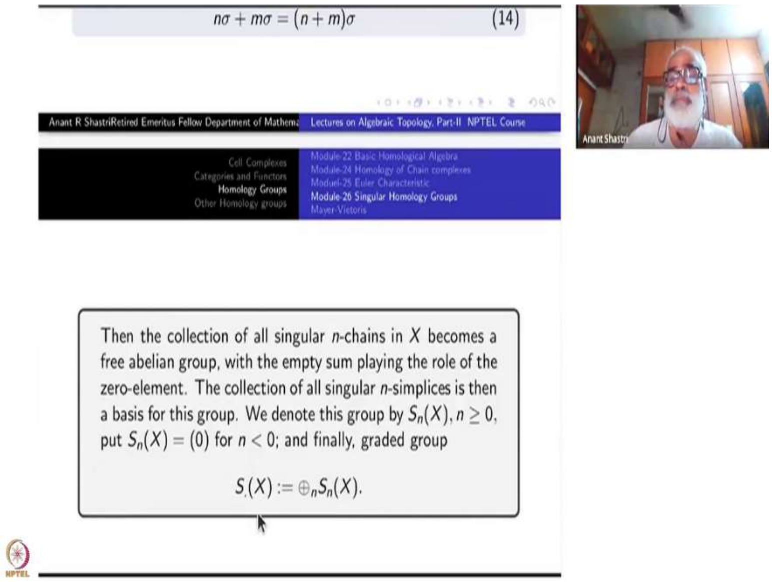

The set of all n-chains in X forms a free abelian group with a basis as the set of all singular n-simplices in X. So, it is a very huge group and we should denoted by $S_n(X)$, (S for singular). And by definition, $S_n(X)$ should be taken to be 0 for n negative. And finally, the total grade group $S.(X)$ is direct sum of $S_n(X)$ over all integers n. So, we have defined a singular simplex and then a chain and then made a group out of these n chains then we have taken the direct sum. So, now we have a graded abelian group.

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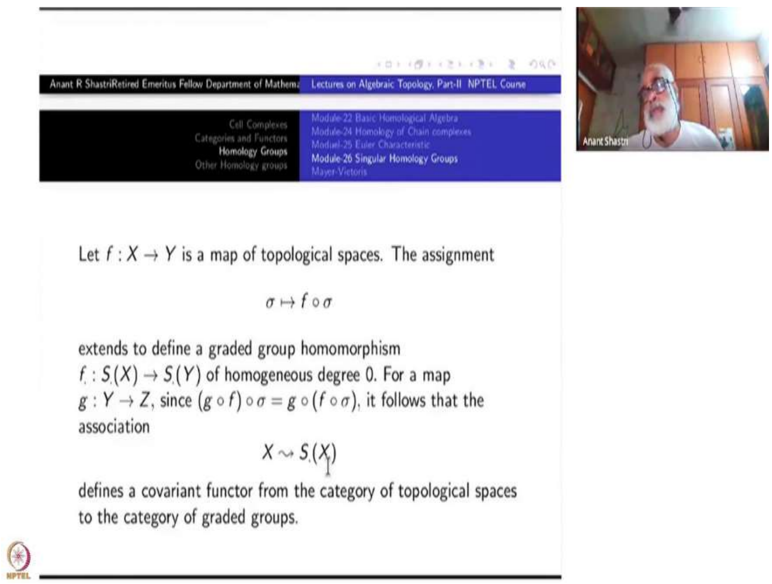

So, now we want to make it into a category. So, given a map from X to Y if you have a n-simplex σ in X, by the very definition, it is a continuous function from Δ_n into X, you can follow it by f, you will get a singular n-simplex in Y. So, σ goes to $f \circ \sigma$ defines a map from the set of singular n-simplices in X to the set of singular n-simplices in Y. Any set theoretic function can be extended to a unique homomorphism of the free abelian groups in the natural way, namely, $\sum n_i\sigma_i$ will go to $\sum n_i f\circ\sigma_i$. That is $f(\sum n_i\sigma_i)$, You can write f outside or inside. It is the same thing that will give you a graded homomorphism $f.$ from $S.(X)$ to $S.(Y)$ and the degree of this homomorphism is 0. n-chains will go to n-chains. Once again it is very easy to verify that if you have map g from Y to Z and a singular simplex in X first you take $f.$ of that then you take g of that it is the same thing as directly taking g circ of that, because $(g\circ f)\circ\sigma$ is the same as $g\circ(f\circ\sigma)$.

