Introduction to Algebraic Topology (Part –II)
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Lecture – 22
Basic Homological Algebra

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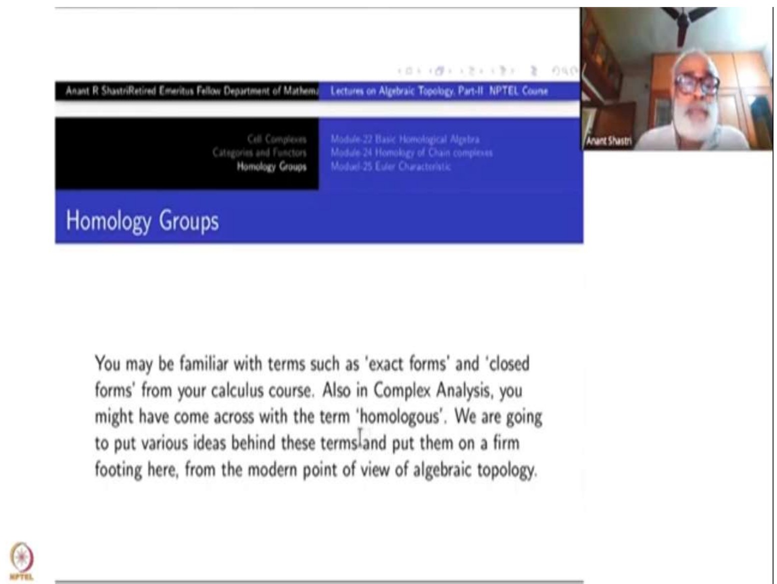

So, we begin with a new chapter, homology groups. This is one single chapter, but for the reasons of proper presentation in the slides, in this Beamer presentation, we will divide it into two parts. First part is mostly algebra and definitions and basic property of singular homology. In the second part we will compare singular homology with other homology theories, discuss some assorted topics, applications and such things for homology.

Like many other branches of modern mathematics, homology and cohomology theories have their roots in analysis, more precisely, in complex analysis of 1-variable. When you started integration theory of line integrals of complex valued functions defined on parameterized piecewise differentiable curves, you realize that the integration does not really require continuity of this function all over the curve. Just piecewise continuity is good enough, but differentiability of the curve is necessary there.

That idea of finitely many discontinuities gives rise to what are called later on, by the name `chains'. Inside a certain domain in ℂ, if you look at all the chains, which share the property that integration of any given complex analytic function takes the same value, those chains are defined to be homologous to each other. That word `homologous' used over there, which is happening inside a domain in ℂ for holomorphic functions, gives rise to the general notion of homology.

So, the topological aspect of the line integrals is brought out by this concept called homology. The integration itself gives rise to cohomology theory. Indeed, integration of the forms, first of all integration of functions functions on curves as integrations of 1-forms, then intergration of 2-forms on surfaces, aand then 3-forms on domains such as cubes in 3-dimensional spaces and so on, that gave rise to what is called cohomology. The very first of this kind is De Rham's cohomology.

Now, for us, we would like to do a more elaborate treatment of these things. So, it is better to separate out cohomology from homology in the beginning, first study homology and cohomology later. So, that's what we're going to do. Unfortunately, because of the time constraint, we will never come to any serious study of cohomology in this course. So, you just end up studying homology a bit more elaborately than what you might do elsewhere.

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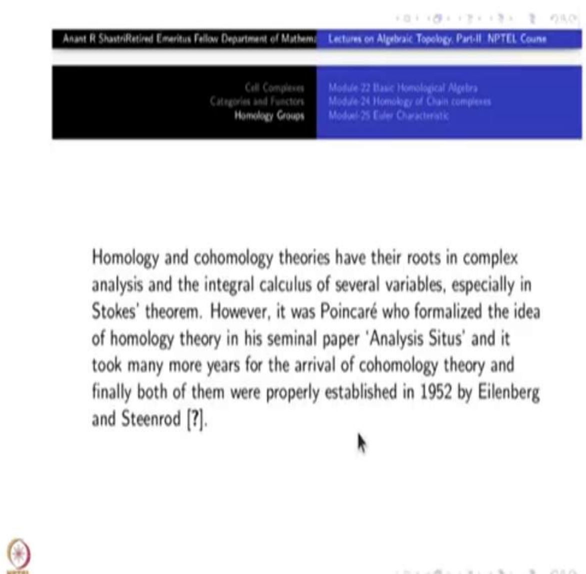

