Introduction to Algebraic Topology (Part – II)
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Lecture – 23
Diagram - Chasing

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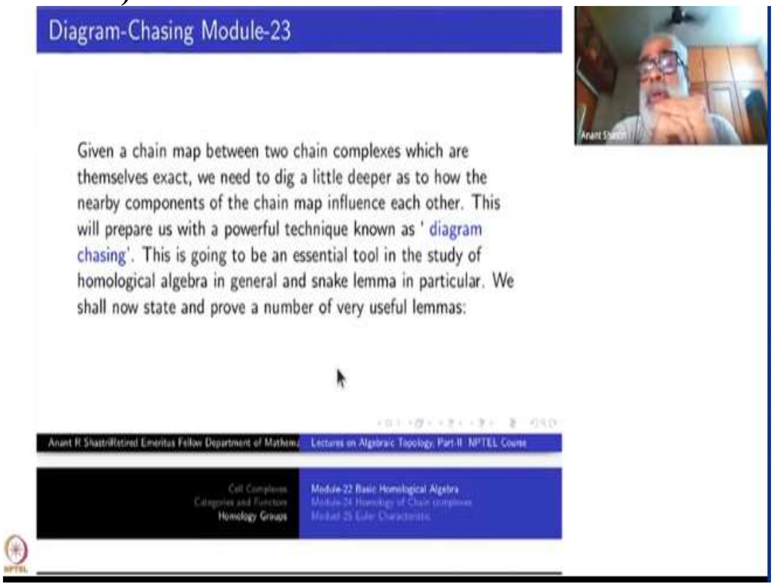

So, continuing the study of chain complexes, given a chain map between two chain complexes which are themselves exact we need to dig deeper as to know how the nearby components are related okay? How will the chain maps influence the nearby graded components? This will prepare us with a powerful technique known as diagram-chasing. What this means? We will see when we actually study this and it is going to be an essential tool in what is known as homological algebra okay?

An important result there, namely, the snake lemma, okay? Before stating with the snake lemma, I will give you a number of simple shorter lemmas which are all proved by this technique, namely, diagram-chasing.

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Consider the following commutative diagram of R-modules and R-linear maps in which the two rows are exact. If f_1 and f_3 are isomorphisms, so is f_2. This is the statement.

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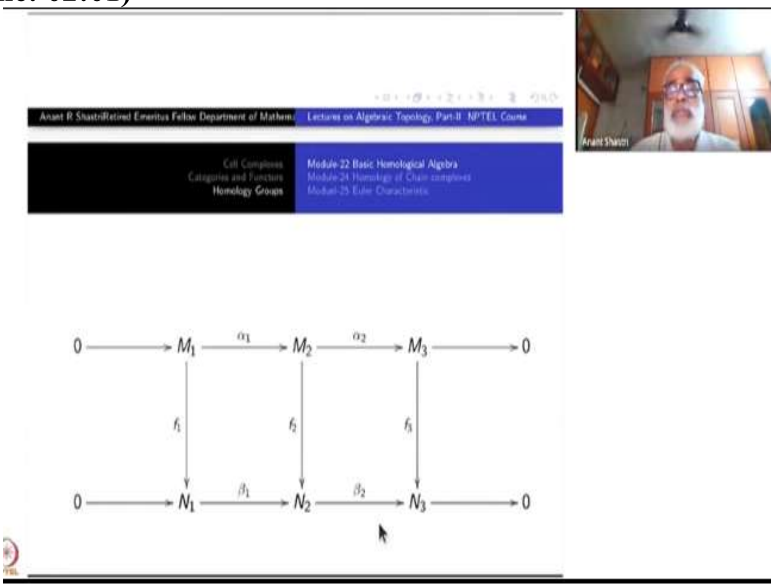

What is the picture? Picture is this one: 0 to M_1 to M_2 to M_3 is one shorter exact sequence 0 to N_1 to N_2 to N_3 is the other shorter exact sequence, okay? Column wise, we have three morphisms here f_1, f_2, and f_3. Of course, I do not have write these two end morphisms, they are 0-maps. The whole idea is that the entire diagram is commutative, which means $f_3 \circ \alpha_2$ is the same thing as $\beta_2 \circ f_2$, similarly, $f_2 \circ \alpha_1 = \beta_1 \circ f_1$. So this is the data. Okay? Now, what is the conclusion? Suppose that f_1 and f_3 are isomorphisms then f_2 is an isomorphism, Okay? So, this is one way, one way conclusion not an `iff' type statement. So, since every map is already a homomorphism of modules, all that you have to prove is that f_2 is bijection assuming that f_1 and f_3 are bijections okay?

