Introduction to Algebraic Topology (Part-II)
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Lecture – 21
Universal Objects

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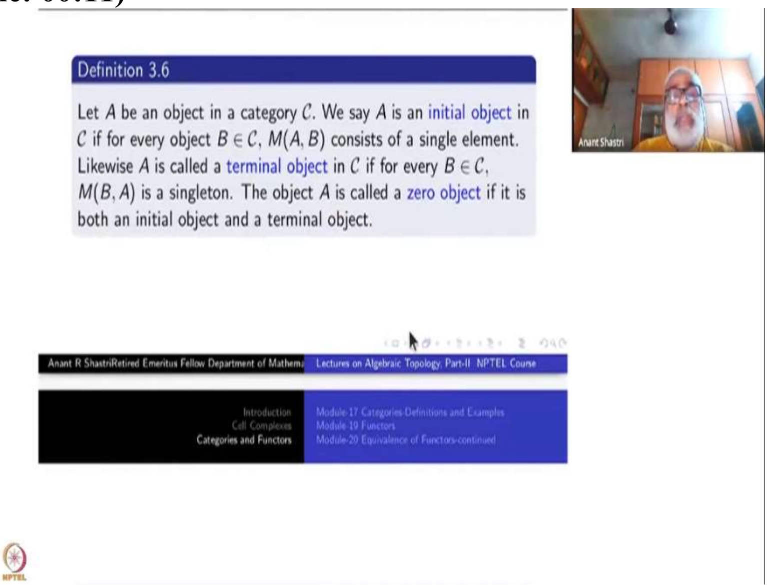

During the last meeting we studied categories and functors. Today we shall demonstrate how certain classical constructions in mathematics can be put in categorical language beneficially.

Take any category $\mathcal{C}$. An object in this category is called an initial object if it admits exactly one morphism into every other object in $\mathcal{C}$. A is an object such that $M(A, B)$ is a singleton every $B \in \mathcal{C}$. Such an object A is called an initial object. Exactly dual to this concept, if A admits exactly one morphism from every member B into itself then you call to be a terminal object, just same thing as saying $M(A, B)$ is a singleton for every $B \in \mathcal{C}$. If A happens to be both initial object and a terminal object, it is possible of course, then A is called a zero-object. So, this terminology is again copied from properties of abelian groups, rings, fields, vector spaces and so on. So, we will see why these three terminologies help us.

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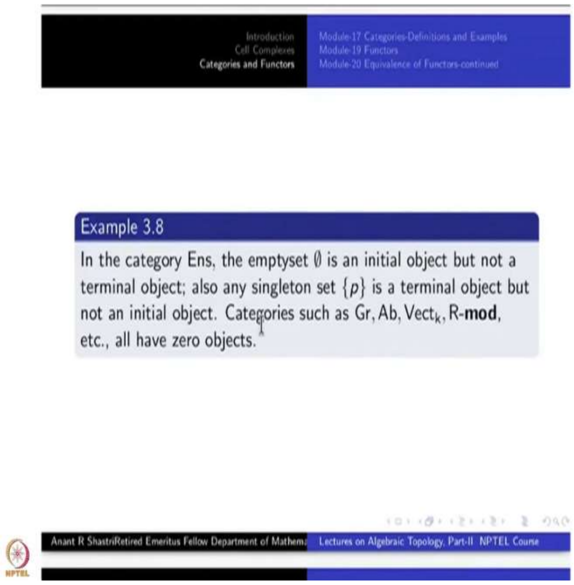

In the category of all sets, the empty set is an initial object but it is not a terminal object. Also, every singleton set in **Ens** is a terminal object but not an initial object. Categories such as **Gr, Ab**, $Vect_k$ etc., they all have 0 objects. The trivial group 0 will have exactly one homomorphism from it to any other group. Similarly, there will be exactly one homomorphism from any group into the trivial group. Though it may be noted that in a non abelian group when multiplicative notation is used for the binary operation the trivial group is often denoted by 1 rather than 0.

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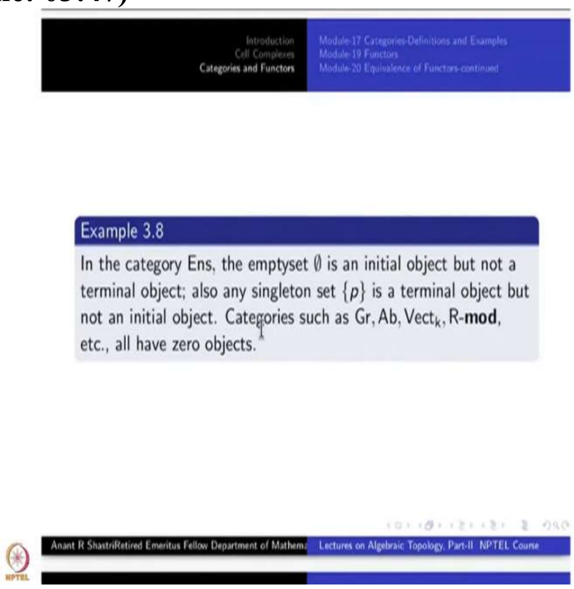

There is a simple way to turn an initial object into a terminal object, by considering the opposite category of the given category $\mathcal{C}$. An initial object in the category $\mathcal{C}$ becomes a terminal object in $\mathcal{C}^{op}$ and vice versa. So, you see **Ens** had the emptyset as an initial object but not a terminal object. But if we take opposite category of **Ens**, it will become a terminal object but not an initial object. It is easy to produce a categories which may or may not have any initial object or may have many of them also.

