Introduction to Algebraic Topology (Part - II)
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Lecture - 24
Homology of Chain Complexes

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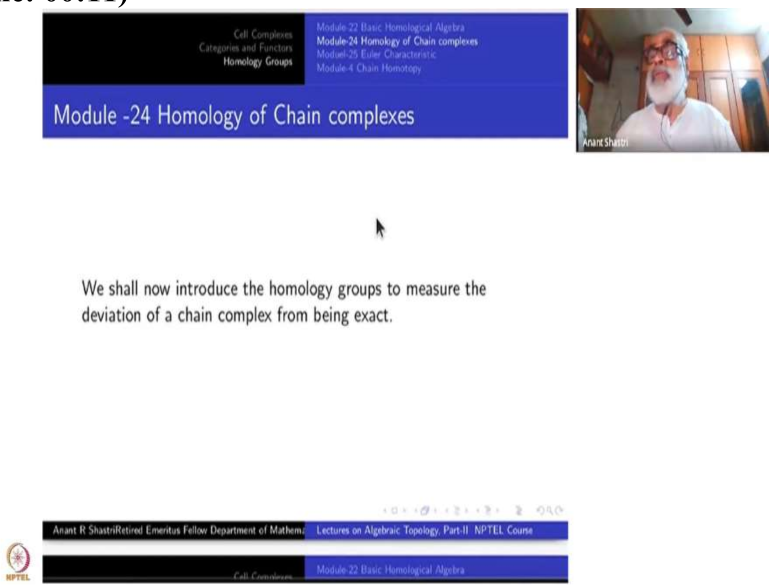

So, far we have introduced the various notions such as graded modules, chain complexes, short exact sequences, long exact sequences, diagram chasing technique etc. then using it, derived Four Lemma, Five Lemma 5 and then the big result, namely, the Snake Lemma, okay? This Snake Lemma is going to give you a good application very soon. We will do that.

So, guided by the experience in complex analysis or in differential equations and so on studying the differential forms and so on, we are led to measure the deviation of a chain complex from being exact. And that is precisely the role of these homology modules, or the homology groups okay? Associated with a chain complex. By the very definition a chain complex has a self operator ∂ of degree -1 such that ∂^2 is 0, which just means that the kernel of ∂ contains the image of ∂.

So, how large is the difference between the image and the kernel? That is what you are going to measure now. Being abelian subgroups or being submodules of one another, the quotient makes a sense as a module on its own. So, a nice way to measure this deviation is to take the quotient. That leads to the definition of homology groups as a graded module now okay?

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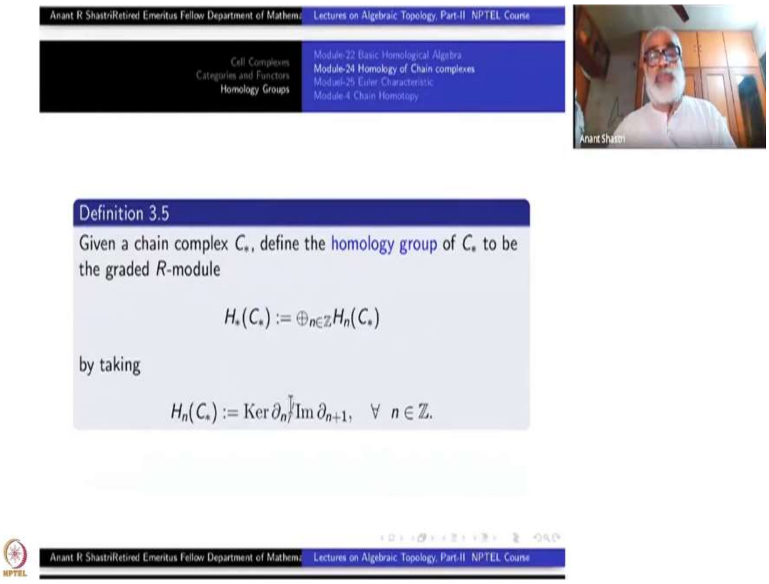

So, let us make a formal definition. Start with the chain complex C_* of R-modules. The homology groups of C_* is going to be a graded R-modules, okay, denoted by $H_*(C_*)$ which is the direct sum of $H_n(C_*)$, where each H_n is a module and I am taking direct over all integers. Recall that C_* itself is a direct sum C_n's right? So, what are these H_n's? I have to define. After that H_* is defined as a direct sum okay.