So, let us prove this by pointwise, namely, pick up a point in N_2 and show that there is a point in M_2 which is mapped onto it by f_2. That's all okay. Somehow I have to use these two facts that f_1 and f_2 are bijective. Start with an element here, there is direct map to go here right? But you can go here under β_2. So pick up the element and look at β_2 of that element.

Now, you see, something nice is happening. This is surjective here, therefore I can pick up an element here, Okay? Which maps onto that one, right? So, I could pick up an element here itself, but after going here, I can come up okay? So, there is an element here, such that f_3 of that is equal to β_2. of this n_2. Now this α_2 is surjective, so, there is an element m_2 here which goes to this element under α_2. So, what we have got is an element here if you come all the way here it is the same thing here as if you come this way same thing as this element just means that if you come this way also it is same element so, I have an element m_2 here such that $f_2(m_2)$ and n_2 are both going to same element under β_2. So, now I am in a good shape. Now I can use the fact that this is an exact sequence of modules. elements are going to same element under β_2 just means that their difference is going to 0 under β_2.

So that is in the kernel of β_2, okay? But the kernel of β_2 = image of β_1 right? Therefore, the difference element $f_2(m_2) - n_2$ is equal $\beta_1(n_1)$, some element here. That is anyway a unique element because this is an injective map. So, that is some element here that is all, we need. Now this f_1 is surjective so, I can pick up an element m_1 such that $f_1(m_1) = n_1$. Now, I can complete the argument nicely. If you look at the image of that here okay.

So, what does that element does under f_2? It will come to the same element here namely, we have $f_2(\alpha_1(m_1))$ is a difference of these two okay. So you add that element to the m_2. That will come to n_2 under f_2. So modify this m_2 by the image of m_1 under α_1 okay that will map, under f_2 to the element n_2.

That f_2 is surjective you see starting with here we went there and then observe that something is happening and then we went this way and modify this element so this kind of technique is called diagram-chasing.

So, you can just glare at it to get the entire proof mentally. Writing down the proof without the diagram, all these homomorphisms, images etc is a bit more difficult in any case. you have to have this picture in your mind if not the picture on the paper. Better to draw the diagram and then everything will be clear okay? That is the point of this diagram chasing. The same technique can be used with slight modification in proving a lot of statements of this type.

The next one. Why f_2 is injective? So what should I do? I take an element m_2 such that f_2 of that element is 0. Okay? I want to show that m_2 itself is 0 right? That is the injectivity. So, come here it will be 0 here also. Because this diagram is commutative, which means $f_3(\alpha_2(m_2)) = \beta_2(f_2(m_2)) = \beta_2(0) = 0$. But f_3 is injective. Therefore $\alpha_2(m_2) = 0$. So the first thing is that I wanted to prove m_2 is 0, at least I have proved $\alpha_2(m_2) = 0$, okay?

Now use the exactness of the top row, $\alpha_2(m_2) = 0$ is 0 means it is in the kernel of α_2 which is equal to image of α_1, so it comes from an element here. So this m_2 is α_1 of some m_1 here. Okay? But if you go this way and come here it is same thing as $f_2(\alpha_1(m_1)) = f_2(m_2) = 0$. Some $\beta_1(f_1(m_1)) = 0$. Both β_1 and f_1 are injective and hence $m_1 = 0$. Therefore $m_2 = \alpha_1(m_1) = 0$.

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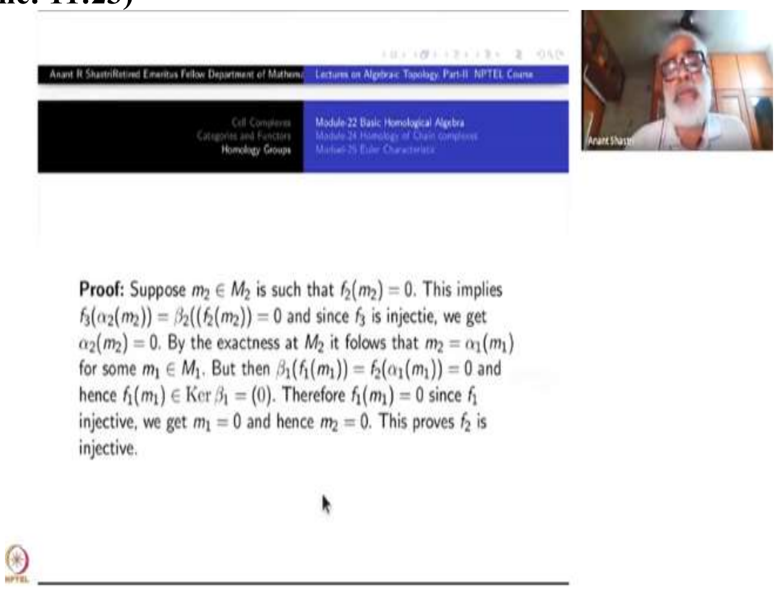

So, first you have seen the surjectivity, which might have taken a little more time. You see the proof of injectivity take less time. Both the cases again and again same technique is used. We shall now propose several such statements.