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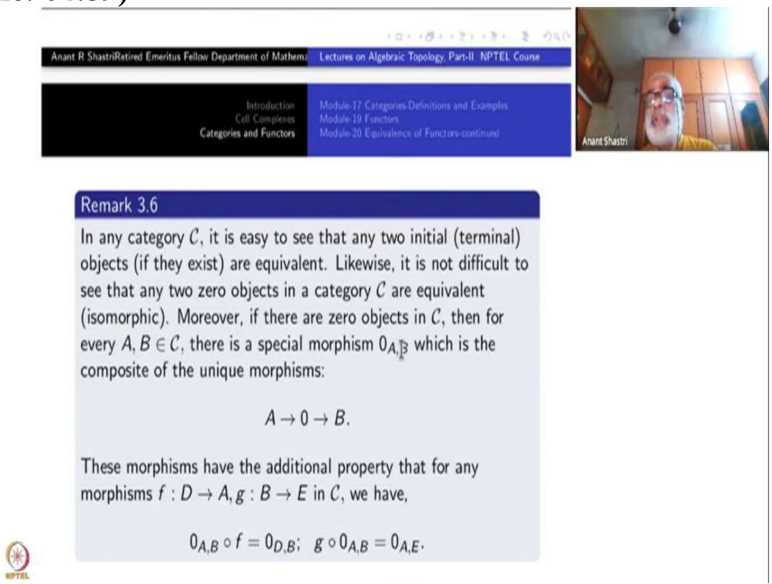

In any category, it is easy to see that any two initial objects are equivalent within the same category. This is a simple observation which helps to understand many part of mathematics very easily. Likewise a terminal object if it exists is unique in the sense of equivalence within that category. Now existence of 0 objects in a category implies some extra structure. Fix a zero object 0 in a category $\mathcal{C}$. For every pair (A, B) of objects, there is a special morphism $0_{A,B}$, which is the composite of the unique morphisms A to 0 to B. Since we have fixed the zero element, there is a unique morphism in $M(A, 0)$ and a unique in $M(0, B)$. Take the composite. That will be denoted by the $0_{A,B}$ and call it a zero-morphism. These 0-morphisms have an additional property: for any morphism f from D to A and g from B to E in $\mathcal{C}$, we have, $0_{A,B} \circ f = 0_{D,B}$. Similarly, $g \circ 0_{A,B} = 0_{A,E}$. Note that there are several 0-morphisms. Composing with a 0-morphism on either side yields again a 0-morphism.

So, this property conforms with our experience with the zero homorphisms of abelian groups, of rings, vector spaces and so on. So, we will come back to the initial object and terminal object later. But right now, I will give you a construction, because that is what my aim was in this lecture.

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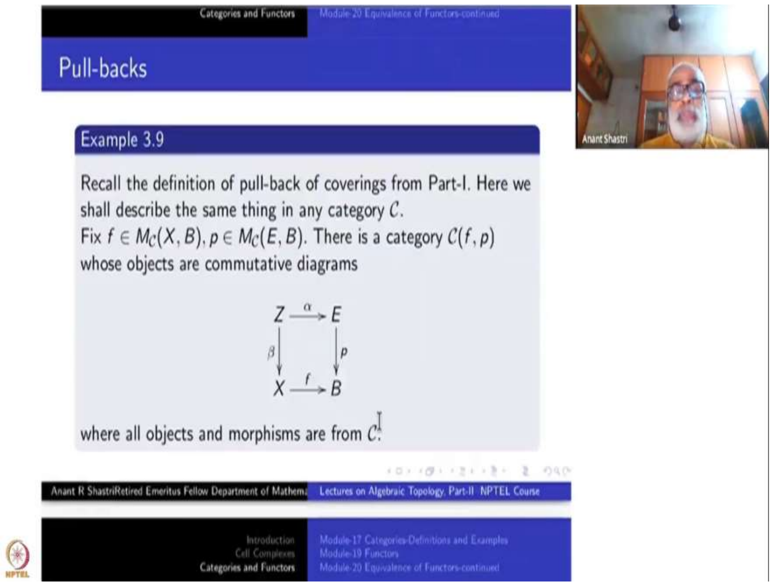

So consider pullbacks. We had introduced this one in part I, in a very special circumstance namely, when we were studying covering spaces or G covering and so on. You have a map from any space X into the bottom space of a covering projection, the base space, then the covering can be pulled-back on to the space X. We would like to do the same kind of thing in any category. We may not know the existence and so on, but at least, we will know that such a concept makes sense.

So, take any category $\mathcal{C}$ and fix a morphism f from X to B and a morphism p from E to B in it. So, both the maps have their codomain B, in this picture. Once you have fixed this, you will define another category itself. In that new category which I am denoting by $\mathcal{C}(f, p)$, which clearly is going to depends upon the category $\mathcal{C}$ where you are working as well as the morphisms f, p you have chosen.

So this is going to be a category I define now. Its objects are all possible commutative squares like this, namely, Z, α and β are all in $\mathcal{C}$, the only condition being $p \circ \alpha$ is equal to $f \circ \beta$. Such a diagram will be an object of $\mathcal{C}(f, p)$.

Next I will define the morphisms and the composition. Various properties such as associativity etc., will be automatic because they will depend upon the corresponding properties in $\mathcal{C}$ being a category. So, what is a morphism from one such object to another such object? We may simply denote these two objects by triples $(Z_i, \alpha_i, \beta_i), i = 1, 2$.