Moreover, if f is the identity map from X to X, clearly $f.$ is identity of $S(X)$. This is all that is involved in saying that the association X going to $S.(X)$ and f going to $f.$ defines a covariant functor. So, from the category of topological spaces the category of graded abelian groups you get a functor. So, graded abelian groups form a category that we already know. So, this is the meaning saying that this construction categorical. So, X to $S(X)$ is a covariant functor from the category of topological spaces to category of graded groups.

Now, we are not satisfied with just that much because we want to do homology. First thing we want to do is to convert $S.(X)$ into a chain complex. That means, we have to define a boundary operator here, a morphism ∂ of degree -1 such that ∂^2 is 0. That is our next task.

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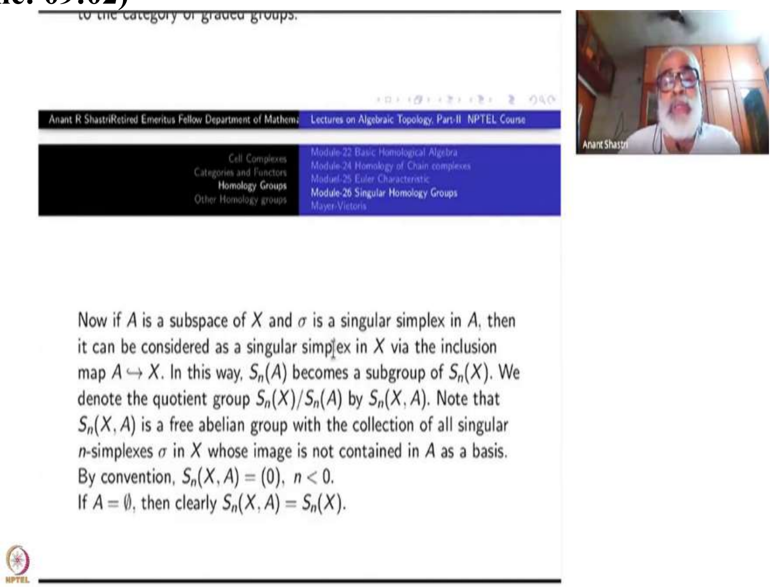

But before that, let us see what happens under subspaces, i.e., some special properties out of functoriality property. If A is a subspace of X, there is a natural way $S.(A)$ will be included in $S.(X)$, namely, if σ is a singular n-simplex in A, by composing with the inclusion map i from A to X, you can just treat it as an n-simplex in X also. So, at the set theoretic level itself n-simplexes in A will form a subset of all n-simplexes in X.

Therefore, the free abelian group over the smaller set will be a subgroup of the free abelian group over the larger set. This way, $S_n(A)$ become a subgroup of $S_n(X)$. This allows us to take the quotient group $S_n(X)$ by $S_n(A)$ and we should denote it by $S_n(X, A)$. Note that $S_n(X, A)$ even though it is a quotient group here, there is a nice about it, namely, it is also a free abelian group. What is a basis? The set of all those n-simplexes in X which are not contained in A. If σ from Δ_n and into X has its image not contained in A then this will be one of the generators for $S_n(X, A)$. So, this is also a free abelian group for each n.

Of course, by convention because $S_n(X)$ is defined as 0 for n negative, similarly, $S_n(X, A)$ is also defined as 0 for n negative. Once again if A is empty then $S_n(A)$ is just the 0 group. So, $S_n(X, A)$ is nothing but $S_n(X)$ in that case. Finally, if you have a map f from one pair (X, A) to (Y, B), that means a continuous function from X to Y, so that $f(A)$ is inside B, then as seen before $S_n(A)$ is mapped inside $S_n(B)$ and hence you would get a morphism from $S_n(X, A)$ to $S_n(Y, B)$.