So, $H_n(C_*)$ is define as that quotient of kernel of ∂_n by the image of ∂_{n+1}. Remember ∂_{n+1} starts from C_{n+1} to C_n and ∂_n starts from C_n to C_{n-1}. So, kernel and image are both sub modulus of C_n okay. So, the image of ∂_{n+1} is contained inside the kernel of ∂_n. Therefore the quotient makes sense and this will be again an R-module okay? Define H_n for every n like this and take the direct sum, that is the entire homology group of C_*.

So, what you have done is that you have associated to each chain-complex a graded module. The beauty of this is that you can make it into a functor that is our next aim here.

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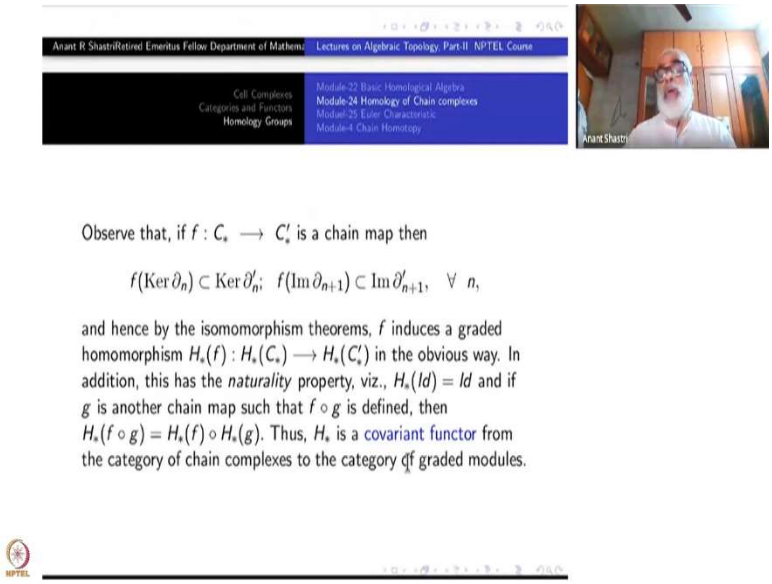

If you have a morphism from C_* to C'_* of chain complexes, (remember a morphism of chain complexes consists of a sequence of module homomorphisms f_n from C_n to C'_n such that they commute with the corresponding boundary operators, ∂ here and ∂' there, i.e., compatible with the structures okay. Therefore what happens is that a chain map has the property that kernel of the ∂_n will be contained inside the kernel of ∂'_n for each n. And image of ∂_{n+1} will be contained in image of ∂'_{n+1} under f. This is an easy consequence of the rule $\partial' \circ f = f \circ \partial$ okay? And hence by the isomorphism theorems of modules or just for abelian groups, we get an induced graded homomorphism which I will shall denote $H_*(f)$. It consists of a sequence of homomorphisms written as $H_n(f)$ from $H_n(C_*)$ to $H_n(C'_*)$.

So all the time, I am using a star to denote the graded components. Namely, f_* restricted to C_n is f_n which takes kernel of ∂_n to kernel of ∂'_n. and image of ∂_{n+1} into image of ∂'_{n+1} and hence induces the R-linear map $H_n(f)$ of the corresponding quotient modules.

So, what happens is that if g from C'_* to C_*" is another chain map then $H_*(g \circ f)$ will be $H_*(g) \circ H_*(f)$. Similarly, $H_*(Id_C)$ is equal to Identity of $H_*(C_*)$. Thus H_* will be a covariant functor from where? From the category of chain complexes to the category of graded modules. When you pass on to the homology, there is no chain complex, you have lost the ∂'s here, okay only graded module structure remains.

So, this functor is called the homology, homology associated to a chain complex. Only later on, we will bring in topological spaces.