So, as I told you, the homology and cohomology theories have their roots in complex analysis. But it took, you know, several years and efforts of various people. Namely, Poincare was the one who came up with a version of homology on spaces called piecewise linear spaces which are slightly more general than simplicial complexes, okay? Spaces which are built-up from convex polyhedrons. That was the beginning. After that, it took several years to truly establish homology theory as we know today, the way we see it now. So, that can be attributed to Eilenberg and Steenrod around 1952.

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So, I repeat once again that we would like to keep the cohomology a little away right in the beginning, though, they seem to be more natural in some sense and more structural. Homology, you know, on the other hand is easier to understand. And as far as we are concerned that itself is a full time job here.

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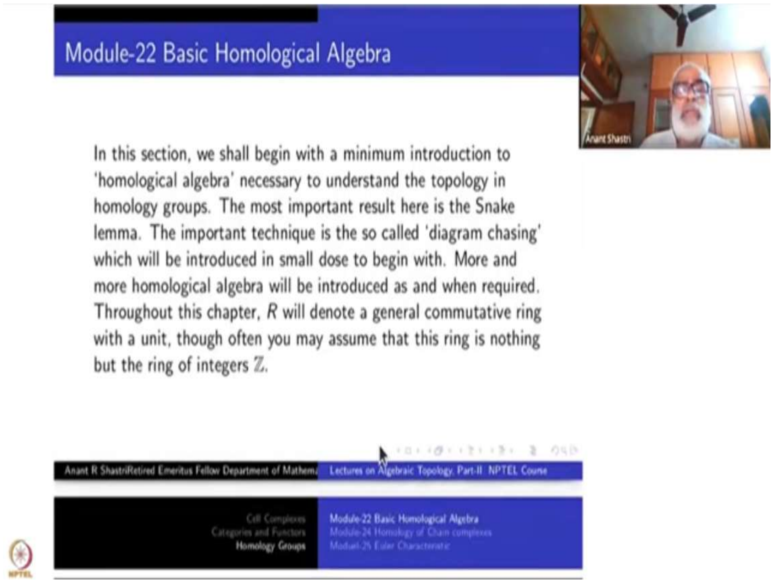

So, we will begin with some unavoidable amount of algebra that is needed for us. It is definitely unavoidable, whether you like it or not. That's why we have to do this thing. We will keep the algebra to a minimum to begin with. And we will keep picking it up as and when we need more and more algebra. Okay? So right now, what we would like to consider is the category of all modules over any commutative ring. But that seems to be a bit too much of algebra, So, we'll just restrict ourselves to only rings which are called principal ideal domains (PID).

If even that is a bit too much for you, all that you can do is that this principal ideal domain whatever it is, just think of this as the ring of integers. And then the modules over it are nothing but abelian groups. However, I will do the treatment for PID's, though I will keep saying modules over the ring R, Okay? So you can just keep thinking about R being the ring of integers, and then the module is nothing but an abelian group. Okay? That is good enough at the starting point. Later on, we'll see whether we need more algebra.

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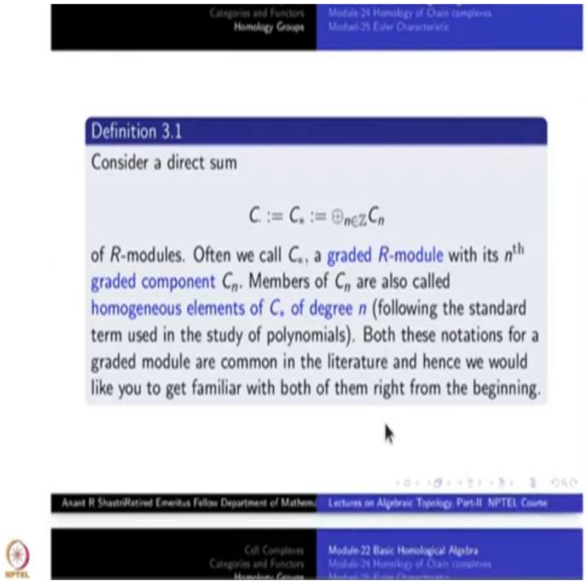

