Introduction to Algebraic Topology (Part – II)
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Lecture – 25
Euler Characteristics

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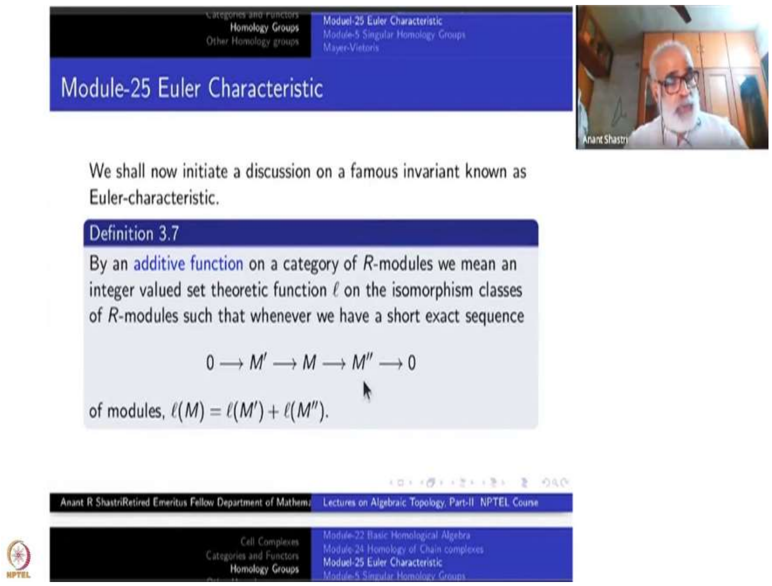

So, we have done some algebra, a preparatory work before studying the homology of topological spaces, we should do one more algebraic preparation though it is not necessary immediately okay? It just fits into the kind of preparation that we are making. Strangely, the title of this talk is Euler characteristic which is very much topological but here we are going to give it a completely algebra treatment.

The Euler characteristic will be discussed again and again in this course. So, we are only initiating a discussion now. So, here is a definition in the category of R-modules, where R is a commutative ring, preferably a PID and if you have difficulty as I told you right in the beginning you can just assume that R is the ring of integers and modules are just abelian groups.

By an additive function on this category we mean an integer valued set function ℓ on the isomorphism classes of R-modules. The collection of all R-modules is not a set but isomorphism classes of R-modules is a set by axiom of choice. So you take a set theoretic function which takes non negative integer values. It should have the following property: whenever you have a short exact sequence of R-modules 0 to M' to M to M'' to 0 okay? You should have $\ell(M)$ must be equal to $\ell(M') + \ell(M'')$ okay? So, you can call such a function a length function which is studied in algebra at various levels.

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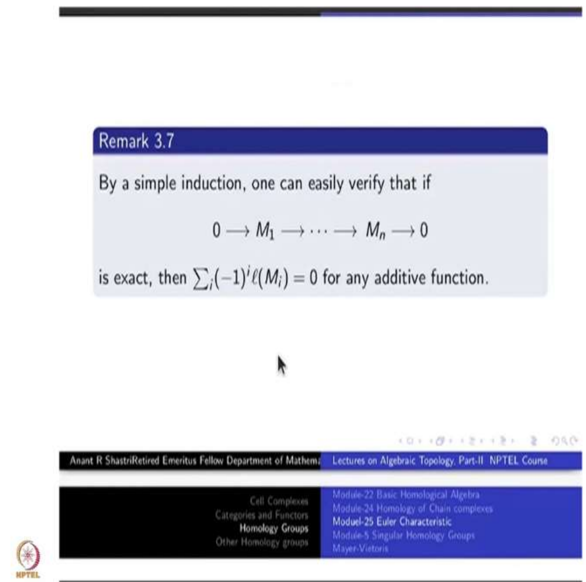

Clearly $\ell(0) = 0$. What I want to say is that by a simple induction, it follows that any additive function will have this property: namely whenever you have a finite exact sequence a finite short sequence 0 to M_1 to M_2 to $\ldots$ to M_n to 0, then the alternate some of $\ell(M_i)$ is zero, $\ell(M_1) - \ell(M_2) + - \ldots (-1)^{n-1}\ell(M_n) = 0$.