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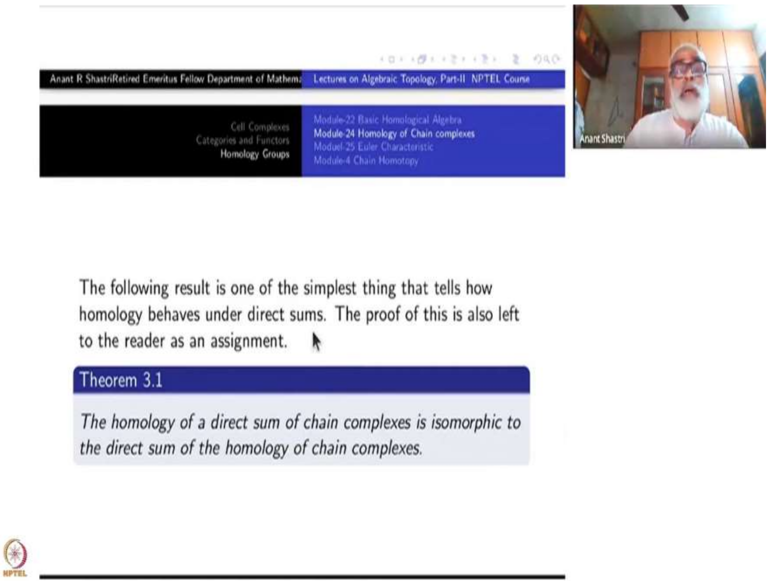

The simplest thing now to prove is the so called weak additivity of the homology functors, namely, homology commutes under direct sum. The homology of a direct sum of chain complexes is isomorphic to the direct sum of homology of chain complexes. It is easy to verify this one. A routine. First do it for direct sum of two chain complexes okay? But then you can generalize it. There is nothing is very special about it, the same argument will go through any number of components, finite or infinite, in the direct sum, okay.

But such verifications, we will leave it to you because these will give you practice with the homology groups, homomorphisms and abelian groups and so on okay.

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So, above property of homology, namely direct sum of chain complexes will give rise to direct sum of the corresponding homologies okay? This property, you can call it additivity of the homology okay. So, what we would like to is to go back to the exact sequences of modules and so on and of chain complexes like this.

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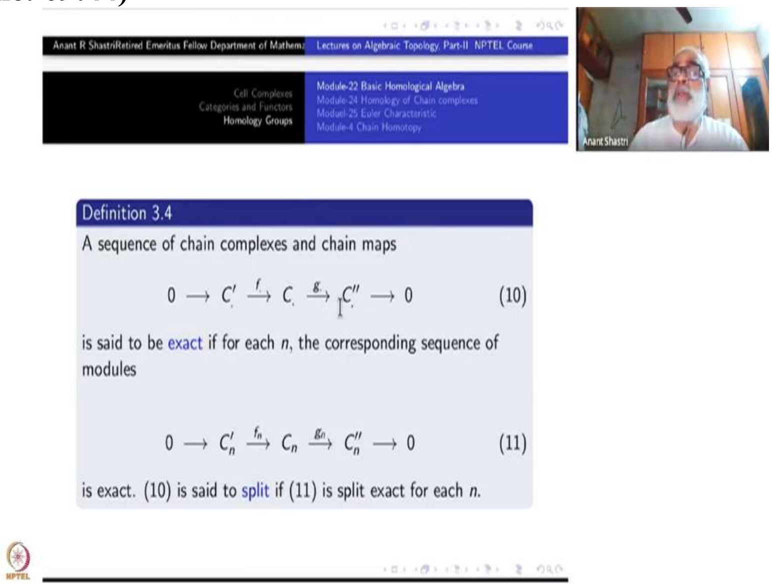

Let 0 to C' to C to C'' to 0 be a short exact sequence of chain complexes. The middle one may not be a direct sum of the other two. Question is how its homology related to the homology of the other two chain complexes? This is what we would like to know. You expect that the three homologies will be also be related by the same kind of short exact sequences like this. In other words I am asking specifically the following:

Suppose you start with a short exact sequence of chain complexes. Then do you get a short exact sequence 0 to $H_*(C')$ to $H_*(C)$ to $H_*(C")$ to 0?