So, the first thing is that we have a bunch of abelian groups or modules, this bunch has been indexed by integers. And then we will take the direct sum of all these infinitely many abelian groups. So, C_n denotes one single abelian group, indexed by an integer and then I will take the direct sum and that I will denote either by C_* or $C.$. Because both these notations are used in the literature, it's better to get familiar with both of them okay?

The n-th group here will be called n-th graded component. So, such a thing is called a gradation of the group $C.$ You can take it to be a graded module okay. And elements of C_n are also called homogeneous elements of degree n, and C_* can still be thought of as just one single abelian group, Okay? More generally, it is a module over R and there are sub modules namely all these C_n's are sub modules of C. And C is made up of these submodules in a very nice way namely, as a direct sum. So, each submodule is some grades component. So, these are just terminologies right now, they have some deeper meaning. Let us go ahead.

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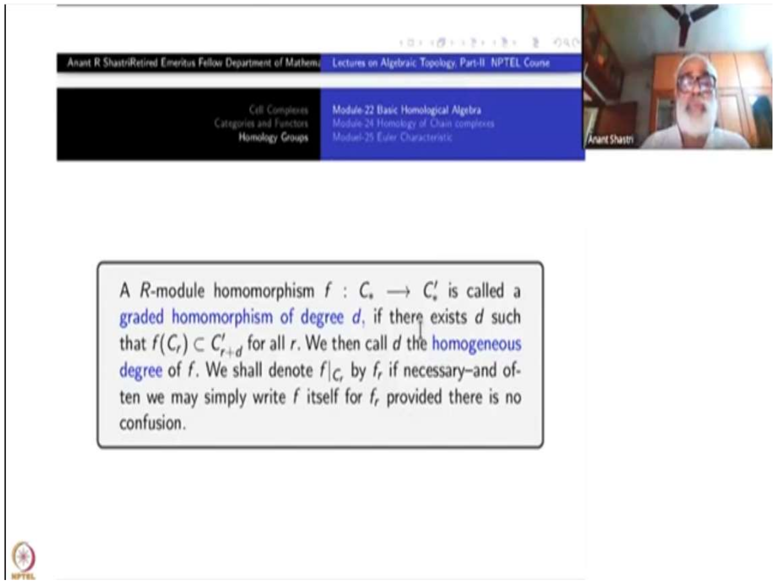

Next, I want to take a R-module homomorphisms, between two such graded modules C_* and C'_*. What is it? It is a R-linear map, but I insist on that they have some more properties. Namely, each C_n is going into C'_{n+d}, for every n, where d is a fixed integer. Then I would like to call such a homomorphism a graded homomorphism of degree d. C_r is carried inside C'_{r+d}, okay? So, this will be a graded homomorphism of degree d, Okay?

Now, if you compose a degree d map with the degree d' map, what you will get is a degree $d+d'$ map, okay? If you fix the degree then it will not form a category, in general, unless $d=0$. Graded modules with graded homomorphism of degree 0 have a special significance for us, namely, they form a category.

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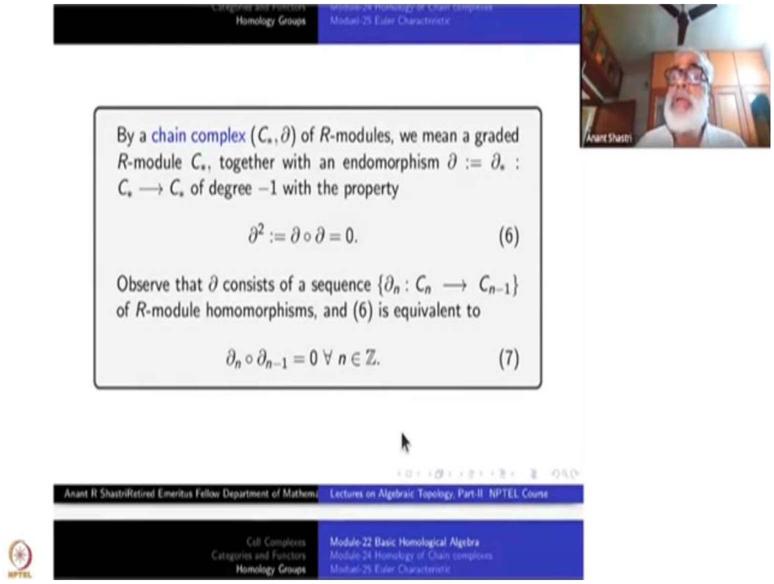