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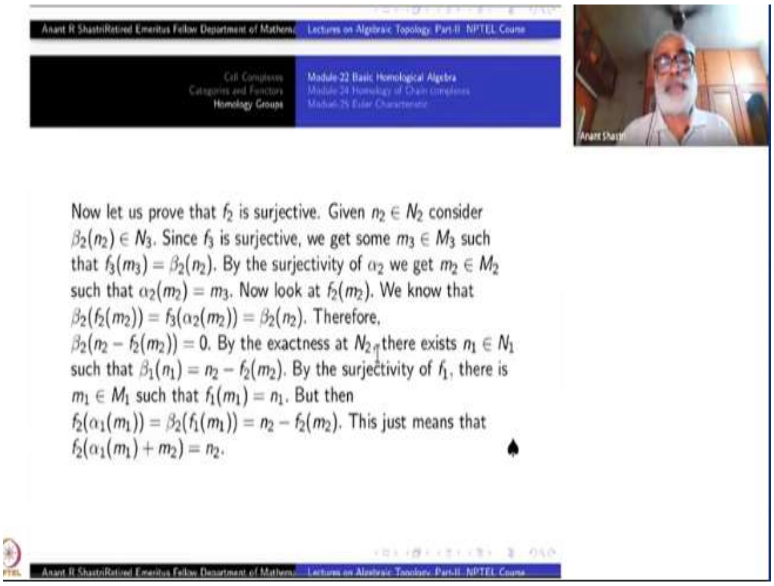

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So, let us leave the next two lemmas as exercises/ assignments to you to work out, but let me state these results, very very important results, called four lemma and five lemma okay? So, in the lemma 3.1 we had two sequences of five terms each out of which two of them were zero. Now, we handling the cases when four or five vertical arrows are involved and none of them is assumed to be zero. terms all.

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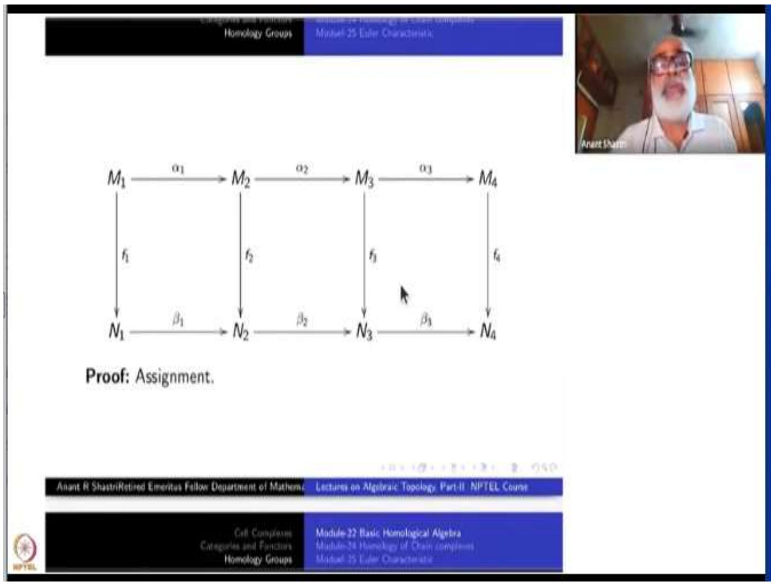

So, I will just show the diagrams, because it is easy to explain with the pictures. In the first lemma 3.2, there are 4 terms in each row, now Okay? Maybe nonzero maybe 0 and so on, there is no assumption that the end terms being 0 and so on. And the two rows are exact. The whole diagram is commutative. These are all R-module homomorphisms. Okay?

The new assumptions here are that f_1 is surjective and f_4 is injective. These are the assumptions on the end arrows. Now, there are statements:

(i) if f_2 is injective then f_3 is injective.

(ii) if f_3 is surjective that f_2 is surjective.