Notice that if α and β are the corresponding chain maps, then we have $\beta \circ \alpha$ is 0 and hence it follows that $H_*(\beta) \circ H_*(\alpha)$ is also 0. But still the exactness at the chain level does not give the exactness at the homology level. Indeed this is precisely what we want to study now.

Starting with the observation that homology of a direct sum is isomorphic to the direct sum of the homologies, we ask same question for homology of a short exact sequence of chain complexes. Expecting that the homology is a direct sum is too much anyway. Even a short exact sequence of homology groups seems to out of question.

However, you relax a little bit and bring your expectations a little lower, then you get a beautiful positive answer. True statements are always beautiful. And that is given by an application of the snake lemma in this case, okay?

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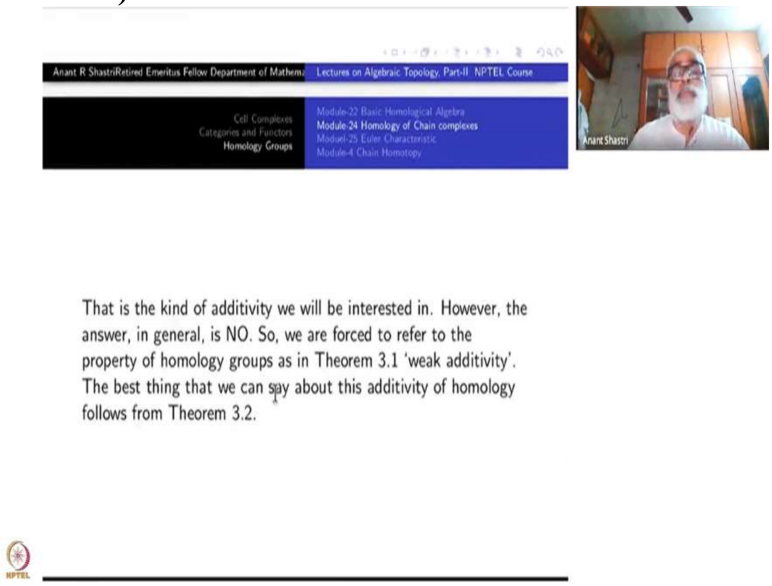

So, this is the kind of additivity we will be interested in. However the answer in general is in the negative. So, we are forced to refer to the additive property of homology groups as in theorem 3.1 okay, as weak additivity. The best thing that we can say about this additivity of homology follows from theorem 3.2 below which is an easy consequence of the snake lemma.

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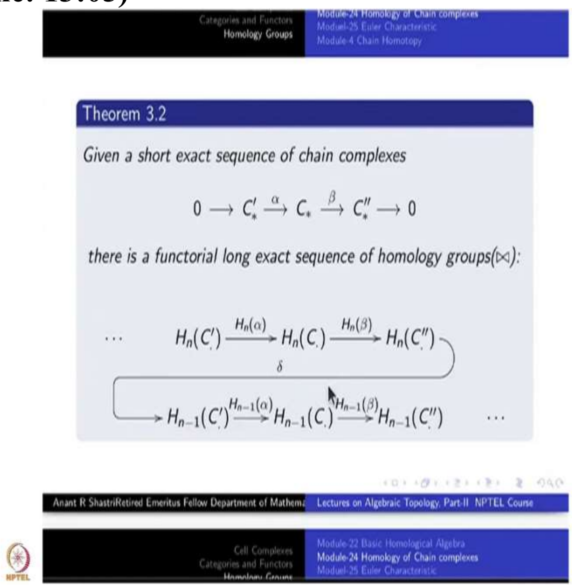

So, here is theorem 3.2 that we are going to present now. This is the best thing that we can show. So, what is the statement? Given the short exact sequence, now I am writing down the maps also here α and β. (Maybe earlier, I used the notations f and g) That is a short exact sequence of chain complexes. Remember this means that there is a whole lot of R-linear maps indexed over integers, for each n, α_n from C'_n to C_n, β_n from C_n to C_n" etc., and there is a commutative diagrams which mean compatibility with the boundary operators. Then there is a functorial long exact sequence of homology groups, and it is represented as follows: . . . indicating it starts from somewhere because this n is going from $-\infty$ to ∞ right? So, at n-th level what happens? $H_n(C')$, the corresponding $H_n(\alpha)$ landing into $H_n(C)$, then $H_n(\beta)$ landing into $H_n(C")$.