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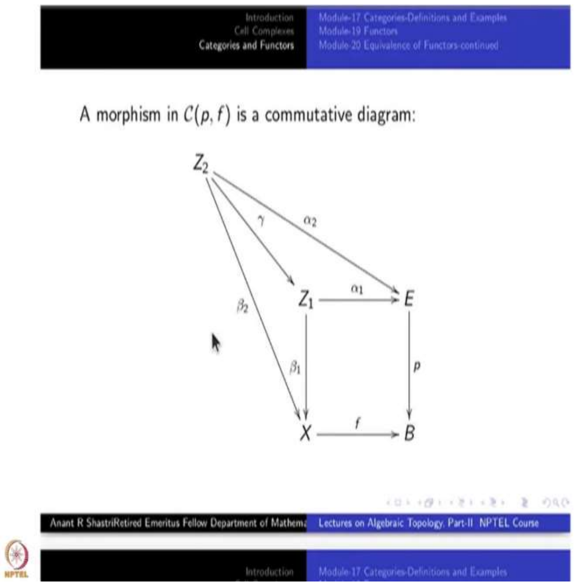

A morphism from (Z_2, α_2, β_2) to (Z_1, α_1, β_1) will consist of a single morphism γ in $M_{\mathcal{C}}(Z_2, Z_1)$, such that the entire diagram is commutative, namely γ followed by β_1 is β_2 and γ followed by α_1 is α_2.

The rest of the commutativity will be automatic. Such a morphism in $\mathcal{C}$ will be a morphism in $\mathcal{C}(f, p)$. If we have another object (Z_3, α_3, β_3) and a morphism γ' in $M_{\mathcal{C}}(Z_3, Z_2)$ which qualifies to be called a morphism in $\mathcal{C}(f, p)$, then the composite $\gamma \circ \gamma'$ in $M_{\mathcal{C}}(Z_3, Z_1)$ will qualify to be a morphism in $\mathcal{C}(f, p)$. That completes the construction of the category $\mathcal{C}(f, p)$.

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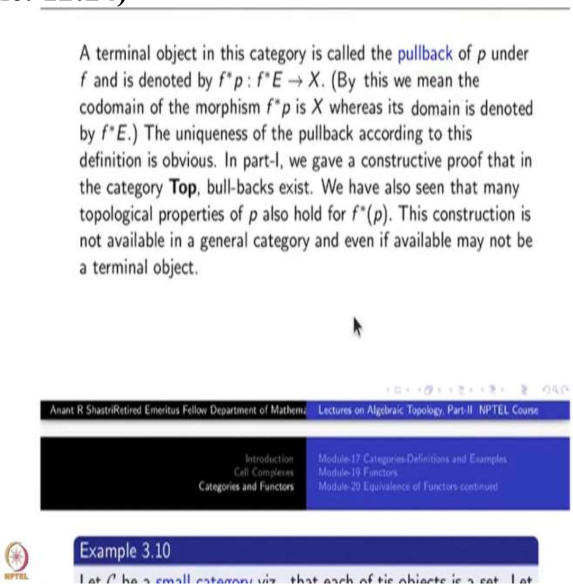

Now, a terminal object in this category will be called a pullback of p under f. Recall that a terminal object means that it is some object with the property that there is exactly one morphism from every other object into this object. In this picture here, if γ is unique for every object (Z_2, α_2, β_2) in $\mathcal{C}(f,p)$, then Z_1 will be called the total space of this pullback, this β_1 will be called the pull back of p under f.

The notation for this will be f^*p. Also, the total space Z_1 will be denoted by f^*E. By this we mean that the codomain of the morphism f^*p is X and its domain is denoted by f^*p. The uniqueness of the pullback upto certain equivalence is obvious, being a terminal object in $\mathcal{C}(f,p)$.

So in part I, we gave a constructive proof of the fact that in the category **Top**, namely, the category of topological spaces, pullbacks always exist. The same construction you can try in various other small categories. But they may fail to give you the full satisfactory answer. The same construction may not be a terminology object. That part we have to re-examine in category separately. But quite often it may work.

So now you see, because of this terminology, I have reduced the work of defining pullbacks what I did for the covering spaces becomes a prototype for so many other categories. You construct this category $\mathcal{C}(f,p)$ in this way and look for a terminal object in that category. One single definition works for all if at all pull-backs exist. So, this is the beauty of categorical language.

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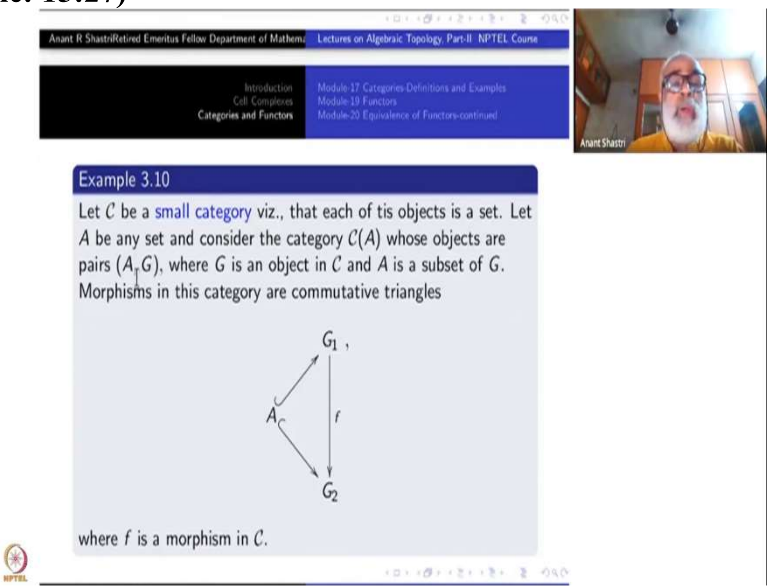

So for example, recall that a category is called a small category if it is objects are sets, each object is a set. Like **Ens** is a small category. **Top** is a small category. Many others like rings, groups, they are all small categories, because the objects in them can be thought of as sets viz., the underlying set. So, you can say that they are all, in some sense, subcategories of **Ens**.