So, additive functions I have this property, viz., alternating sum whenever makes sense is zero. So, how to prove this one? I have told you that a long exact sequence like this can be always split up into short exact sequences and then for each short exact sequence you have the property. So here is a chance to illustrate that principle.

So, let us name the first morphism in the sequence α_1 from M_1 to M_2, just for the sake of writing down the proof. Okay?

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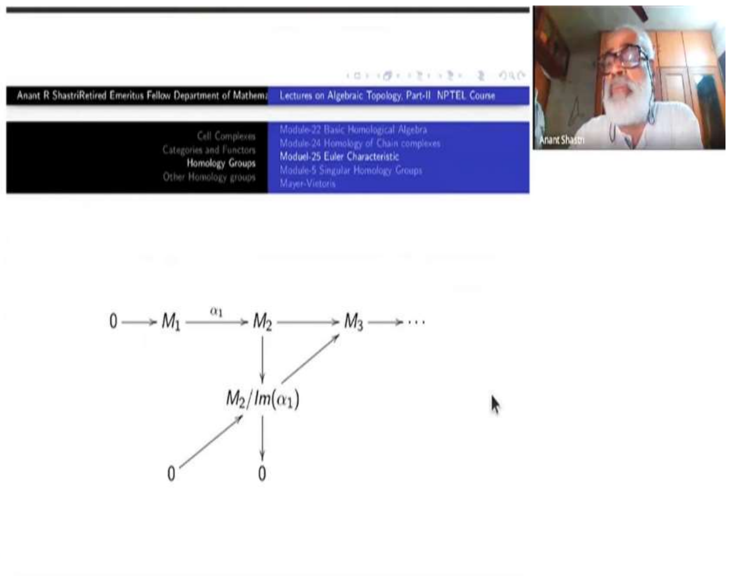

I do not care what other morphisms are, the first one is α_1. So, I split up this one I do not go directly to M_3 but I go to M_2 modulo image of α_1. So, that is the cokernel of α_1 right? So, this quotient is surjective so if you look at the sequence 0 to M_1 to M_2 to M_3 to 0, this is a short exact sequence okay? Therefore, $\ell(M_1) - \ell(M_2)$ +ℓ(cokernel of α_1) =0, right?

So, I know this much. Now I want to go here M_2 by image of α_1, this map M_2 to M_3, whatever it is, on image of α_1 it is 0, therefore it factors down through the quotient map M_2 to cokernel of α_1 and gives you this morphism here. And that morphism is injective because of the exactness of the original sequence at M_2.

This means that at this point cokernel of α_1 the new sequence is exact. From here onward the image of this α_2 is same thing as the image of this $\bar{\alpha_2}$ whatever and beyond that the sequence is not changed okay? Now look at the length of the new sequence, it has come down by 1 okay, which you may relabel as 0 to M_2' to M_3 ... apply induction on this one to get that the alternate sum is zero. Adding these to equations with correct signs, observe that the two extra terms viz., ℓ(cokernel of α_1) cancel out giving you that the alternate sum of $\ell(M_i)$ is zero for the original sequence.

So, an additive function has this property whenever you have finitely many terms in an exact sequence, then the alternate some is 0 okay? So, this is this should be a very good measure to measure of deviation of a chain complex from being an exact sequence. You see at least in the case when you have only finitely many terms in a graded module.

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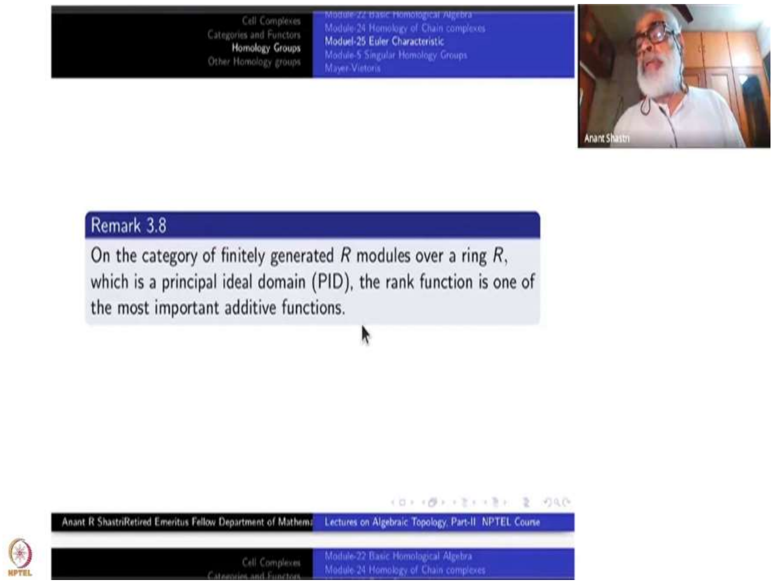