Okay, so you may get confused about these two statements, which one is what and so on. To remedy that best thing is that you write down the proofs. Practice a little bit Okay? Method of proof is exactly similar to what you have done in the lemma 3.1. But unless you try it yourself, you would not learn. Okay?

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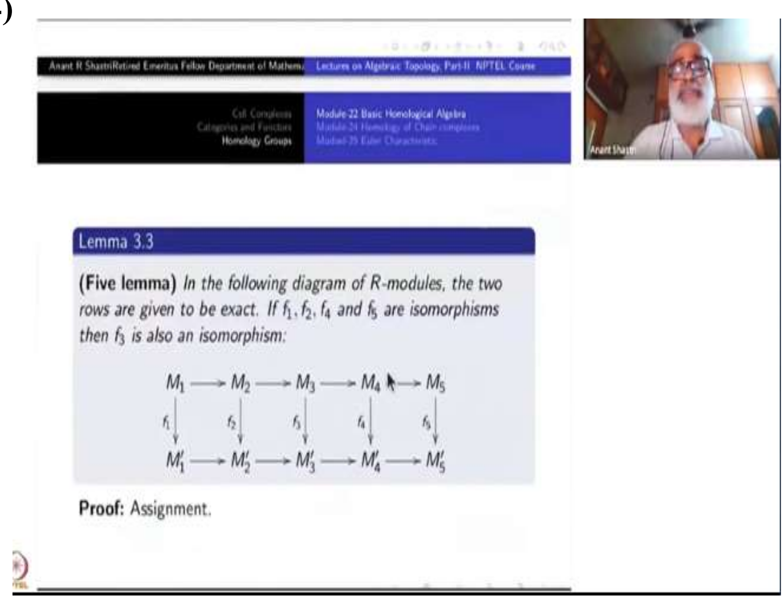

The five lemma is easier. It is a very powerful and result that is given in a ready to use form. Here there are two exact sequences of 5 terms and the terms are arbitrary, none of them assumed to be zero, okay. And the whole this is a commutative diagram of R-modules and homomorphisms. The statement is that if f_1, f_2, f_4 and f_5 are all isomorphisms then f_3 is an isomorphism.

So, this is a direct generalization okay? One step generalization of the lemma 3.1 that we had in which the end terms were 0. So, automatically this f_1 and f_5 automatically isomorphisms. In this five lemma, I have made the terms arbitrary and put the condition on the morphisms to be isomorphisms, okay. You can proof this directly by diagram-chasing or applying the four lemma twice.

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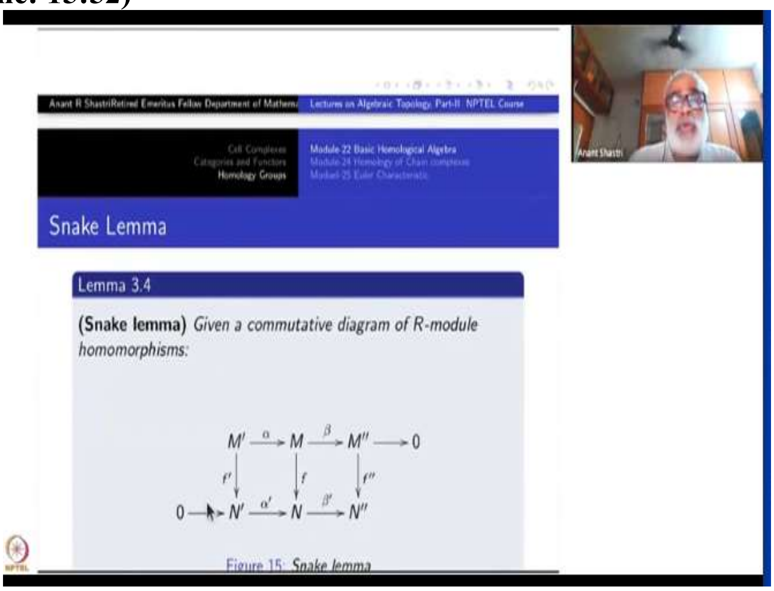

Now, I come to the central result of this section, namely, what is called a snake lemma which is going to be a strong input from homological algebra. So, this time you have two four-term exact sequences (rows) top and bottom slight different in shape. top on sending in 0 and bottom, the other way around starting at 0.