But that is not necessary. A small category need not be a subcategory of **Ens** always. Just the objects are sets, that is enough. Fix a small category $\mathcal{C}$ and let A be any set. Now I am going to define another category $\mathcal{C}(A)$. Its objects are pairs (A, G) (you understand, $\mathcal{C}$ is a category and I pick up any set A and then I am taking pairs (A, G), where G is an object in $\mathcal{C}$ and A is a subset of G.

For example, suppose $\mathcal{C}$ is the category **Top**. G is an object in it means that G is a topological space. But I can talk about subsets of G and A is just a subset of G. Or $\mathcal{C}$ may be **Gr** and then G will be a group but I can talk about just a subset of that group G, so this A must be just a subset of G, and not necessarily a subgroup. So, take such pairs (A, G) where G is an object in $\mathcal{C}$ and A is a subset of G. They are the objects of this $\mathcal{C}(A)$.

Now, I am going to define morphisms. Morphisms are again commutative diagrams as shown in this picture, namely, I have inclusion maps A to $G_i, i = 1, 2$. Also I have a morphism f from G_1 to G_2 in $\mathcal{C}$. However, composition f with the inclusion map of A to G has to make sense and must be the inclusion map A to G_2. This makes sense only if we put some extra assumption such as that f is a set theoretic function. Such as assumption is missing from the slide).

So along with that curcly is a small category let us also assume that $M_{\mathcal{C}}(G, H)$ is a subset of $M_{\mathbf{ENS}}(G, H)$, and the binary operation in $\mathcal{C}$ is the composition of functions. You have to put such a hypothesis explicitly. So, this diagram makes sense and it must be commutative. Such diagrams will be called morphisms of this category $\mathcal{C}(A)$. Objects are like including maps.

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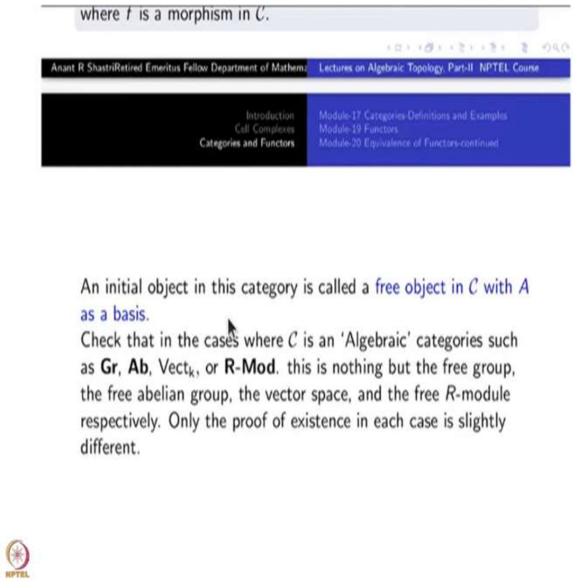

An initial object if it exists is unique upto an equivalence. (I do not know whether it exists). Such as initial object is called, a free object in $\mathcal{C}$, with A as a basis. That is a definition. Let us look at the examples which I have been telling often. Namely, **Gr, Ab**, $Vect_k$, $\mathbb{R}$-mod etc. they are all small categories in which the extra condition that morphisms are also set functions and the binary operation is the composition of functions.

What is free object in each case? A free group, a free abelian group, vector space with A as a base or a free module over $\mathbb{R}$ with A as a basis etc. All these 4 different things you may have studied very thoroughly at different places. All of them and more can be studied in one single go by this concept of an initial object. They are free objects in the specific category. So, if you prove a theorem for free objects, it will be true for all of them.

Even the proof of the existence which is constructive will be similar if not exactly the same. So, you may have to keep modifying it a little bit. For the existence part, you will have to use some special properties of the particular category that you are working in and that cannot be generalised.

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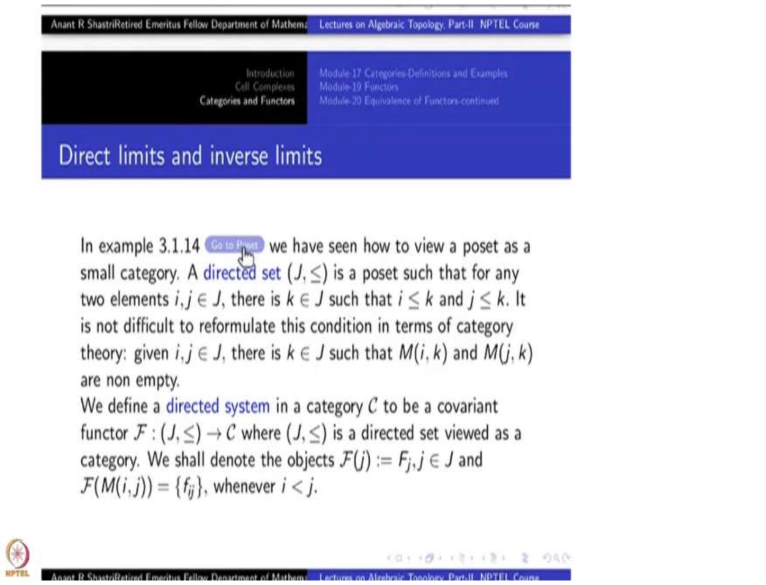

So, pullbacks etc, are new but these free objects were somewhat old. I will ask you one simple question. If you take $\mathcal{C}$ as **Top**, what is the free object there. Think about it. The direct limit and inverse limit are the next topics that I want to discuss.