So, we have got our starting point example now, namely, in the category of finitely generated R-modules over a principal ideal domain. There is a structure theorem there just like in the case of finitely generated abelian groups, modules over the integers okay? Every finitely generated module can be written as a direct sum with a free module which is of finite rank and the torsion submodule.

So, for every finitely generated R module over a ring, the rank function is one of the most important functions and that function is an additive function. The rank of a direct sum two modules is equal to the sum of the ranks. For this additive property, you can produce a little more general one, namely, if you have an exact short exact sequence of finitely generated R-modules okay over a PID, then the alternate sum of the ranks is 0.

The simplest case is when R is a field. Then what are finitely generated modules over a field? They are finite dimensional vector spaces. If you have a short exact sequence of vector spaces, 0 to M_1 to M_2 to M_3 to 0, the rank-nullity theorem says nothing but that the dimension is an additive function, namely, alternate sum of the dimensions is 0.

So, this is a very nice name, called rank nullity theorem rank for a linear map: the dimension of the image and nullity of the linear map (i.e., dimension of the kernel) add up to the dimension of the domain. This term `dimension' is replaced by the term `rank' in the case of arbitrary rings. For arbitrary commutative rings this is a difficult notion. Sometimes the rank may not be defined properly okay? In general it is not defined alright. But for a PID, it is well defined. There are other cases also where it is defined, not necessarily for PIDs only. But we do not want to get into that kind of algebra here okay?

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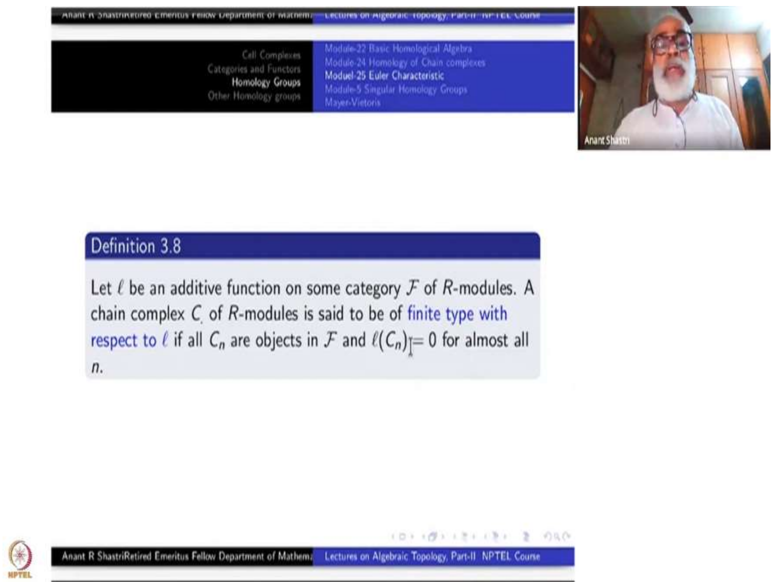

So, let ℓ be an additive function on some category of R-modules. A chain complex $C.$ of R-modules is said to be finite type with respect to the function ℓ, (ℓ is fixed, and all C_n are objects in this category, okay?) If $\ell(C_n)$ is 0 for almost all n, that means, only finitely many terms may be nonzero.

So, I am cooking up this definition, because I want to take alternate sum, so if infinitely many non zero terms then the sum does not make sense okay? Values of ℓ are integers, there is no question of convergence here. Indeed, convergence means that after a finite stage all term must be zero. must be 0, okay? So that is why I have put this condition in this definition. So, such chain complex $C.$ will be called finite type with respect to ℓ. If you change ℓ this may not be finite type okay? The given $C.$ may not be finite type with respect to one function ℓ and may not be so with respect to some other function ℓ'. Be careful about that.