So, now, the next thing is we want to put some more structure on these graded modules. So, this structure comes from another morphism viz., ∂, called the boundary operator. This is a self operator on $C_\cdot$, i.e., an R-linear map from C_* to C_*, but this time it is of graded degree minus 1. And it has one extra property, namely, ∂ composite ∂ is the 0-homomorphism, okay?

In particular, this means that for each $n, \partial(C_n)$ goes into the subgroup C_{n-1}. Therefore for each n, the restriction of ∂ over C_n itself is a R-linear map, which we denote by ∂_n. Then the extra condition can be written as ∂_n composite $\partial_{n+1} = 0$ for each n. A grades module with such a structure will be called a chain complex okay?

So, usually, classically as well as presently, we write this as follows:

$$\ldots C_n \to C_{n-1} \to C_{n-2} \ldots$$

so that it looks like a chain. So, the name chain complex is stuck to it. It's nothing special about this name chain. Like the modules are nodes or vertices and the arrows are like edges. One edge followed by another edge, starting and end points are R-modules. So, it does look like a chain. So, these are called chain complexes.

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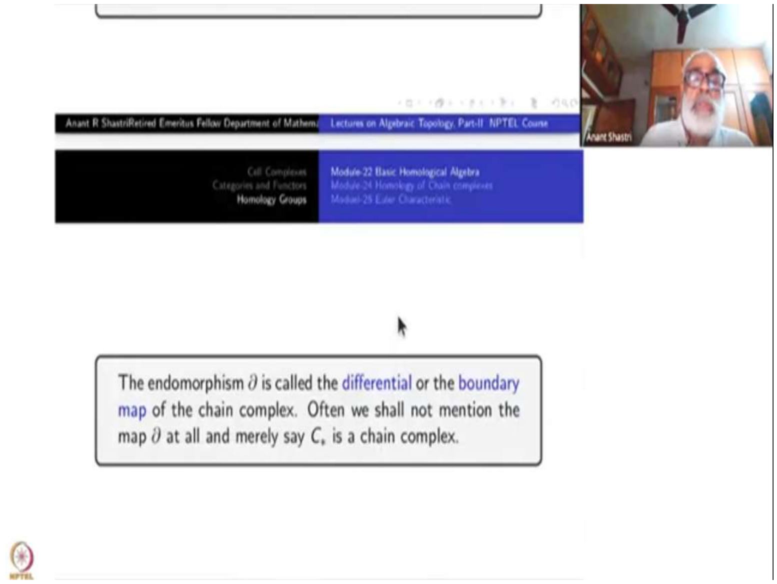

The endomorphism ∂ is called a differential or the boundary operator on C. So there are both theses names okay?

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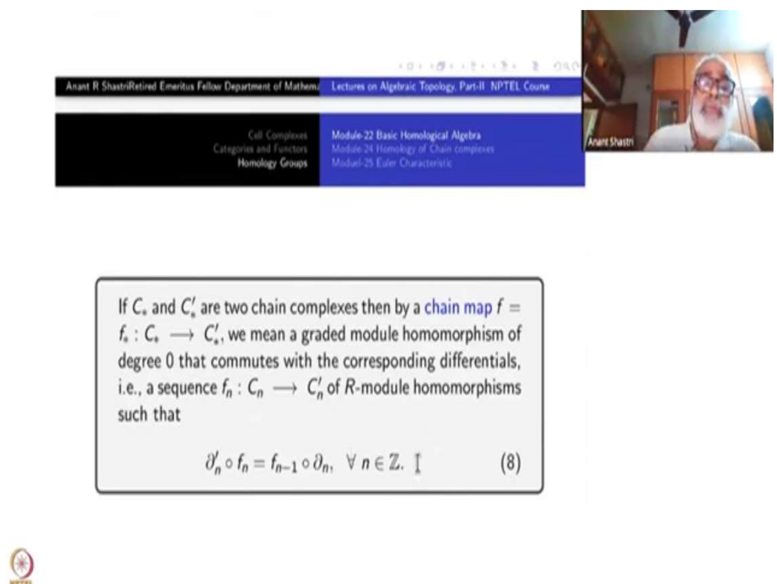