So, this consideration will tell you that the association (X, A) to $S_n(X, A)$ also forms a covariant functor. $S_n(X)$ can be thought of as a special case of the when set A is empty. This is another way of looking at it. So, simultaneously we have defined two functors here one from the category of the pair of topological spaces and other one on the category topological spaces.

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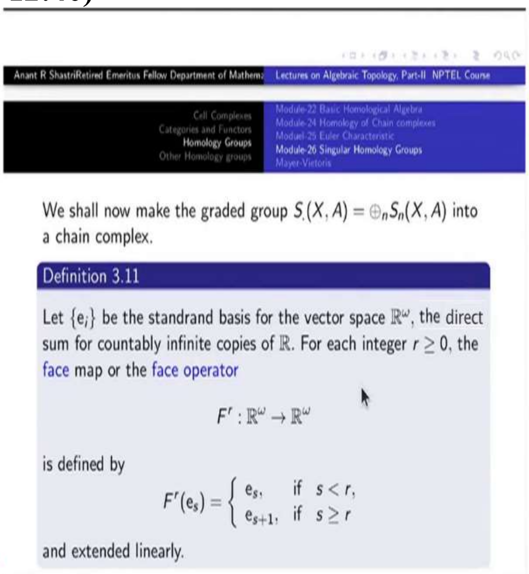

Similar to $S.(X)$, we define $S.(X, A)$ to be the direct sum of $S_n(X, A)$'s. Now, let us make them both into a chain groups. The construction for this one is the same for the cases. So, recall that we use $\mathbb{R}^\omega$ to denote the direct of countably infinite copies of $\mathbb{R}$. This is a standard inner product space.

So, for each $r \geq 0$, there is a face map or face operator F^r from $\mathbb{R}^\omega$ to $\mathbb{R}^\omega$ which is some kind of a shift operator. If you have not heard of shift operators, don't worry, here is the definition. We use the standard basis elements basis elements of $\mathbb{R}^\omega$ to express it. For example, the operator F^r does not shift the basis elements e_s for $s < r$, first $r-1$ elements, are not affected. The r-th element e_r itself and all other $e_s, s \geq r$, are shifted by one place, So, e_s is going to e_{s+1} for $s \geq r$. Of course, we extend this to a linear map over all of $\mathbb{R}^\omega$. So, in effect, each F^r is an injective linear mapping which will miss one of the basic elements in its image viz., e_r itself. So, once you define it on the basis elements you extend it linearly. These are called the face operators why they are called face operators?

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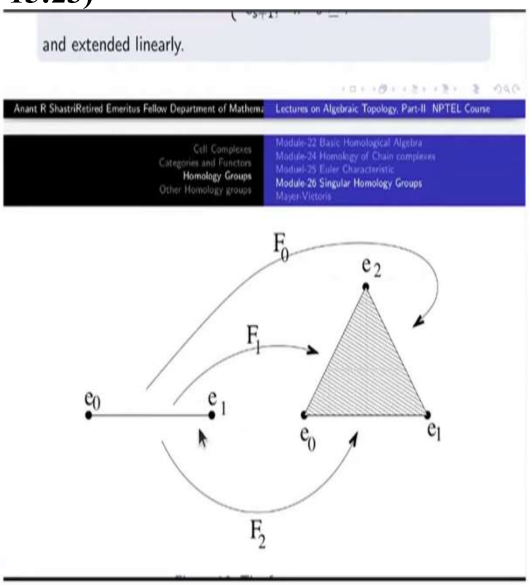

Look at this picture. The line segment from e_0 to e_1 is my Δ_1. This this edge. Here Δ_2 is the standard 2-simplex, which is the convex hull of e_0, e_1 and e_2. So, what does F_0 do? F_0 starts shifting e_0 to e_1 and e_1 to e_2 etc. and is linearly extended. So, this edge goes into the edge $[e_1, e_2]$. That is a 1-face of the 2-simplex Δ_2. You can see that this is opposite to the vertex e_0. So, F_0 maps the 1-face here to the face opposite to e_0. Like this $F_1(e_0) = e_0$ and $F_1(e_1) = e_2$. So the 1-face is mapped onto the 1-face opposite to e_1. Similarly, F_2 will map the edge $[e_0, e_1]$ to itself. No shifting at this stage. So, this edge is opposite to is e_2.