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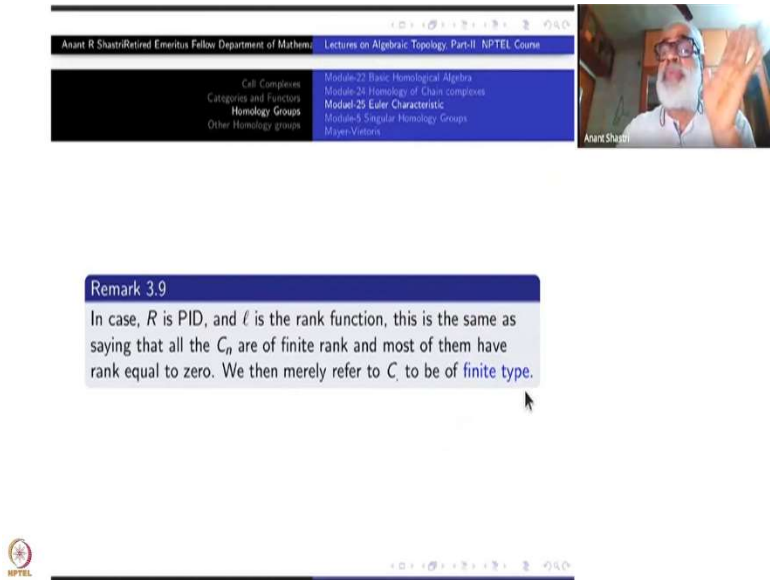

So, what is of very importance for us is that if R is a PID and ℓ is the rank function okay? Then saying that $C.$ is of finite type with respect to the rank function is the same as saying that all C_n are of finite rank and most of them have rank equal to 0. For this special case, we simply refer to $C.$ as of finite type. When you just say it is finite type that means we are we are interested in R as a PID and the ℓ is the rank function. For examples, R could be ring of integers ℓ is the usual rank function. Otherwise, i.e., in the general case, I have to mention specifically the additive function with respect to which we are taking finite type. That is just a convention.

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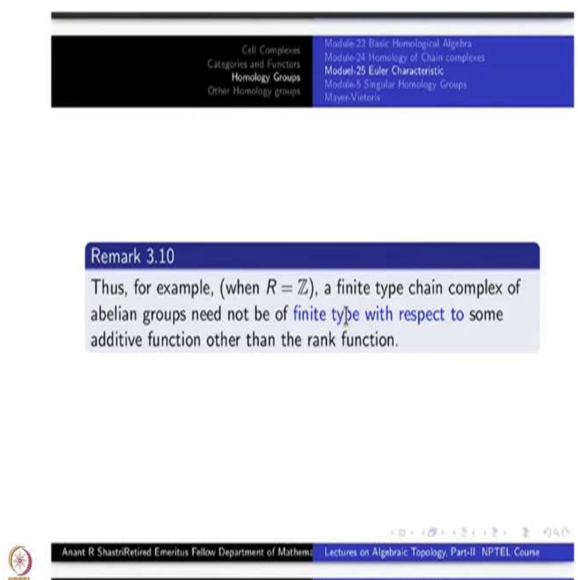

Thus, for example when R is $\mathbb{Z}$, a finite type chain complex of abelian groups need not be finite type with respect to some other additive function other than the rank function. Can give or think of another such additive function?

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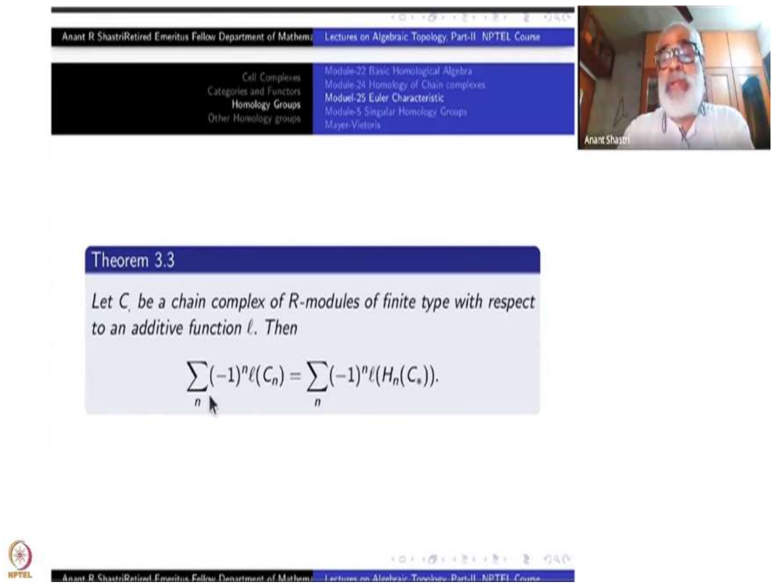