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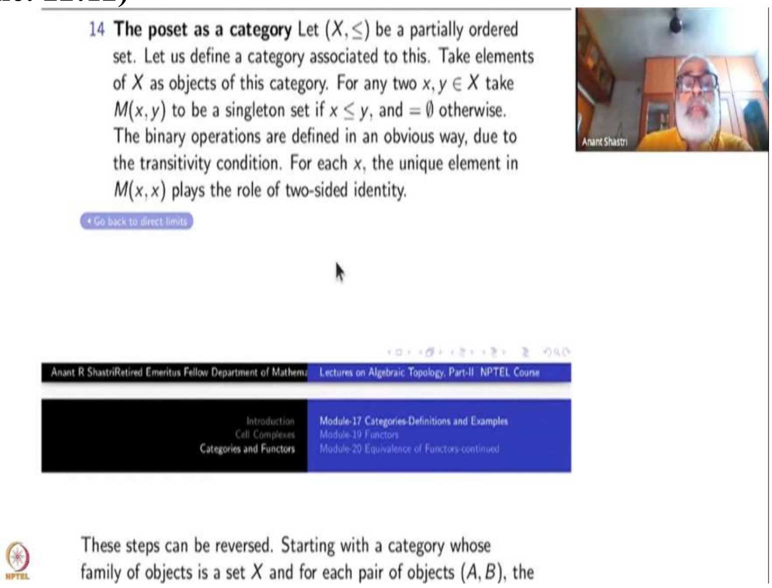

So, go back to this example wherein, we started with a partially ordered set and then converted it into a category. That conversion was recoverable. Namely, the category associated to a partially order set has objects which are points of the set and morphisms are precisely singletons only if $x \leq y$. If x and y are incomparable, or if $x > y$, then $M(x, y)$ will be empty. So, that was the category that associated to its poset. So, we will use that now. We have seen how to view a poset as a small category.

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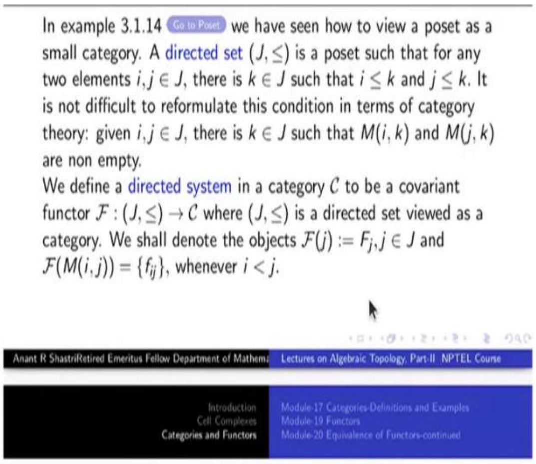

A directed set $(J, \leq)$ is a poset such that for every two elements i and j in J, there is a $k \in J$ such that $i \leq k$ and $j \leq k$. The elements i and j may not be comparable. If they are comparable, then I do not need another k to satisfy that above condition, one of them is bigger than or equal to both of them.

But if they are not comparable, then of course the above condition is an extra assumption on the directed set. It is not always true for an arbitrary poset. Given any two elements there is an element which is bigger than both of them. So, that is the meaning a directed set, a partially ordered set with this extra condition. It is not difficult to reformulate this in terms of the corresponding category.

All that it means is that given i and j inside J, there is $k \in J$ such that $M(i, k)$ and $M(j, k)$ are non empty. That's all. You can talk about the same thing in slightly different language category. Now, given a directed system $(J, \leq)$ and a category $\mathcal{C}$, I am going to define a directed system in $\mathcal{C}$. It is just as a covariant functor $\mathcal{F}$ from this category $(J, \leq)$ to $\mathcal{C}$, where $(J, \leq)$ is viewed as the associated category.

So, a directed system is nothing but a covariant functor from a directed. What does it constitute? For each element $i \in J$, you have an object $F_i \in \mathcal{C}$ and for each $i \leq j$, there will be a morphism f_{ij} from F_i to F_j in $\mathcal{C}$. Then because it is a directed set, given F_i and F_j, there will be some F_k such that there is a morphism f_{ik} and a morphism f_{jk}. So, that is the picture of a directed system.