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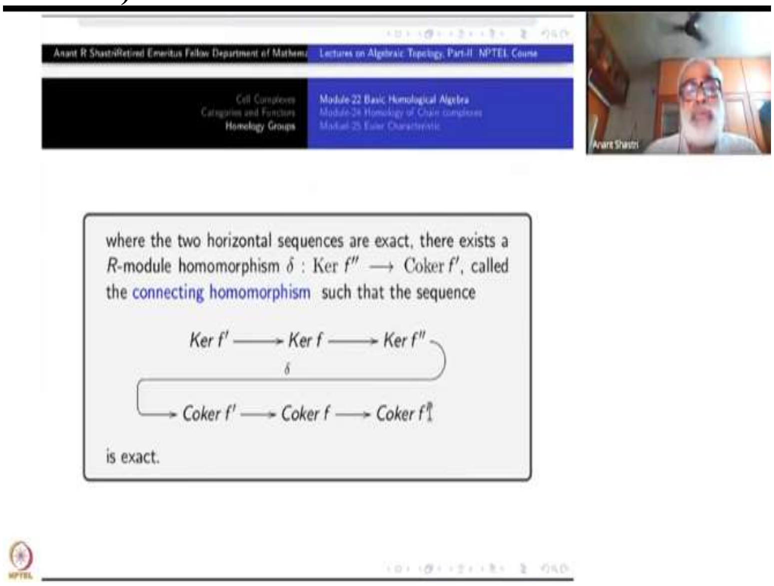

Given a commutative diagram of R-modules homomorphisms as shown in the diagram, there exists a R-module homomorphism δ, from the kernel of f'' here which is a submodule of M'' to the cokernel of this map f', (Remember kernel is just set of all points wherein f'' is 0 but the cokernel is the quotient module N' divided by the image of f' by definition). This δ is called the connecting homomorphism.

(What does it connect that is important thing here?) Look at the six modules obtained by taking kernel and cokernel of f', f and f''. Because this diagram is commutative it follows that α and β restrict to define morphisms from kernel of f' to kernel of f to kernel of f'', which we shall denote by α and β only. Similarly, α' and β' induce morphisms from corkerel of f' to cokernel of f to cokernel of f'', which we denote by α' and β' respectively. You may use you elementary knowledge of first and second isomorphism theorems for abelian groups (as well as for modules). So, there are morphisms like that okay. So, these morphisms are not shown in this picture because they are obvious morphisms they are there α, β. You can write here, and you can write here $\bar{\alpha'}$,$\bar{\beta'}$ if you want to be more careful.

So, the point is that you have these two obvious sequences of three terms each being exact and then all of a sudden there is the morphism δ from kenel of f'' to cokernel of f' which connects them to give a 6-term exact sequence.

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Moreover this assignment from the given data to δ, the connecting homomorphisms has a naturality property? What is the meaning of that? Of course, we know how to express naturality property. So, you must make a category and another category and the assignment must be a functor. So, what are the these categories here? I will explain that later.

First let us understand how such fantastic morphism arise, and then various exactness parts, namely, kernel of δ equals image of β and image of δ equals kernel of α'. Exactness at other two places are easy okay, because they are true in more general situation. The new and mysterious things are about the morphism δ. So, this is a fantastic theorem. Even the definition of this is fantastic, okay? But having seen the diagram chasing as a technique in the previous lemmas, the surprise will be a bit less. Now we will see how it is done okay?

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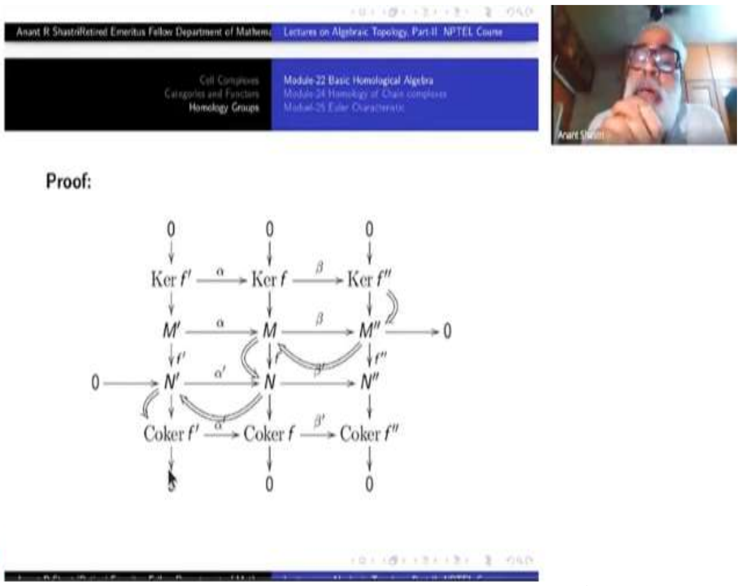

So first take a look at this diagram. Come to the central rows: M' to M to M''. N' to N to N'', Okay? For each morphism f', f and f'', I am taking kernel and cokernel to get a 6-term exact sequences vertically, the two end modules being zero. What is elementary is that now all the rows and columns are exact and the entire diagram is commutative.