If C_* and C'_* are two chain complexes then by a chain map f_* from C_* to C'_*, we mean a graded module homomorphism f of degree 0, that much you have to admit, but one more condition is there. Note that though I merely say C_* is a chain complex, what I mean is that it is the pair (C_*, ∂). This is just like our practice of saying X is a topological space, without explicitly mentioning the topology.

So, the extra condition is that f should respect the two boundary operators in the domain and the codomain. This just means that for every $n, \partial'_n \circ f_n$ is equal to $f_{n-1} \circ \partial_n$. You can just write it as $\partial' \circ f_* = f_* \circ \partial$, a single equation, without writing any indexing. It makes sense and the correct decomposition into equations with n-th graded components can be recovered easily, if you just note that ∂ is graded map of degree -1 and f is a graded map degree 0.

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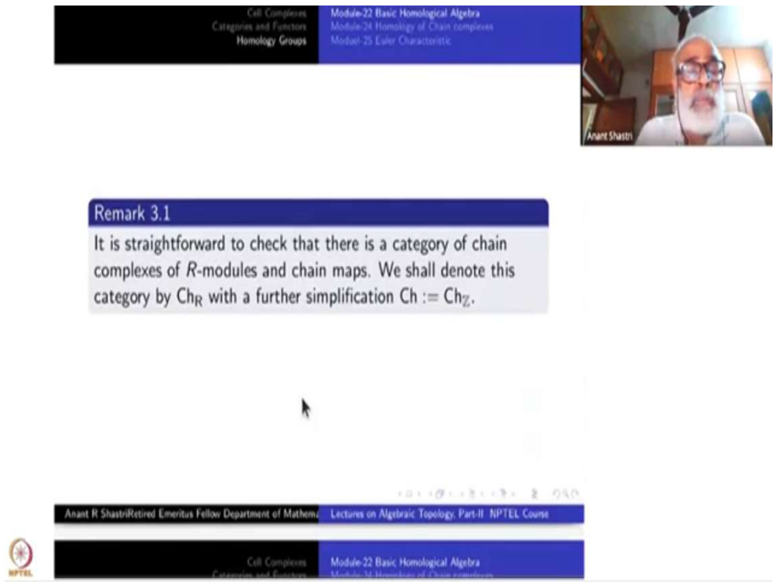

The important thing here is that now we have a subcategory of the category of graded chain complexes and graded morphisms, namely, the category of chain complexes and chain maps. Chain complexes along with the degree 0-morphism forms a category, okay? All that I have to say is that the usual composite of two morphisms is again a morphism in this sense, okay. So, we denote this category by Ch_R or just Ch when the ring R is understood. We rarely use this notation but this being such an important category, you can have a notation for it, that's all.

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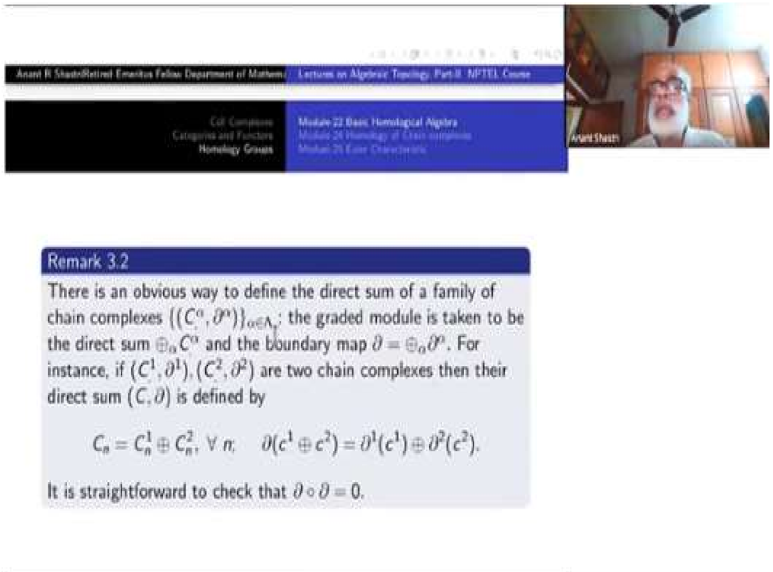