Now here is the theorem that we are interested in okay? What does it say? The rank function defined for a finite type chain complex now gives you something interesting on the homology okay? Let $C_\cdot$ be a chain complex of R-modules and is of finite type with respect to an additive function ℓ. Then the alternate sum of ℓ of C_n's is equal to the alternate sum of ℓ of $H_n(C)$'s, okay? Indexing must be carefully chosen okay?

You cannot change the indexing otherwise there will be a sign change here. Just by shifting all indices by 1, the whole sum will change the sign right? So be careful about that. So, for a chain complex of finte type, the alternate sum of ℓ's is the same that taken over homology. If the chain complex is actually exact then all the homoogy modules would be zero and hence the alternate sum would also zero for homology as well as cor the chain complex. That is a very weak special case of the statement. The present theorem though not so string is very very useful theorem.

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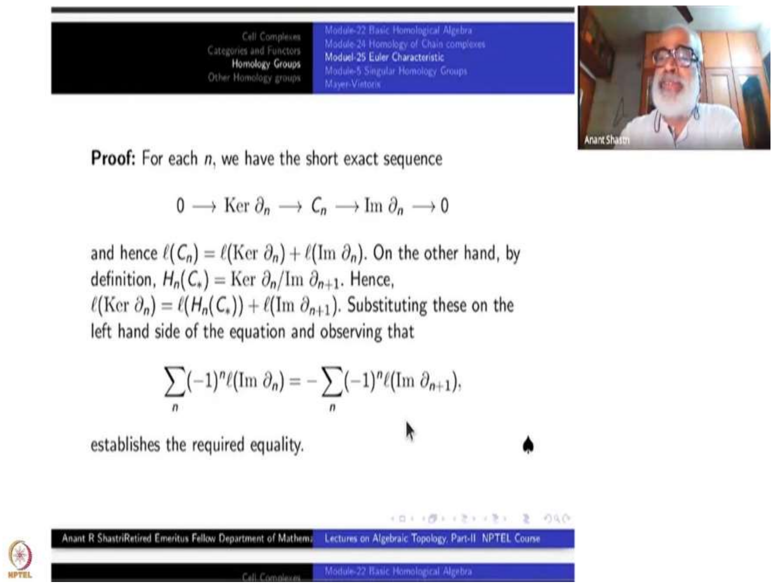

So, here is a proof which is simpler than the for the earlier lemmas where we used splittings. For each n, concentrate at one C_n. If you take the image of ∂_n as the codomain of ∂_n, and not the whole of C_{n-1}, you get a surjective map. It follows we have a short exact sequence 0 to $Ker(\partial_n)$ followed by the inclusion map to C_n followed by ∂_n to its image to 0. Therefore we have $\ell(C_n)$ is equal to $\ell(Ker(\partial_n)) + \ell(Im(\partial_n))$. This is the first identity.

Similarly, since $H_n(C)$ is $Ker(\partial_n)$ modulo $Im(\partial_{n+1})$, we have $\ell(H_n(C))$ is equal to $\ell(Ker(\partial_n)) - \ell(Im(\partial_{n+1}))$. Now take the alternate sum over n and substitute for $\ell(Ker(\partial_n)$'s from the second into the first identity. Note that the terms involving $Im(\partial_n)$ cancel out and we get the required formula.

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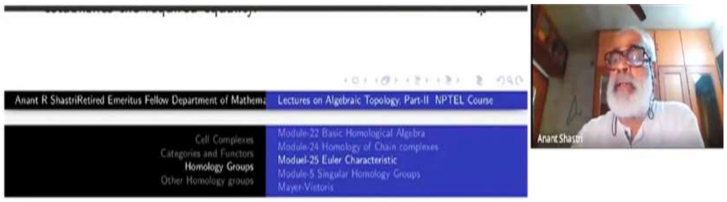