But from here where do you go? It is not the zero module here. This is the point okay? What you will get is $H_{n-1}(C')$, then two more terms again repeat but this time indexed by $n-1$, upto $H_{n-1}(C")$. Again the next module at the $n-2$ level and so on. So that is the meaning of this dot, dot.

So, this is the long exact sequence of homology groups in the statement. And then there is also this word `functorial'. Now it is easy to guess the meaning of the functorial in this context. You have already made a category whose objects are short exact sequences. Also there is the category of long exact sequences. This association is a covariant functor from one to the other.

All this is a consequence of the snake lemma. Now for each n, you have a snake appearing here. You see this is going to be our connecting homomorphism δ and the corresponding to that we are having the δ there also. We are not going to introduce too many notations. such as δ_n etc. So, δ was there to remind you that this is also something to do with that, we are putting the same δ here, So, just to keep reminding about its origin in the snake lemma. Let us see clearly what we got here.

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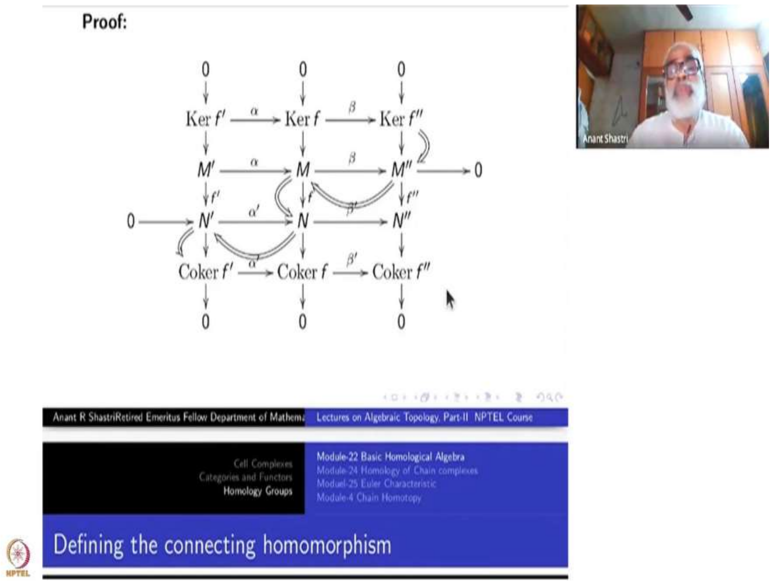

The short exact sequence of chain complexes easily gives rise to a sequence of snakes by taking $M = C_n$ and $N = C_{n-1}$ etc for each n. We can then pass on the kernels and cokernels etc, to get an associated 6-term exact sequence as given in Snake lemma. If we piece them together we get the homology long exact sequence. The functoriality of this association is also built is the snake lemma itself.

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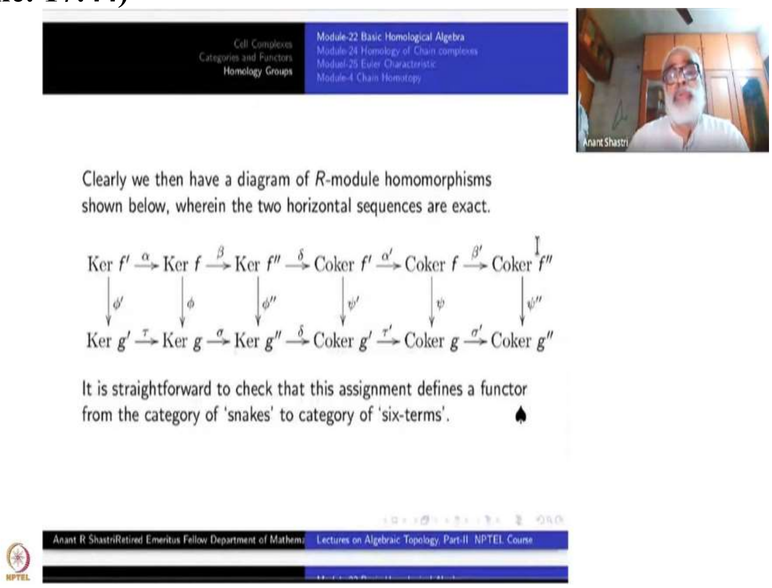