So, this simple example illustrates the behaviour of the face operators. We hope that now you understand why they are called face operators. This simplex is put as a face in the next one. You can compose two face or more operators, that will put smaller simplexes into larger and larger simplexes as faces.

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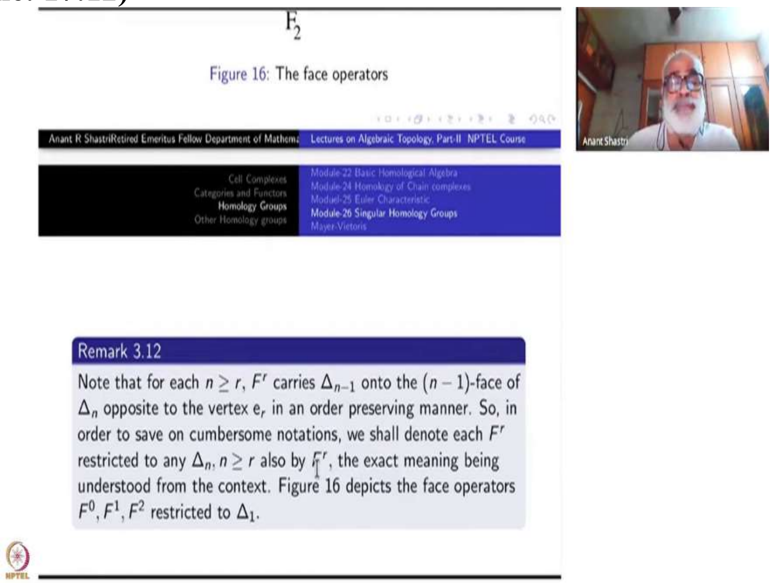

In general, note that F^r carries Δ_{n-1} on to the $(n-1)$-face of Δ_n opposite to the vertex e_r. All F^r are injective and order preserving.

So, we shall denote each F^r restricted to any Δ_n also by the same symbol. Its image will be in Δ_{n+r}, but we will not use separate notations for this that will be too much of cumbersome notation, the exact meaning being understood by the context.

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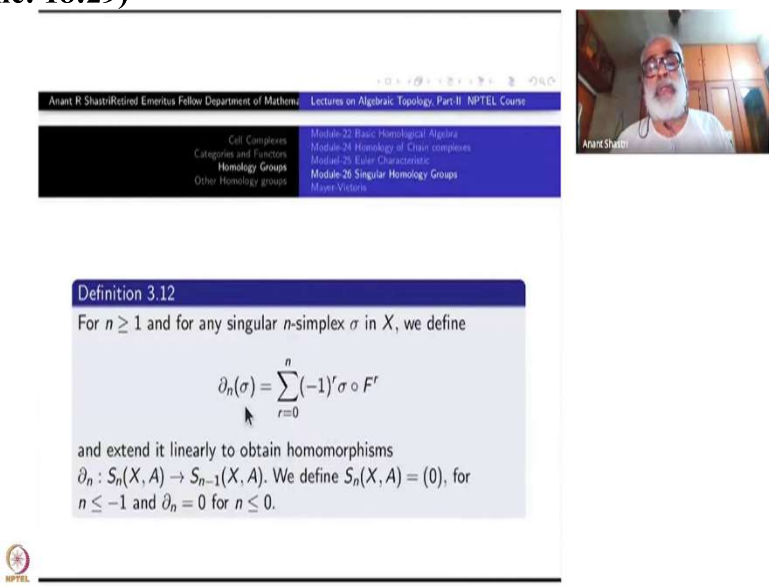

Now, the boundary operator on $S.(X)$ can be defined easily. The face operators from Δ_{n-1} cover the entire of the boundary of Δ_n. Taking a clue from the integral calculus, we can express this fact in an oriented fashion.