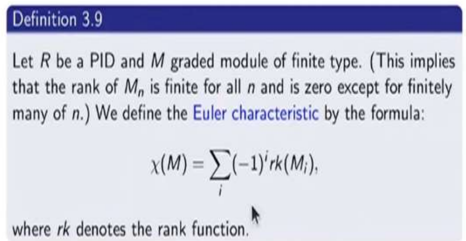

So, now we make a definition, following this great observation okay? Let R be a PID, and M be a graded module of finite type. I recall what is the meaning of this, namely, most of the graded components of M are zero and the rank of the remaining finitely many of them are finite okay? We then define the Euler characteristic of M by this formula. This is the standard notation $\chi(M)$ okay? M is graded module, not just one single module, $\chi(M)$ is the sum over i where i ranges from $-\infty$ to ∞ (but any way, it is a finite sum okay?) of $(-1)^i \mathrm{rank}(M_i)$, okay?

So, this is called the Euler characteristic of M. You could have defined $\chi(M;\ell)$ also with respect to any other additive function. So, you can call that as of Euler characteristic with respect to ℓ, but without any qualifier, just the Euler characteristic just means summation of $(-1)^i \mathrm{rank}(M_i)$.

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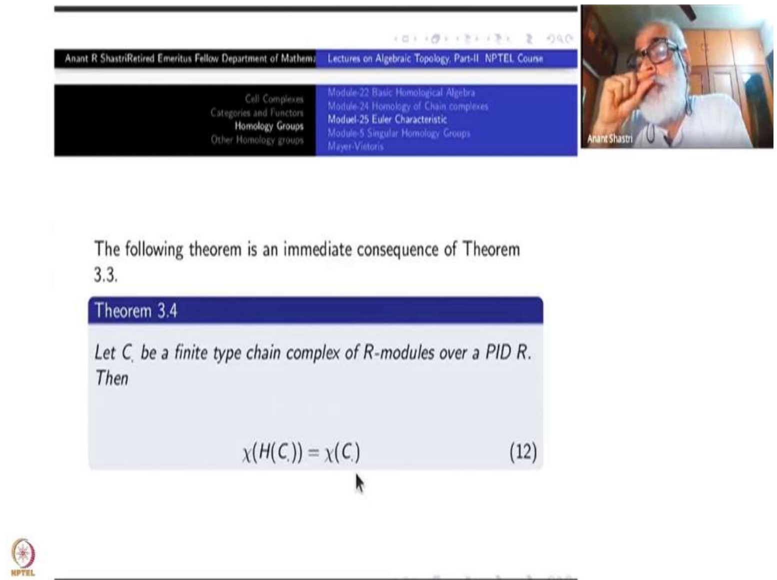

So, whatever you have proved just now, the theorem says that for a finite type chain complex of R-modules over a PID, the Euler characteristic of the homology is equal to Euler characteristic of the chain complex itself. So, this is the algebra that we needed later on. so we have established that one.

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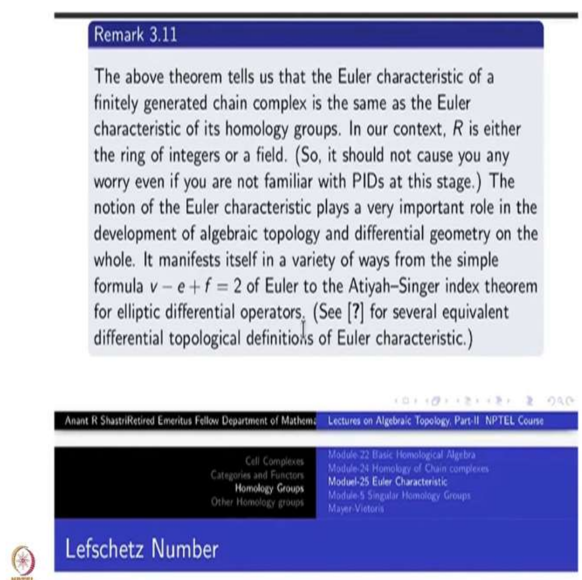

The above theorem tells us that the Euler characteristic of a finitely generated chain complex is the same as the Euler characteristic of its homology groups. In our context, R is either the ring of integers or a field. We could have taken R equal to $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Then the rank is nothing but the dimension of those vector spaces. So, you should not worry much even if you do not know what are PIDs and what are modules over PID etc.