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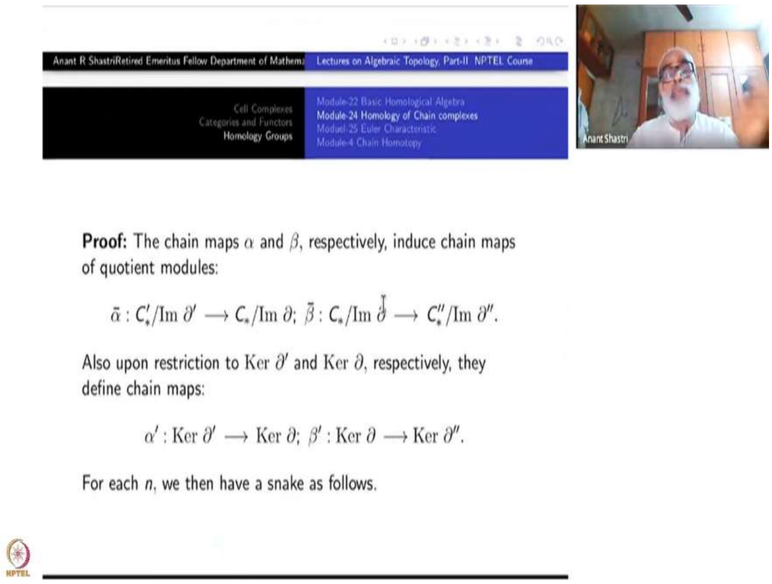

But you have to guess how to get these homology terms out of this kernel and co-kernel. That turns out to be an elementary module theory (or elementary abelian group theory), okay? Instead of directly taking C_n's, I am going to take certain quotients and submoudles. The chain homomorphisms α and β in the original short exact sequence induce chain maps of submodules and quotient modules. I am not directly going to the homologies here okay.

So, for each n, this is my snake as show in this diagram. The terms in the first row are the co-kernel of ∂', ∂ and ∂" respectively, viz., the first term is for example, the same as C_* by the image of ∂. Similarly the morphisms are induced by α and β respectively. The terms in the second row are kernels of ∂', ∂ and ∂", but one index lower. And the morphisms are restrictions of α and β.

I have written α' and β' for these restrictions. We then have a snake for each n as follows. Thus out of one one single short exact sequence, I am getting these snakes one for each n. Okay? From the snake lemma, I will get a 6-term exact sequence. Putting all the 6-term sequences together, I will get the long homology exact sequence. So these are the two steps in the proof.

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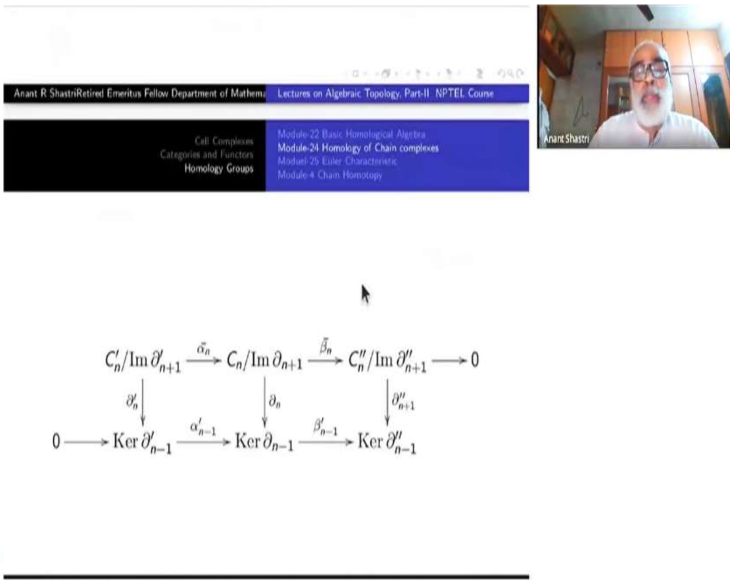