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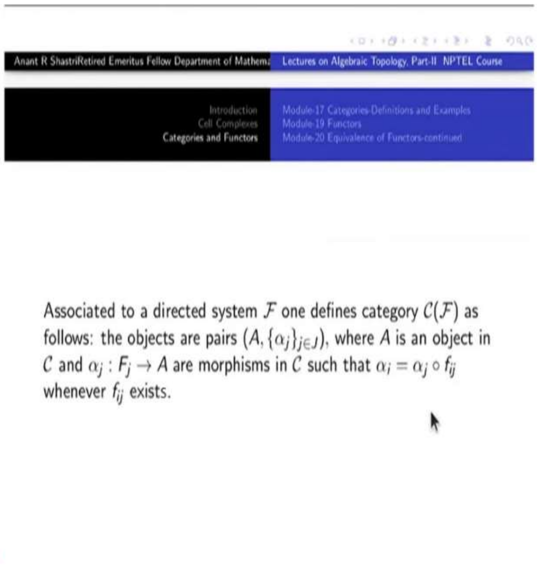

Associated to a given directed system, we shall define a category $\mathcal{C}(\mathcal{F})$ now. Just like while defining free objects, or while defining the pull back after fixing two maps f and p and so on. That kind of game we playing again.Whatever you want to do, you first make up your category appropriately. Look at the data to begin with and then out of that you have to construct a correct category and then look for initial or terminal object in it accordingly. So, now what I am going to do? I am going to define another category which depends upon this directed system. And every thing is happening inside the given category $\mathcal{C}$.

Objects of $\mathcal{C}(\mathcal{F})$ are pairs $(A, \{\alpha_j; j \in J\})$, where A is an object in $\mathcal{C}$ and $\{\alpha_j\}$ is a family of morphisms in $\mathcal{C}$ from F_j to A, indexed over J, such that $f_j \circ f_{ij} = f_i$, whenever it makes sense, i.e., whenever $i \leq j$. So, we can represent this by a diagram as shown here. For each pair (i, j), whenever $i \leq j$, you have such a commutative diagram of morphisms in $\mathcal{C}$.

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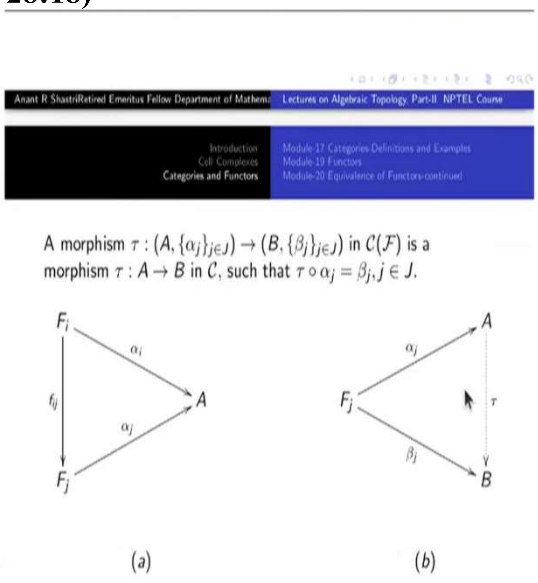

So, these are objects of your category. So, what are morphisms? Again from one pair to another pair? Suppose you have another object $(B, \{\beta_j : j \in J\})$ here. So, there must be a morphism in the category $\mathcal{C}$, let us say τ from A to B which fits all these diagrams, that means for all j, we must have these commutative diagram shown on the right here, viz., the same tau should satisfy $\tau \circ \alpha_j = \beta_j$ for all j. That completes the description of the category $\mathcal{C}(\mathcal{F})$.

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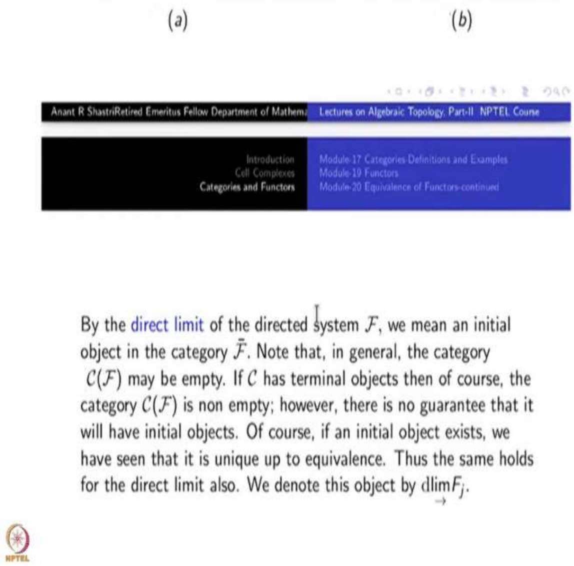

By a direct limit of the directed system $\mathcal{F}$, we mean an initial object in this category $\mathcal{C}(\mathcal{F})$. Note that in general, category $\mathcal{C}(\mathcal{F})$ may be even empty. Why? you see an object in $\mathcal{C}(\mathcal{F})$ means a lot of data you have. The category $\mathcal{C}$ must have morphisms like this, for each i some f_i from F_i to A, with a specific properties. If not there would not be any objects in $\mathcal{C}(\mathcal{F})$. So, there is lots of conditions here. This category $\mathcal{C}(\mathcal{F})$ itself may be empty. Then there is no question of having an initial object. Even if it is non empty, there may not be any initial object. If it exists, of course, upto a strong equivalence, it will be unique. So, a direct limit A means an initial object in this category $\mathcal{C}(\mathcal{F})$, we note that in general, the categories here may be empty.

If $\mathcal{C}$ has terminal objects, suppose I take a terminal object $A \in \mathcal{C}$, then automatically all these morphisms will be there and they will be commutative also, because terminal object has this property as a unique map here the composite has to be there. So the composite will be automatically will be there. So, $\mathcal{C}(\mathcal{F})$ is non empty. However, there is still no guarantee that the direct limit will exist. Of course, an initial object exists, we have seen that it will be unique. So we can have some notation for it viz., direct limit of F_j's.