There is an obvious way to define several operations on Ch, similar to operation for abelian groups or for modules. Like you can take direct sum of a family of chain complexes. Direct product also or Hom and so on. For example, take a family $\{C^\alpha, \partial^\alpha : \alpha \in \Lambda\}$ of chain complexes. I am indexing then with a super-script α and Λ is the indexing set, each of them is a chain complex. Then the direct sum (C, ∂) is defined by taking C equal to the graded module with its n-th graded component equal to the ordinary direct sum of modules $\oplus_\alpha C_n^\alpha$ and ∂ to be the direct sum $\oplus_\alpha \partial^\alpha$.

For instance, let us just take two chain complexes, (C^1, ∂^1) and (C^2, ∂^2). The direct sum of these two is the chain complex (C, ∂), where $C_n = C_n^1 \oplus C_n^2$ and $\partial_n = \partial_n^1 \oplus \partial_n^2$ for each n. A generic element here looks like an element c^1 direct sum an element c^2 okay, then $\partial(c^1, c^2)$, (direct sum of ∂^1 and ∂^2) is mapped to direct sum of the elements $\partial^1(c^1)$ with $\partial^2(c^2)$. Okay? So it is pretty straightforward to verify that composite of ∂ composite ∂ is zero. Almost all algebraic operations permissible with R modules can be imitated for chain complexes also.

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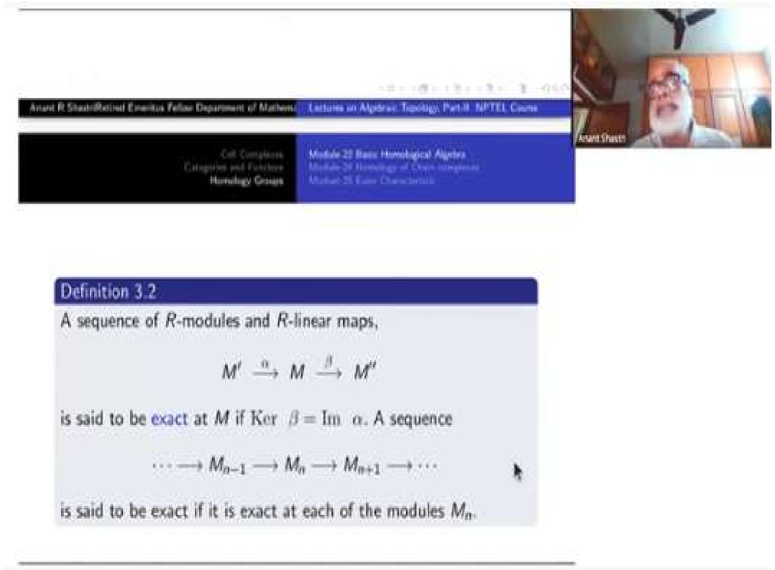

Now, for any sequence of R-modules, you may or may not know this, concept called `exact'. So consider a sequence, $M'$ to $M$ to $M''$, at least three terms here. It is to be called exact at $M$, if not only $\beta$ composite $\alpha$ is $0$, but, the kernel of $\beta$ should be equal to image of $\alpha$. Then we call it as exact at $M$. Okay? This word `exact; is actually borrowed from calculus, where you consider exact forms or exact differentials, or exact equations etc. Okay?

Suppose now that we have a sequence of several terms, may or may not be a chain, but a have sequence of several terms of modules and homomorphism of modules. Such a thing will be called an exact sequence if it is exact at each of its terms M_n. Okay? All exact sequences are chain complexes.