Remember that a snake has a 0 term at the end of the top row and a 0 term at the beginning of the bottom row. Temporarily, you can name them $M', M, M"$ and $N', N, N"$ and apply the snake lemma to that take the kernels, and co-kernels to pass on to the six term exact sequence. Finally, you have to see what are those 6 groups. They happen to be the 6 terms which I am interested in, namely these 6 terms and they will be connected by this δ okay? So the snake always gives you the 6 term exact sequence right.

So, notice that first vertical arrow is the morphism induced by the original ∂' from C'_n to C_{n-1} . Since it takes the submodule image of ∂'_{n+1} to 0, we get a well defined morphism from quotient of C'_n with Image of ∂'_{n+1}. Also, since it takes its value inside kernel of ∂'_{n-1}, we can replace the codomain by $ker(\partial'_{n-1})$. Similar explanation (ii) valid for the second and third arrows also.

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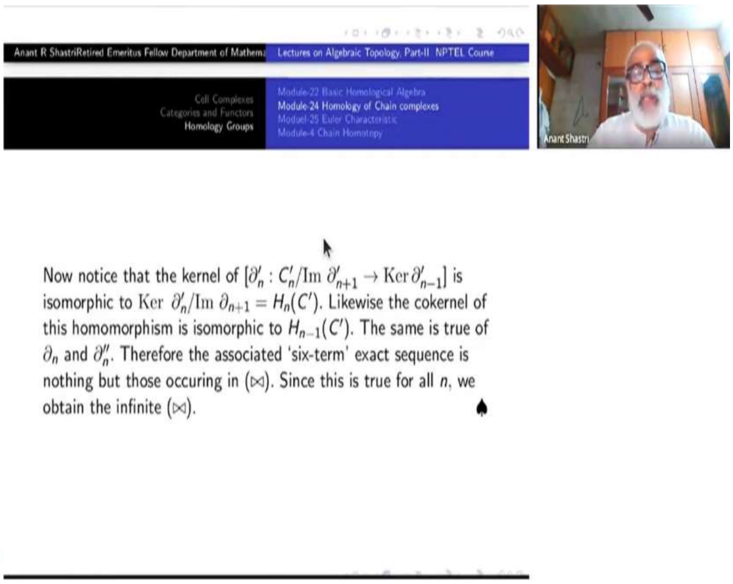

By the second isomorphism theorem it easily follows that the kernel of this first arrow is nothing but $H_n(C')$. Similarly, the kernels of second and third arrows are nothing but $H_n(C)$ and $H_n(C")$. Exactly similarly, we check that the cokernels of these three arrows are nothing but $H_{n-1}(C'), H_{n-1}(C)$ and $H_{n-1}(C")$, respectively. Thus all the six terms are as we wanted. That completes the proof of the theorem. Thus the hard part of this theorem is taken care by the snake lemma.

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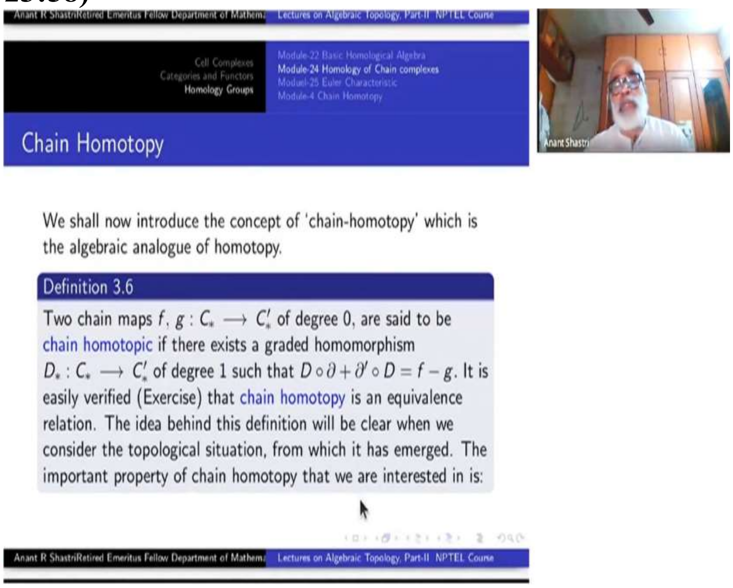