The notion of Euler characteristic plays a very important role in the development of algebraic topology, differential geometry, analysis etc. It manifests itself in a variety of way from the simple observation, namely, for any planar graph, number of vertices minus number of edges plus number of faces is equal to 2. This is the famous formula of Eular for a planar graph: number of vertices minus number of edges plus the number of domains is always equal to 2. So, this was the observation of Euler. We can say Euler was the great grandfather of topology.

But this simple thing has now become so great you know, it manifests in so many other ways. There are at least half a dozen different definitions and then you can prove this is equal to that and so on, at various places and then you take one of those definitions and generalize it and so on. So, you get things such as Atiyah Singer index theorem for elliptic differential operators etc.

So, many other things they are all interrelated so you take one aspect of it, generalize it and do something and so on. So just this one single concept of Euler has created a lot of mathematics.

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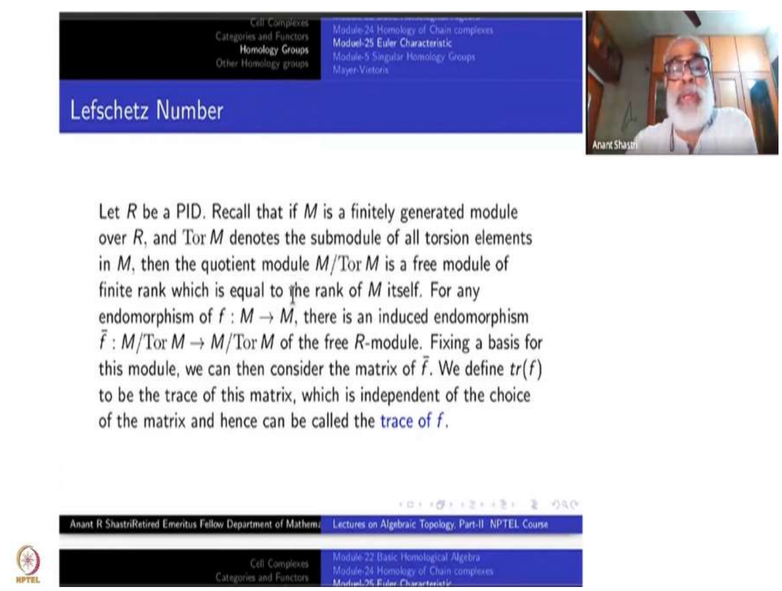

So, let us do one more slight variation of this one, if not all. This is called Lefschetz number. This will also be used later in this course okay. So, we fixed the ring R to be a PID, M is finitely generated module over R. Then tor M denotes the sub module of all torsion elements in M, m in M is called a torsion element if there is r in R such that $rm = 0$ and r not equal to zero.

It is just like finite elements of finite order r in an abelian group. The set of all torsion elements denoted by $\mathrm{Tor}\ M$ is a submodule of M and the quotient M by $\mathrm{Tor}\ M$ is a free module of finite rank. This is not a general result but it is true for PIDs anyway, okay?

The rank of this quotient is equal to the rank of M itself. You can take this as a definition. Any morphism f from M to N induces morphism $\bar{f}$ from M by $\mathrm{Tor}\ M$ to N by $\mathrm{Tor}\ N$ because the torsion elements always go to torsion elements under f.

You can fix a basis for a free module, since there is always a basis right? You fix a finite basis for M. Then an endomorphism $\bar{f}$ can be written in terms of a matrix. So, if the rank is r, then the matrix will be of type $r \times r$, okay? The trace of f is defined to be the trace of corresponding matrix for $\bar{f}$, okay? And one can verify that it is independent of a choice of the basis. This is elementary linear algebra okay? Trace of a matrix ABA^{-1} is the same thing as trace of B, because $tr(AB)$ equal to $tr(BA)$ that is what you have verified okay. So, same thing works here with trace will be independent of what basis you chose okay. So, we can define trace of endomorphism f itself alright now we are in business. So, now instead of M we are looking at endomorphisms of M. Instead of the module we are looking at endomorphisms.