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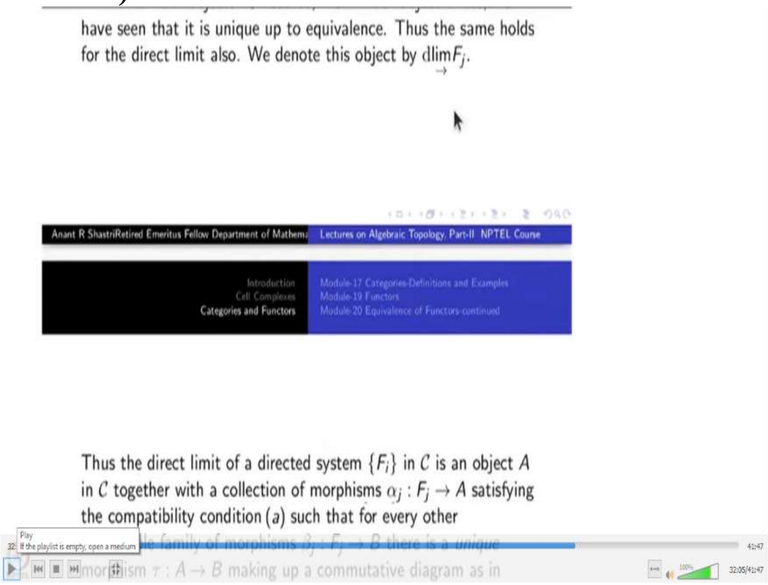

Thus the direct limit of a directed system in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with the collection of morphisms α_j from F_j to A (because after all it is an object in this $\mathcal{C}(\mathcal{F})$ the new category that I have defined) satisfying compatibility condition, this condition and then it must satisfy this other thing condition also viz., for every other compatible family β_j from F_j to B, there is a unique morphism τ from A to B making these commutative diagrams.

So, this is the way a directed limit is defined every time whether it is a direct system of groups, direct system of topological spaces, direct system rings, direct system of algebra and also. So, we can be talked all of them in one single go using this language.

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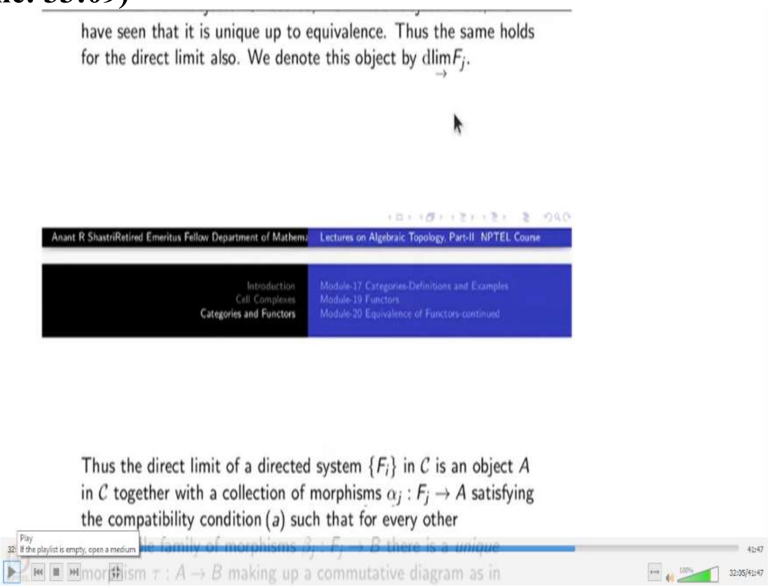

So again, come back to the category of sets itself. It turns out that that every directed system has a direct limit. That may be due to the fact that there are lots of terminal objects viz. every singleton is a terminal object. This fact may have some impact on the existence question of the direct system. I am not saying that this is the proof of the existence okay?

So, what how do you do that, take B to be the quotient of the disjoint union of F_j's by an equivalence relation. Now F_j are just sets, the disjoint union is also a set. Introduce an equivalence relation: x equivalent to $f_{ij}(x)$ for all $x \in F_i$ for all i. Let q be the quotient function the set B of equivalence classes. Given any object (B', β_j) in the category **Ens**($\mathcal{F}$), the identity function on the disjoint union will factor out to give a function τ from B to B', which will automatically fit the bill.

Same kind of construction will work in the categories **Gr, Ab**, $Vect_k$ etc., When I say same it is not meant to be exactly same because you have to worry about group operations, vector space operations and so on. But similar construction works, in many other categories, called abelian categories. There are various concepts which generalise many many algebraic objects, topological, analytic, computer scientific, and all such together. So, that is the whole idea of of category theory perhaps. We are coming to a close now, we will start talking about other things now.

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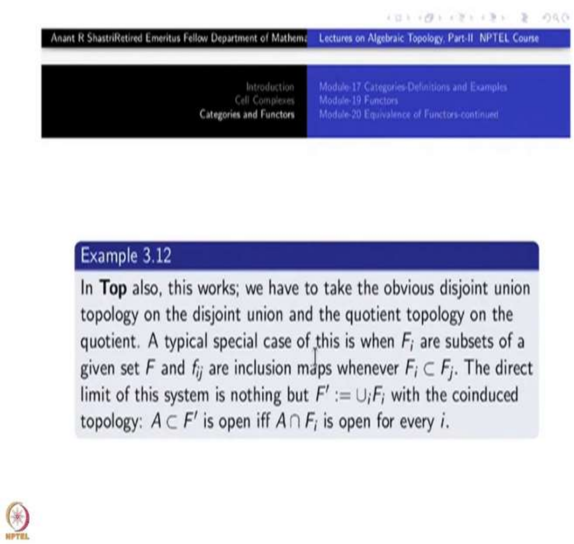

So, let me just give you one more example. One example in the category **Top**. The direct limit construction works in **Top** also. What I will do? Take the disjoint union just like we did in **Ens**, and do the identification also. But what is the topology? On the disjoint union, take the disjoint union topology and on the quotient set you take the quotient topology. Thus direct limit is a very special case of coinduced topology.