Let us now introduce an important concept here which is guided by the homotopy in the topological spaces. But you may wonder how one got this. To understand this one you have to wait a bit. (You will understand it if you have done something like Poincare lemma in your differential topology okay?) So these things were cooked up from experience with proving Poincare lemma for example, which tells you what to do using integration. Here we are converting everything purely algebraic way. So, if you just see this without knowing all these background, you may lose the motivation. Why one does such a thing? So you have to just mug it up till you see a little more and then you will start seeing more, and more, okay?

Given two morphisms f, g of degree 0 between two chain complexes you say they are chain homotopic to each other if there exists a graded homomorphism D_* from C_* to C'_* of degree 1, these are degrees 0 maps this degree 1 map such that $D \circ \partial + \partial' \circ D$ is f$-g$ okay.

It is easily verified (when I say easily verified there is work for you to do okay?) the map D is a called a chain homotopy. If there is such a D then f and g are siad to be chain homotopic okay? So, this relation chain homotopic is an equivalence relation. The important thing to see is that if f is homotopic to g, and g homotopic to h implies f is homotopic to h, the transitivity, okay. So, this is not at all difficult. You have to just take the sum of the two chain homotopies.

The idea behind this definition will be clear when we consider the topological situation from which it has emerged. So, that is why you had bring in this important concept of chain homotopy, as illustrated by the following consequence in homology.

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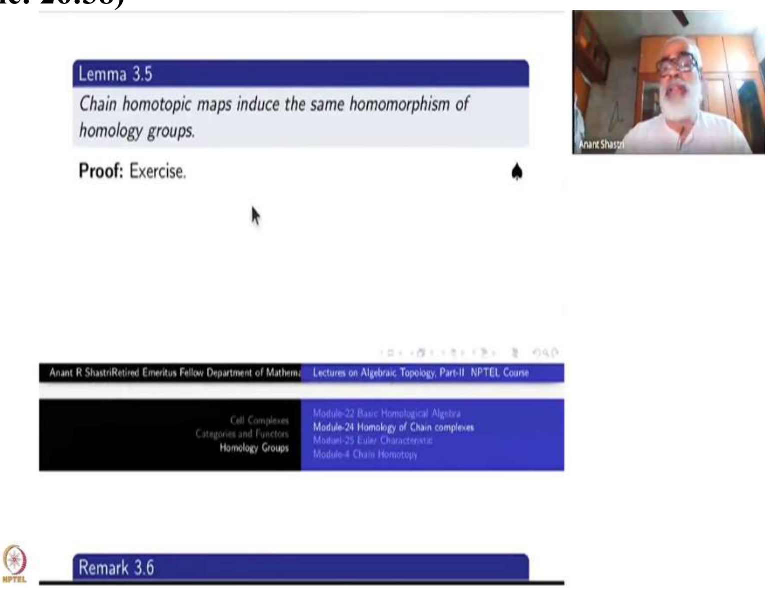

Chain homotopic maps induce the same homomorphism of homology groups. Once you have said this, verification is totally easy okay? So, you have two morphisms f and g you have to take the n-th level groups and see what happens to $H_n(f)$ on an element in $H_n(C_*)$. An element here is represented by an element c in the kernel of ∂_n modulo the image of ∂_{n+1}. So, take $f(c)$ okay? modulo the image of ∂'_{n+1} in C'_* right?

You have to show that $g(c)$ is also the same that is what you have to show okay? So, if you look at this one $f - g$ for any cycle (c) here, (cycle means an element c in the kernel of ∂, okay?) Therefore on the left hand side the first term is zero and the second term is clearly in the image of ∂'. Therefore, $f(c) - g(c)$ represents the same element in $H_n(C'_*)$. So, I have proved it for you anyway. So you write down the details now that is all so exercise has been taken away already just write down. Let us stop here we will pick it up from here next time.