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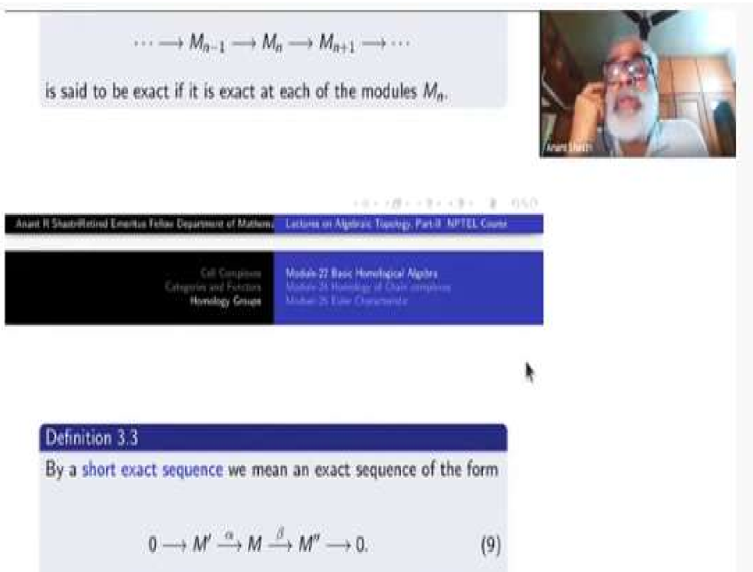

A chain complex may not be an exact sequence, it may just fall short of that? The composites of consecutive arrows are all zero, implies that the image of the previous arrow is contained in the kernel of the next arrow. If that is an equality everywhere then it would become exact. So, exactness is much more stronger than being a chain complex.

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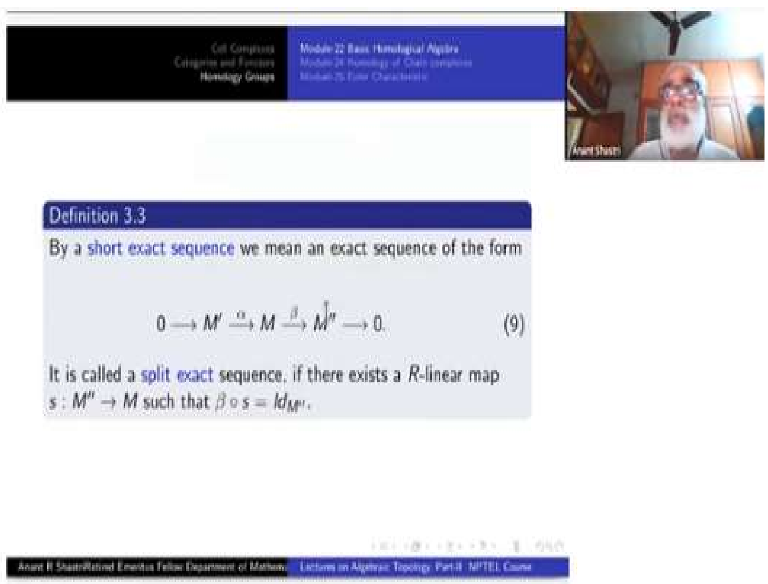

Now we can talk about these terms in the category Ch as well. Infact, I may be really just recalling them. You may be familiar with them already, Take a sequence of R-modules like this. There are 5 terms here, the starting and ending terms are 0. So, essentially there are only three terms, okay? If this sequence is exact then it is called a short exact sequence, Okay. There are obviously even shorter sequences which are exact exist but they are of lesser importance. In practice, we mention on the three middle terms and morphisms the two end modules are assumed to be zero.

But usually the name for short exact sequence exists frequency is this 5 term sequence which the endpoints or end modules are 0 okay. Suppose you say (14) is an exact sequence. That means kernel of β is equal to image of α, α is injective and β is surjective.

It is called split exact sequence, if you have a morphism s from M'' to M called a splitting of β that means, if $s \circ \beta$ is the identity of M''. (So, s is a left inverse of β, or equivalently β is a right inverse of s.)