The statement of the theorem is the existence of this delta with certain exactness properties, connecting the first row with the fourth producing a 6-term exact sequence. Okay? So, here you have to do some kind of diagram chasing but this time very systematically, go down, go left, go down, go left, go down, or down-back-down-back-down. Okay, So these steps have been indicated by double arrows.

So, I have defined this map delta right from here to here. Take an element in kernel of f” and go down via the inclusion map to M”. Since β is surjective, therefore, I can pick up an element in M which is mapped onto that by β. So here, I have to pick up something, I do not have a homomorphisms here that is why I have double arrows here. The single arrows are homomorphisms, they are R-linear maps.

So I have to pick up some element here, which goes to that one Okay? Now, push it down here come down to N, via f. If you look at the image of that under β', it is the image of this under f” but that is 0 because it is coming from kernel of f”. Therefore, it is in the image of α', so pick up an element in N' which is mapped onto that under α'.

This time there is no ambiguity because this α' is injective. (So I get a unique element here. But ambiguity is here already anyway.) Finally to push it down under the quotient map to get an element of the cokernel of f'. Here also there is no ambiguity.

In any case, in arriving at this last element you do not know whether you will get the same element each time. If somebody else follows the same steps does he get the same element? That is the first surprise. The answer is YES. You will get the same element no matter who picks up what element at various stages. Note that the ambiguity occurs only at the module M. From here to here there is no ambiguity.

More precisely, we must look at the difference between two pick-ups, right? I mean one person picks up one element and another picks up another element or you may have picked up 2 different elements at 2 different times, the difference would be such that under the map beta it would go to zero. Therefore, the difference itself comes from M', it is equal to α of some m' in M'. Now, look at its image under f in N. That is equal to $\alpha'(f'(m'))$. It follows that if you look at the two elements that you have got in N' corresponding to the two different pick-ups, their difference is in the image of f'. Therefore, when you pass on to the cokernel of f', they represent the same element.

So, whatever element you get here, there is a unique element. So call that δ of this element. So function δ is well defined, okay? So, you see with the picture, it is easy to explain than writing down symbols for so many elements etc., But you have to write down all this at least once, okay.

Now, the well definiteness implies a certain freedom in the choice of elements you pick up at the module M. That can now be used to show that this δ is a homomorphism. So, to show that this is a R-linear map, start with an element here, multiply by a scalar pick up the original thing here and multiply by a scalar, that will go to the element multiply by the scalar. That is at the module M. The rest of the steps are already linear maps. That proves easily that $\delta(\lambda x) = \lambda\delta(x)$.

Let us look at the sum. Suppose you have x_1 and x_2, two elements here in the kernel of f'. You come here okay, to M'' and pick up say m_1 and m_2 going to x_1 and x_2 respectively under β. What will pick up for the sum? Just pick up the sum $m_1 + m_2$. Under β, it will be mapped to $x_1 + x_2$. After this everything is linear and so it follows that $\delta(x_1 + x_2) = \delta(x_1) + \delta(x_2)$. So, δ is a homomorphism of R-modules alright.

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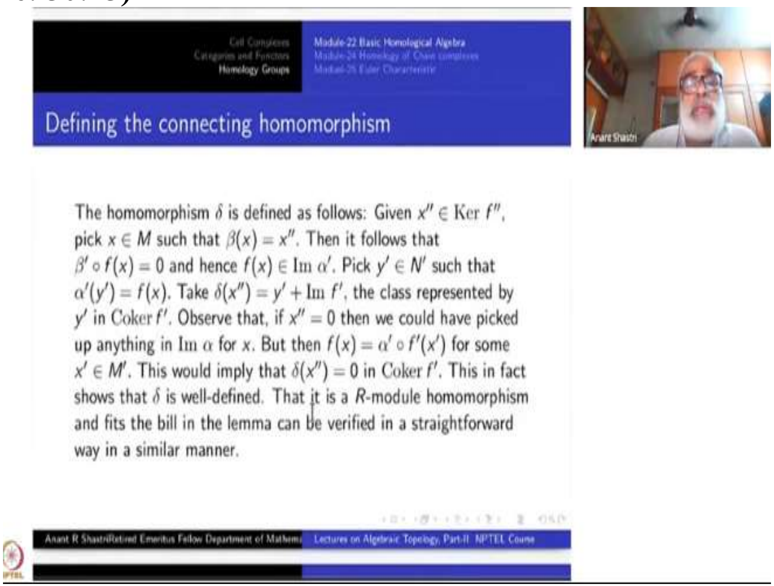

So, verification of exactness of the 6 terms sequence has to be done now. Let us go back to this picture. The exactness at Kernel of F and cokernel of f are general consequences of the exactness at M and M'. So, we need to check only that kernel of δ equal image of β and kernel of $\bar{\alpha}'$ is equal to image of δ. The two proofs are identical. So, I will prove only one of them whatever you want Okay? They are all similar. So, let us say take the first one which is happening in the submodule kernel of f".