For example, here look at the boundary of Δ_2. I can take first the edge $[e_0, e_1]$ followed by the edge $[e_1, e_2]$ and then followed by the edge $[e_2, e_0]$. Note that the last one is not exactly the image of Δ_1 under F_1 but in the opposite direction. Therefore I should put a negative sign on this. So, this motivates the definition of the boundary of the identity n-simplex Δ_n and thereby the boundary of any singular n-simplex in general. Whatever you have to do, do it for the identity simplex take its image under σ. Thus we can define $\partial(\sigma)$ to be $\sigma(\partial(\Delta_n))$ which should be the alternate sum $\sum(-i)^i F^r$.

We apply σ for the whole sum or equivalently σ taken inside the summation. So, this is a definition of the boundary of any singular n-simplex σ as a $(n-1)$-chain. On the right we have just one singular simplex but on the right we have a $(n-1)$-chain. And then we extend this one linearly over the whole of $S_n(X)$ to obtain a homomorphism from $S_n(X)$ to $S_{n-1}(X)$.

So, that will also give you ∂ from $S_n(X, A)$ to $S_{n-1}(X, A)$ also, because if σ were taking value in A, then all the maps $\sigma \circ F^r$ for all face operators also takes value in A. Easy to see that. Hence the same ∂ factors down to give a morphsim on the quotients. So simultaneously we have this morphism both for $S.(X)$ as well as $S.(X, A)$.

So, remember we have defined $S_n(X, A)$ is to be 0 for n negative therefore, the boundary of another you defined for a $n < 0$, as soon as you hit this index with negative then there is no question this has to be 0, so we will define this also 0. So, that completes definition as a homomorphism of degree -1, but we have yet to verify why $\partial_n \circ \partial_{n-1}$ is 0, ∂^2 is 0 is what you have to define that is a straightforward computation. Nevertheless, we will do it you know in a systematic way.

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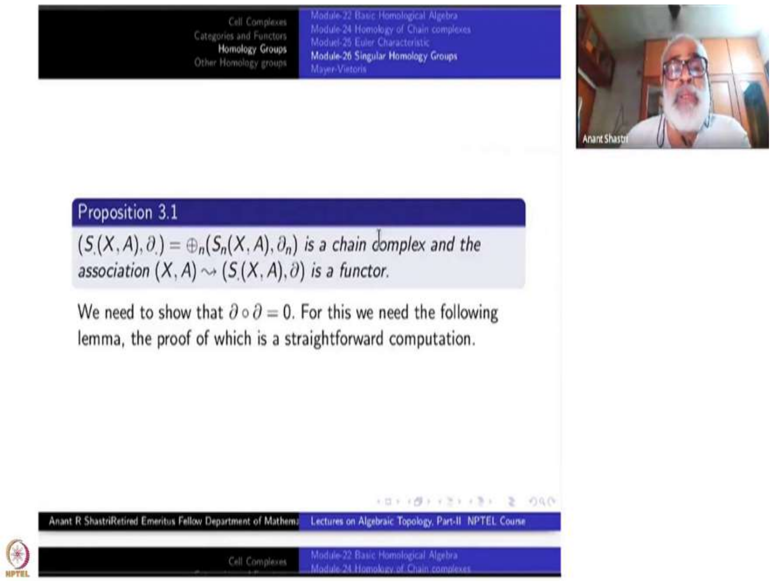

So, the statement of this proposition is that $(S_{\cdot}(X, A), \partial)$ which is by definition the direct sum over n of $(S_n(X, A), \partial_n)$ is a chain complex and the association (X, A) leads to $(S_{\cdot}(X, A), \partial)$ is a functor. The last part is already verified, property we have already verified. So, it remains to verify that $\partial \circ \partial$ equals to 0, which is a straightforward computation.