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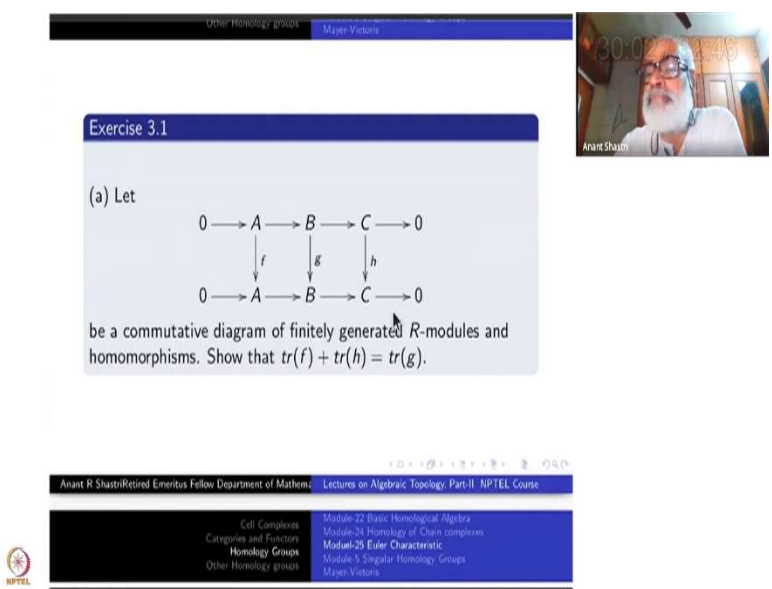

So, what we will do is a simple exercise here okay? Similar to what we have for exact sequences. Given a short exact sequence of finitely generated R-modules, as shown, the additivity of the rank function tells you that rank of B is equal to $rank(C) + rank(A)$, right? So, we generalize this here. Take a short exact sequence here and an endomorphism of the exact sequence okay? That measn a commutative diagram consisting of three vertical arrows. okay? The trace of this can be defined trace of this can be defined trace of this can be defined right? These are endomorphisms, okay?

The exercise says that the trace the central one g is equal to $trace(h) + trace(f)$ okay? So, first look out for the special case of the vector spaces. Then it is simple linear algebra. The general case is exactly the same because you are doing matrix theory okay. That is hint. You can write down the details.

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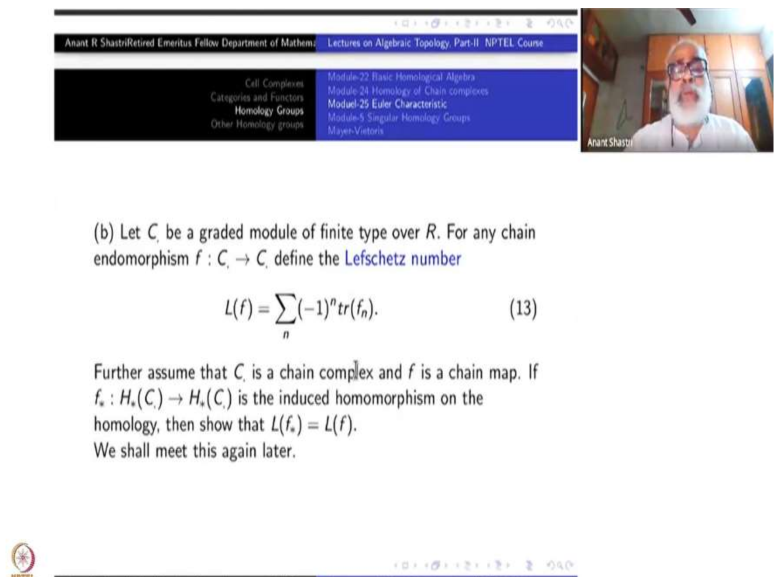

Next let us take $C.$ a graded module of finite type and f from $C.$ to $C.$ an endomorphism of degree 0, okay? Define the Lefschetz number $L(f)$ as the alternate sum of the traces of f_i okay? Note that $C.$ is any graded module, no structure of chain complex or exactness is used in this definition.

Now assume further that $C.$ is a chain complex also and f is a chain map of degree 0 and let f_* denote the induced graded module endomorphism on the homology $H(C)$. Then the claim is $L(f_*) = L(f)$. Exactly same as Euler characteristics of H_* and of C_*. So try out these exercises.

If you do not get, we are there to help. This is just straightforward exercise I have not done it for chain complexes.

This is not a difficult thing okay. So, thank you we will stop here now. Next time we are going to introduce the most serious part of this section, namely, the construction of singular homology groups. Thank you.