So, look at an element here in the kernel of δ. In the definition of δ, when you come here to N' and take its image under the quotient map to cokernel of f', what is the meaning of saying that element in the cokernel is 0? That it belongs to image of f', okay. So, you get an element in N whose image and α goes to the same element as the one that was you have picked up under f. Therefore the difference of these to is the kernel of f. Now check that β of that element is equal to the original element you started with. That proves kernel of δ is contained in image of β. So the argument is again diagram chasing.

Conversely, if you take an element x here in kernel of f and take β of that. In the definition of $\delta(\beta(x))$, we are free to pick up x itself and then $f(x) = 0$ and hence we are now forced to pick up 0 in N' and hence it follows that $\delta(\beta(x))$ is 0.

So, this completes the proof of exactness at kernel of f”. Similar is the proof of the exactness at cokernel f'. Okay? So, that is exactly similar again and left to you as an exercise, Some practice.

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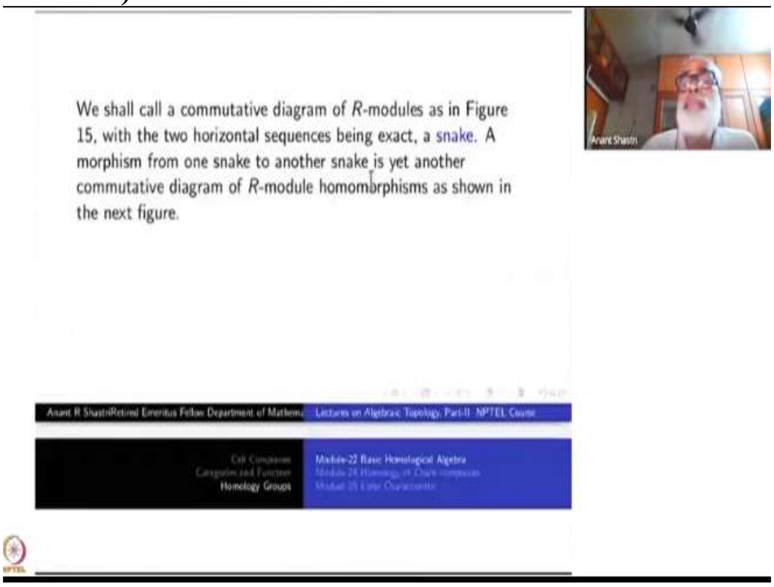

Now, there is one more thing I have to explain, namely, what is the meaning of the naturality of this map? Okay, So look at this, this picture which I call the data, okay? I want to make a category out of it. Every object of that category looks like this, where M, M' etc R-modules morphisms are also in R-mod. Okay? Let us name such an object, call it a `snake'. Take the collection of all snakes in R-mod. They form the objects of this category. So I am going to define a category which I call snake. (You may denote it by $\mathcal{S}$).

What are the morphisms here? By now you know how to do all that. You take another picture like this, as I have done here, take a morphism from each of the six terms in the first object to the corresponding term in the second object, so that the whole lot of diagram is commutative. That forms one single morphism in this category snake.

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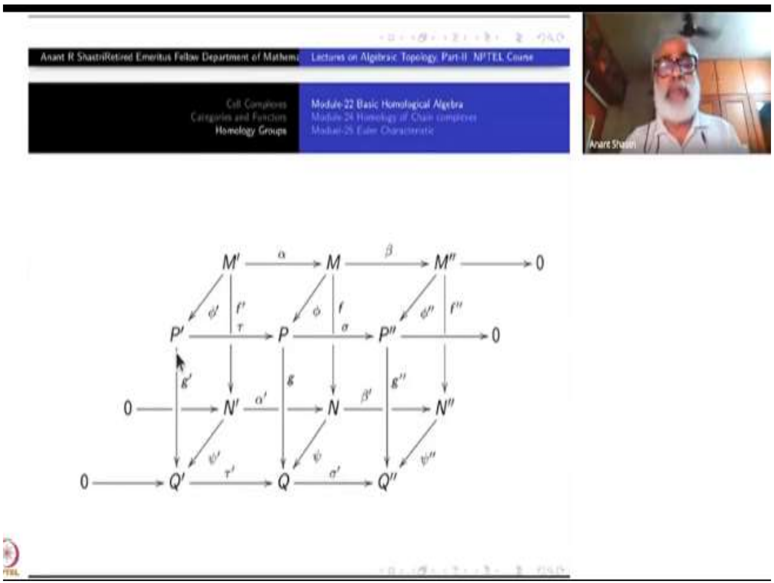