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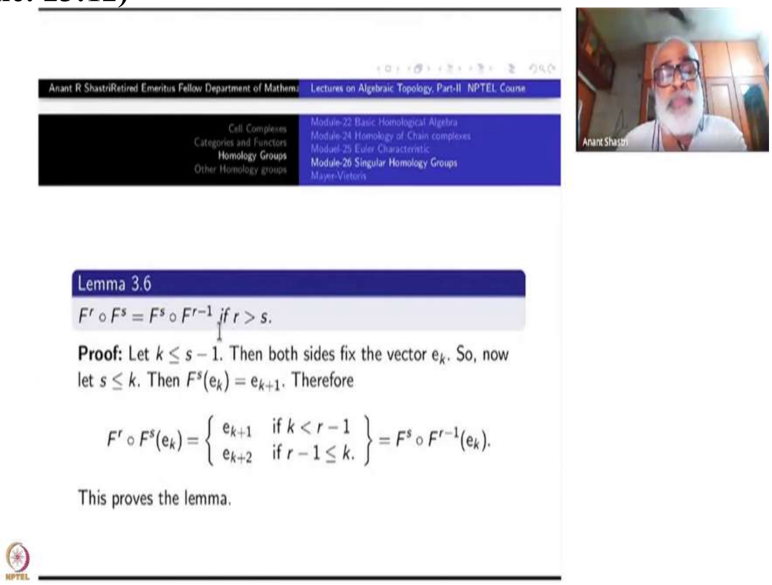

For which you need one of the interesting properties of this face operators. The relation between the composites of two face operators take in different order, namely we have this lemma: If $r > s$ then $F^r \circ F^s = F^s \circ F^{r-1}$. Thus all the composites can be re-expressed so that the indexing is always increasing. So, now the proof of the lemma.

If $k \leq s - 1$, then F^r, F^s and F^{r-1} all of them fix e_k and hence both sides fix e_k. Next consider the case $s \leq k$. Then $F^s(e_k)$ is equal to e_{k+1}. Now apply F^r on e_{k+1}. There are two subcases. If $k+1 < r$ which is the same as $k < r-1$, then $F^r(e_{k+1}) = e_{k+1}$ itself. Otherwise it will be e_{k+2}.

So, this is the combined formula for $F^r \circ F^s$. I want to say this is equal to $F^s \circ F^{r-1}(e_k)$. So, what is $F^{r-1}(e_k)$ in the first subcase when $k < r-1$? $F^{r-1}(e_k)$ is just e_k. But then $F^s(e_k)$ will be equal to e_{k+1}.

On the other hand, if $r-1 \leq k$, i.e., in the second subcase, $F^{r-1}(e_k) = e_{k+1}$ and then F^s of that will be $e_{k+2)}$, because $s < k-1$. This proof is usually left to you as an exercise, but here I have proved it.

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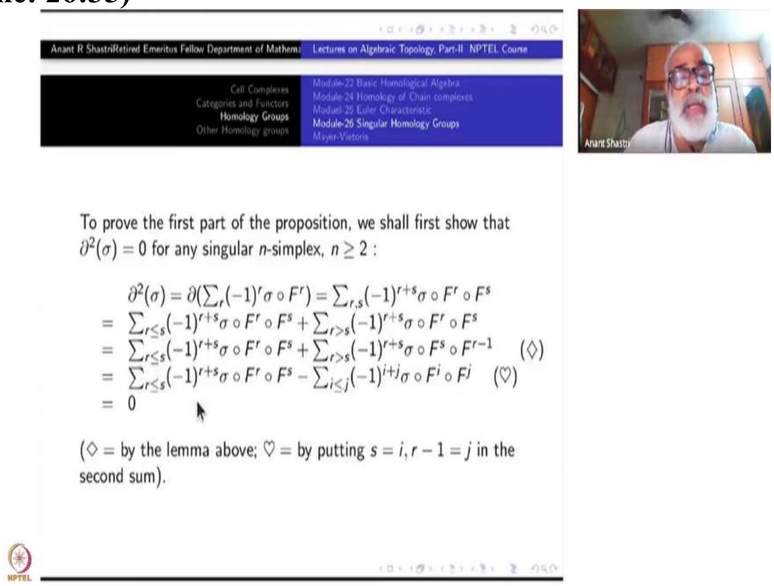

