Introduction to Algebraic Topology (Part-II)

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Lecture - 22
Equivalence of Functors - Continued

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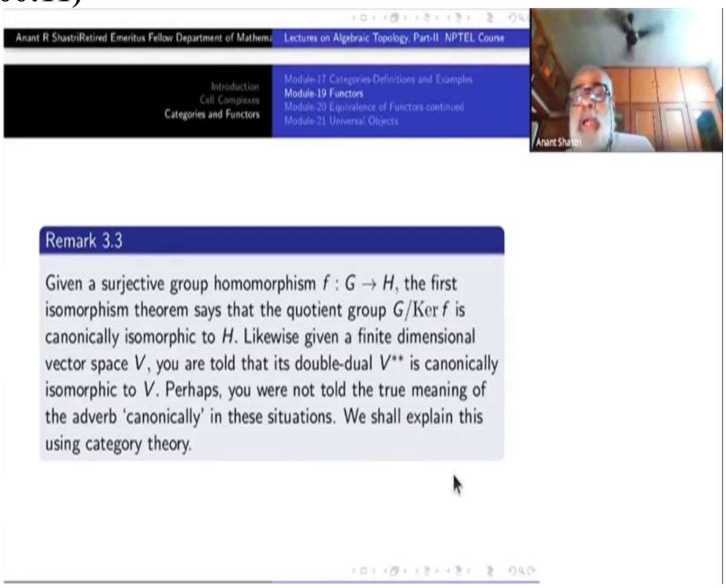

So, let me begin with the last remark that we did last time namely, given a surjective homomorphism from one group G to another group H, the first isomorphism theorem says that the quotient group $G(f)$ is canonically isomorphic to H. Likewise given a finite dimensional vector space V, you are told that its doubled dual V^{**} is canonically isomorphic to V.

Perhaps you were not told the true meaning of the adverb \`canonically', what is the meaning of the term \`canonically' in these situations. So, I would like to explain this to some extent whatever possible, nobody can explain fully after all, using the category theory. Before that I have to introduce another important notion in the category theory, namely, the notion of natural transformations and natural equivalences. The word \`natural' is there or you may replace by the word \`canonical'. So, that is what it comes to now. So that is the topic today. And later on, we will even generalise this concept of natural transformation itself.

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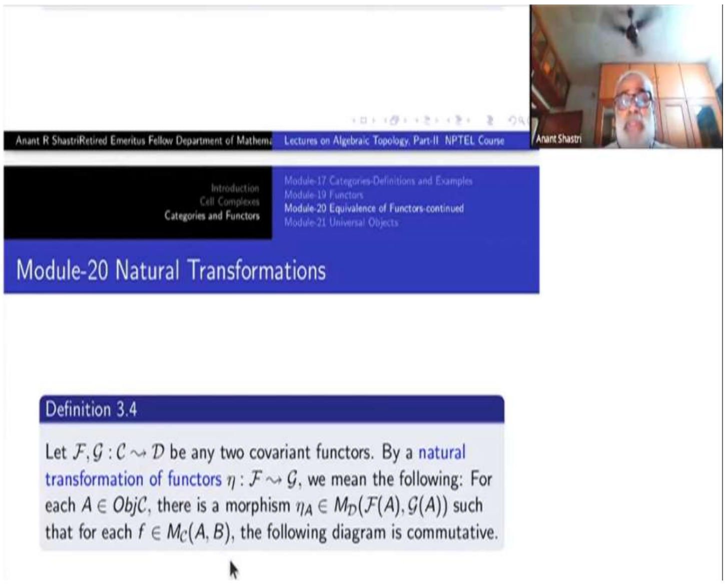

So let $\mathcal{F}$ and $\mathcal{G}$ be any two functors from the same a category $\mathcal{C}$ to $\mathcal{D}$. There is a way of defining this concept when both these functors are contravariant, but I am taking, to begin with, $\mathcal{F}$ and $\mathcal{G}$ to be covariant functors. Exact similar definition is valid for contravariant functors also.

So, by a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are functors from the same category $\mathcal{C}$ to the same category $\mathcal{D}$, this η is also indicated by a twisted arrow like this, we mean a whole set of data as follows: What are they? That is what I am going to define. The entire thing below is one single definition:

(i) for each object A of $\mathcal{C}$, there is a morphism η_A in the category $\mathcal{D}$, i.e., η_A is in $M_{\mathcal{D}}(\mathcal{F}(A), \mathcal{G}(A))$. Remember A is an object in $\mathcal{C}$ and $\mathcal{F}(A)$ and $\mathcal{G}(A)$ are objects inside $\mathcal{D}$. Therefore, $M_{\mathcal{D}}$ of that pair makes sense. So for each A, I am having a morphism in this category.