So, many of these topological things that you have studied could have been worked out with general category theory. On the other hand, the topological ideas themselves will be generalised all these categories what are called as topas. So, there are so, many other things to do within the category theory. In the beginning they closely resemble algebra. But later on, they become topological. In modern algebraic geometry, all these things are a must. They have to study all these things.

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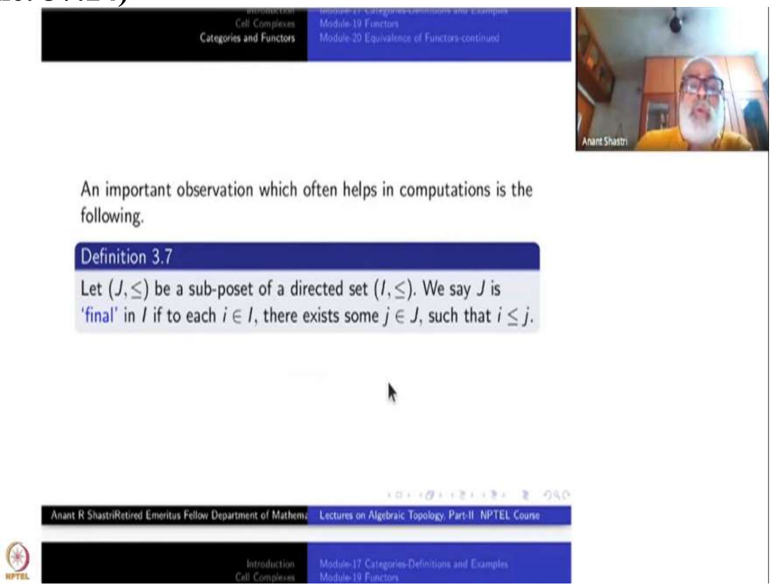

So, for a directed system there is one small technical thing. We remark that this has nothing to do with the category theory. So, it is better to know that since I am talking about the system. Suppose you have a poset I and a sub poset J. So, $(I, \leq)$ is a partially ordered set and this J is a subset of I with the induced relation. We say this J is final in I, J is a final family in I, if for each $i \in J$ there is a $j \in J$ which is bigger than i.

For example, if you take natural numbers for I and J take to be only the odd numbers, this J will be a final in I, because given any natural number there is an odd number bigger than that. This is similar to the concept of a subsequence of a Cauchy sequence. Indeed it is a generalization. You do not need to go to category theory for it. This is done even in the classical set-up.

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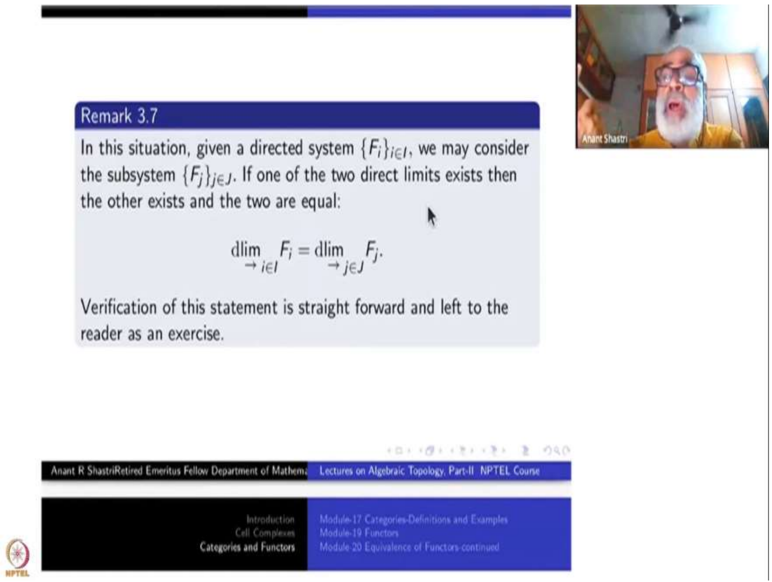

So, what happens is that suppose you have a directed system $\{F_i\}$ and then you consider the subsystem $\{F_j\}$. If one of the direct limit exists, then both of them will exist and they will be equal. So, this as I said it is similar to the subsequence. Now, for a subsequence, the limit may exist but for the given sequence limit may not exist, in general. So, there is a problem there. But with direct limits, that will not be a problem.

So, directed systems are already like Cauchy sequences, not just general sequences. That is why this works, you can verify this one. This is just a remark. You need to prove. Assuming $(B, \{\alpha_j : j \in J\})$ is the direct limit for the subsystem, we take β_i from F_i to B as follows. Pick any $j \in J$ such that $i \leq j$ and take $\beta_i = \alpha_j \circ f_{ij}$. Verification of the details is left to you as an exercise.

The entire discussion about directed system is valid for contravariant functor also instead of a covariant. Then the name will be an inverse system. Instead of initial object, you take the terminal object, that will be called the inverse limit. So that is easy exercise for you to think about it. So, after that I have a number of exercises here you can keep trying them, some of them will be discussed in in assignments and so on. So, our next topic will be homology theory. That will be for another five weeks. Thank you.