Now, we can now prove the proposition in a very elegant way. First we shall show that ∂^2 is 0 for any singular n simplex. Then by the linearity it will be follow that ∂^2 of any chain is also 0. So, this makes sense for $n > 1$ only. If n is 1 or 0, then $\partial^2(\sigma)$ is already 0 simplex and $\partial \circ \partial$ will be automatically zero. So, there is nothing to prove. So, you can assume $n \geq 2$.

So, by definition $\partial^2(\sigma)$ is $\partial(\sum(-1)^r \sigma \circ F^r)$. The summation has certain limits depending on n. but we need not bother about them, except that it is a finite sum. Infact, $n+1$ terms will be there. So, if we apply ∂ once again we get a double summation over (r, s) of $(-1)^{r+s} \sigma \circ F^r \circ F^s$.

We now break this summation into two parts, first the sum of all those (r, s) where $r \leq s$ and second one is the sum of all those (r, s) where $r > s$. In the second summation, I interchange

$F^r \circ F^s$ with $F^s \circ F^{r-1}$ Now, what happens is that the first index is again less than or equal to the second index. So term by term there will be a one one correspondence in the set of terms occurring in the two summations However, the signs are opposite one being $(-1)^{r+s}$ and the other corresponding term having $(-1)^{s+r-1}$.

So, do that systematically as follows: put $s = i$ and $r - 1 = j$ in the second summation. Then $r > s$ is the same as saying $i \leq j$ and hence I can write the second summation as the summation over (i, j) of $(-1)^{i+j-1}\sigma \circ F^i \circ F^j$. This is precisely the negative of the first term.

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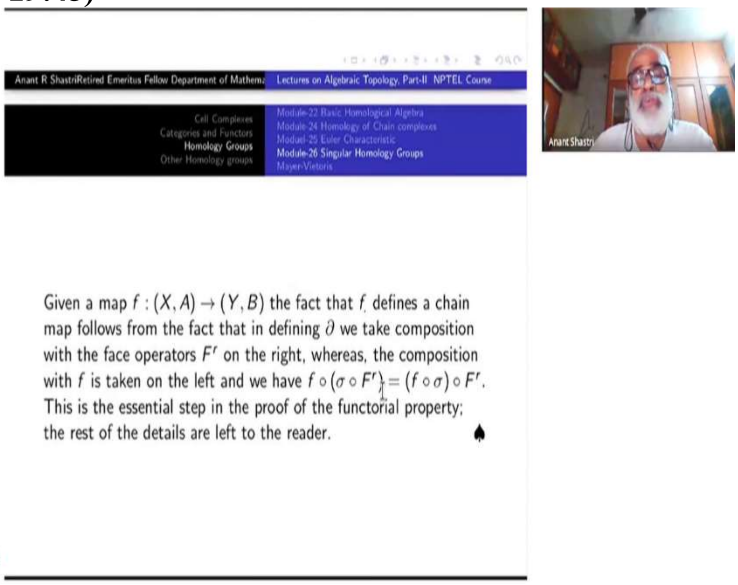

Given the map f from (X, A) to (Y, B) the fact that f defines a chain map follows from the fact that in defining ∂, what we have done is to compose σ with F^r on the right side? And how is $f(\sigma)$ is defined? $f(\sigma)$ is defined by taking composition with f on the left side. So the law of associativity of the composition of functions is at work here. That takes care of functoriality.