So, here $M', M, M"$, $N', N, N"$ form the first object, the other object is formed by P', P, P'', Q', Q, Q''. From one object to another object, a morphism will be set of six morphisms in R-mod like this. Okay, $\phi', \phi, \phi'', \psi', \psi, \psi''$, they together make one single morphism in the category of snakes, provided the entire diagram is commutative.

The important thing is that all these squares, these squares, these squares must be commutative. Okay, The entire box here okay, there are two boxes here, they are all commutative. So, for any starting point to some end point if there are two different ways, then the result of compositions along the two different ways must be the same. That is the meaning of commutativity of the entire diagram. So check it yourself that this forms a category. Okay?

We now form another category for the codomain. This will be called the six-term-sequence category. The objects are six term exact sequences in R-mod and a morphism is a bunch of six morphisms in R-mod which form a ladder as shown in the picture.

Now the claim is that assignment to each snake, the six term exact sequence given by the snake lemma be comes a covariant functor. Given a morphism in the category of snakes, you have a bunch of morphism in R-mod, they induce the corresponding morphisms of R-mod which form a single morphism in the category Six-term-sequence.

It is elementary module theory that and does not need any extra proof except wherever the connecting morphism δ is involved, viz., the commutativity of the central square out of the five squares.

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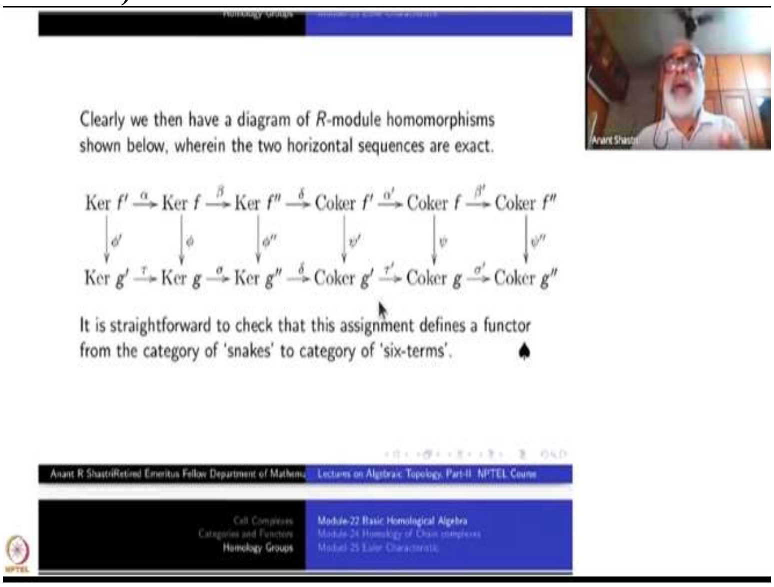

So this is a picture, we have used the same notation δ to denote the connecting homomorphisms corresponding to two different snakes, which is indeed justifiable only after the verification of the functoriality of this assignment.

But the point is that in the definition of delta there is one choice involved and hence we need to verify the functoriality carefully. In this situation, as observed above, we need to prove that ψ" $\circ\, \delta = \delta \circ \phi$" from kernel of f'' to cokernel of g'. The hint is to use the freedom involved in the choice of the definition of δ on the RHS, viz, whatever you have picked up in M in the definition of the first δ, take its image under ϕ to be the element in P. With this hit, we leave the rest of the details to you to figure out.

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So, the student is advised to go through the definition of connecting homomorphism carefully and memorize it. Why? Because despite the so much of developments in theory, while dealing with the connecting homomorphisms, quoting all those theorems and lemmas may not help in certain situations, often determining the image of a particular element under δ. I have met with such a situation while doing my own research work Okay? So my I am telling you that sometimes you have to actually see the definition of a certain thing while comparing it with another formula, so better to know how this is defined. So it is better to get things clarified completely at this stage. Okay? Thank you.