I'm just recalling these thing if you already happen to know them. Otherwise, you will have to work out a little bit, these things are not very difficult. The split exactness is equivalent to say there is a morphism t from M to M' such that $\alpha \circ t = Id_{M'}$.

The third thing is that you can write M as a direct sum of a copy M' and a copy of M'', viz., $\alpha(M') \oplus s(M'')$. So, $\alpha(M')$ and $s(M'')$ are submodules of M, they span the entire M and they will have intersection 0. So, that is the meaning of direct sum. M can be written as a direct sum of these 2 submodules. That is pretty easy to see okay?

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Conversely, you can start with a direct sum M' and M'', take the inclusion here and take the quotient map there that will give a short exact sequence, which is obviously a split exact sequence. So, we realize that giving a short exact sequence is not the same as giving a direct sum. Because direct sum requires one more extra condition, either the existence of or the existence of t as in the last condition, which are equivalent. The direct sum of two modules always gives you a certain short exact sequence but not conversely. Okay?

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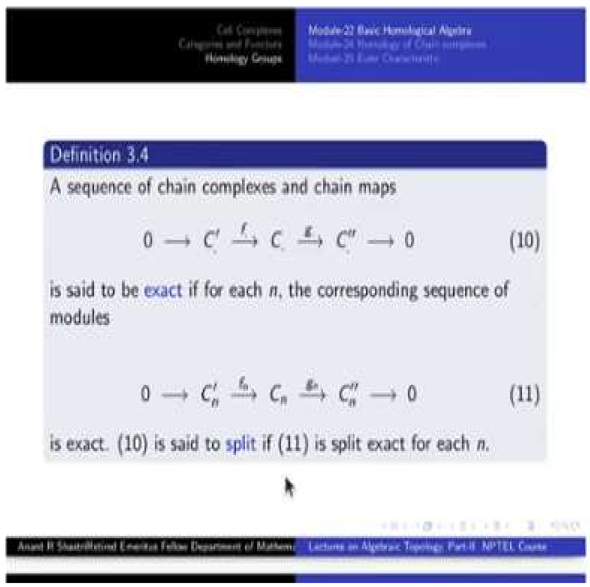

Now, we want to imitate that with chain complexes. Now suppose we have a short exact sequence of chain complexes, f and g are morphisms of degree 0, okay? Unless stated otherwise, chain maps will always of degree 0, okay because that is the one that forms a category. So, take 0 to $C'_\cdot$ to C_0 to $C''_\cdot$ to 0, a short exact sequence, which is equivalent to say that for each the corresponding five term sequence of modules is a short exact sequence. Similarly call it a split exact sequence, if for each n, the corresponding five term sequence of modules is split exact.

So, this is slightly an unconventional definition, but this is the definition I would like emphasis. You are having a category of modules and you would like to have a splitting which is also categorical. No, but this word splitting is not categorical. First, we are talking about the exactness of the sequence which is the same thing whether you consider it in the category Ch directly or separately for each n in the category of modules.

However, while talik about spliting, you demand that for each n, from C_n'' to C_n you have a morphism which is a left inverse, these morphisms collectively may not commute with the boundary operator and hence may not form a spliting of the chain complexes. So, that is the difference. So, definition of splitting is weaker here because what happens is in practice, that is what you get and that itself is useful to some a great extent.

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So, there is a subcategory of short exact sequences of chain complexes of R modules of category of all chain complexes. This is a full subcategory. Okay? Obviously, these will be used to split up the study of longer chain complexes. Suppose you have a chain complex, a very long one, you can always break it up into a number of shorter sequences and then study each of them. Later, we'll study to do that. Okay, At this stage, I just want to say why these short exact sequences are considered. Okay? There are like stop-gap steps, like inductive steps, concentrating at one module at a time. Okay? Since you can't take isolated single module there, you have to take the previous one and the following also. So that is the that's the minimum to take, that is the best thing you can do. So one by one, we will study a lot short exact sequences and that will give you information on original chain complex. So that's the idea. Okay.

Now I want to introduce maybe I should stop here, it is time for today. We will take this one next time. Okay, thank you.