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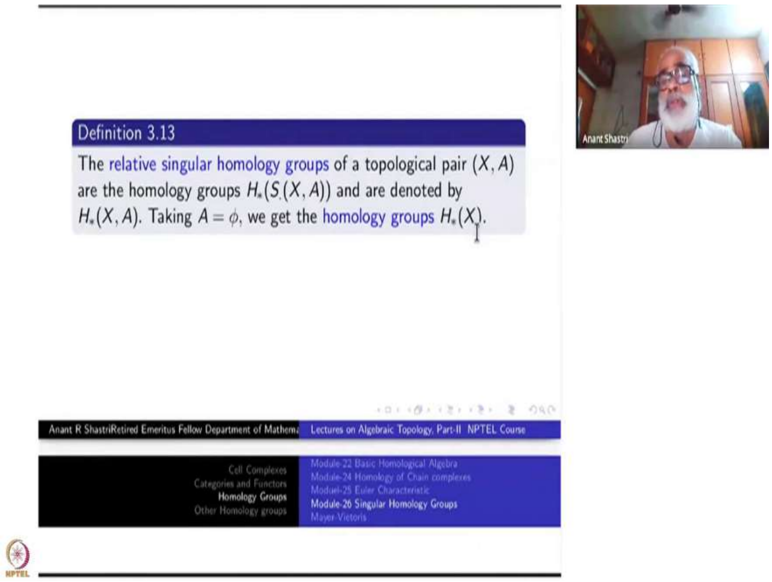

We now make a definition. The relative singular homology group of a pair (X, A) is the homology of the chain complex $S.(X, A)$. We will just write it as $H_*(X, A)$, by-passing the notation $S.(X, A)$. When A is empty, we just write it $H_*(X)$. These are called the homology groups. Thus the construction of the relative homology groups of a pair (X, A) is over.

Once again, if we have morphism f from (X, A) to (Y, B), i.e., if you have continuous map f from X to Y which takes A to B, then we have the chain map $f.$ from $S.(X, A)$ to $S(Y, B)$, which in turn gives you a homomorphism of the homology modules. Therefore, this association becomes functorial.

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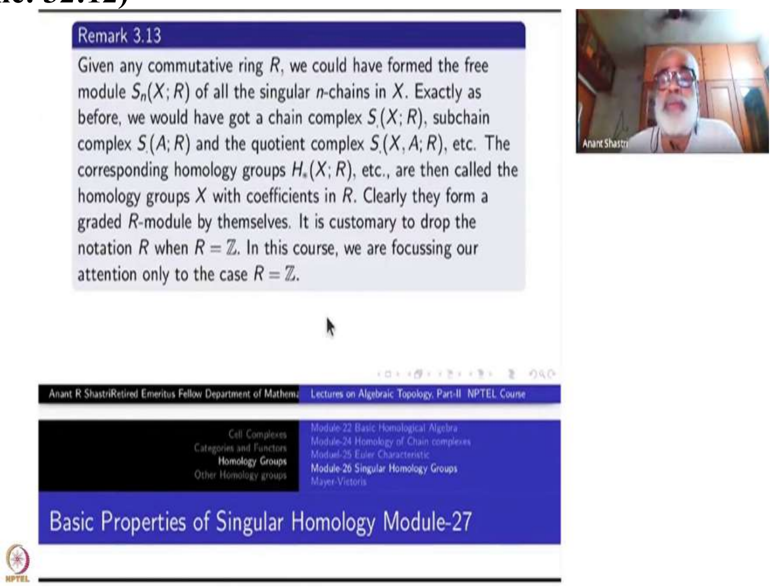

One more thing. I can I make a remark. If we had started with an arbitrary commutative ring R, instead of $R = \mathbb{Z}$, then in the $\sum n_i\sigma_i$ you allow n_i to vary over the ring R, the scalars from R. What you get is the free module over R with the basis consisting of all singular simplexes. So, then we would write $S.(X), S.(X, A; R), H_*(X, A; R)$ etc., Instead of $\mathbb{Z}$, you would have R here. When the ring R is $\mathbb{Z}$, the way I have done, we are not going to expressly write the $\mathbb{Z}$, that is all. So the entire construction of this one would have been possible over any ring. So, this is where we stop. Next time we will start the properties of homology. Thank you.