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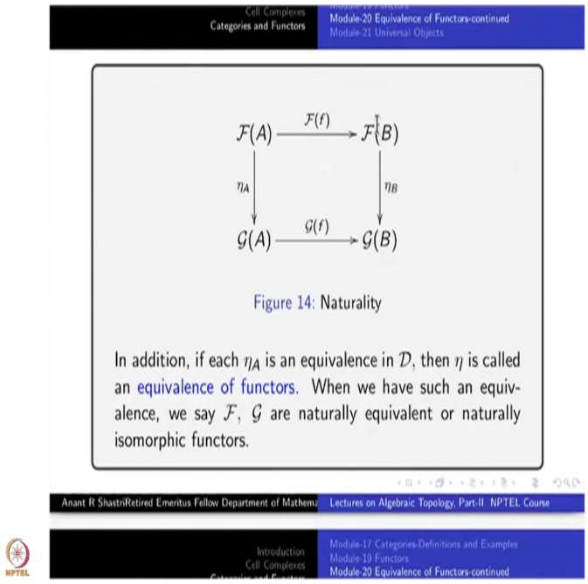

(ii) And this association is such that if f is a morphism in $\mathcal{C}$ itself, say from A to B, then you must have the following commutative diagram of four morphisms in $\mathcal{D}$, namely you have A gives rise to η_A from $\mathcal{F}(A)$ to $\mathcal{G}(A)$ and B gives rise to η_B form $\mathcal{F}(B)$ to $\mathcal{G}(B)$, the morphism f gives rise to $\mathcal{F}(f)$ and $\mathcal{G}(f)$, these are related by this commutative diagram. Remember, $\mathcal{F}(f)$ and $\mathcal{G}(f)$, by the very definition, are morphisms from $\mathcal{F}(A)$ to $\mathcal{F}(B)$ in the category $\mathcal{D}$. η_A and η_B are also in the category $\mathcal{D}$. So, this entire diagram of is in the category $\mathcal{D}$. Commutative means $\eta_B \circ \mathcal{F}(f)$ must be equal to $\mathcal{G}(f) \circ \eta_A$.

That completes the definition a natural transformation of covariant functors. If you reverse all the arrows in the above diagram, you will get the definition for the case when $\mathcal{F}$ and $\mathcal{G}$ are contravariant functors. So, this is the meaning of `naturality',

You have to have one more definition, namely, suppose further that each η_A is an equivalence in D, and not just a morphism, i.e., if η_A is invertible, then we say that η is a natural equivalence of these two functors. When such equivalence exists then $\mathcal{F}$ and $\mathcal{G}$ are said to be naturally equivalent, or naturally isomorphic. Sometimes when you are doing category theory, you do not want to keep saying `naturally' or `canonically'. Just saying that two functors are equivalent or isomorphism is enough.

So, once you are in category theory, just the isomorphism always means natural isomorphisms, that there will be commutative diagrams like this of isomorphisms.

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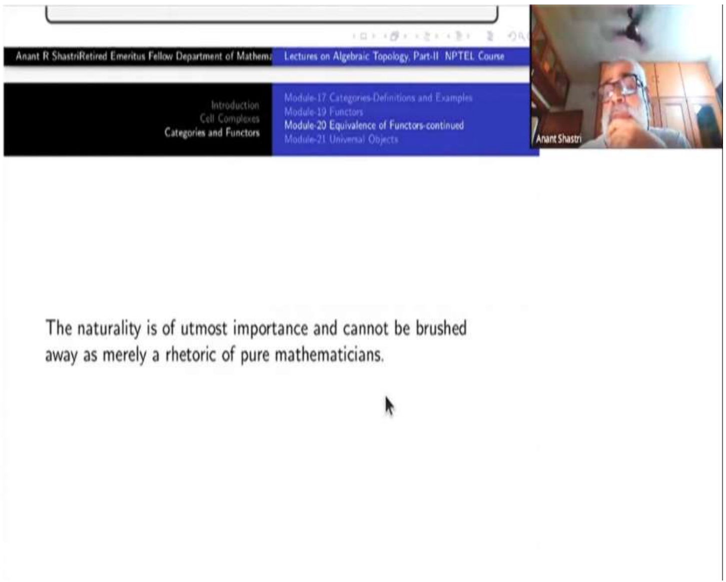

So, here is a quotation. I do not know where I read it. Definitely, it is not mine, but a quotation. I mean I do not remember from where got it. Since I keep repeating it often, for my colleagues/students etc, it has become my statement now. (Read it from the slide.)

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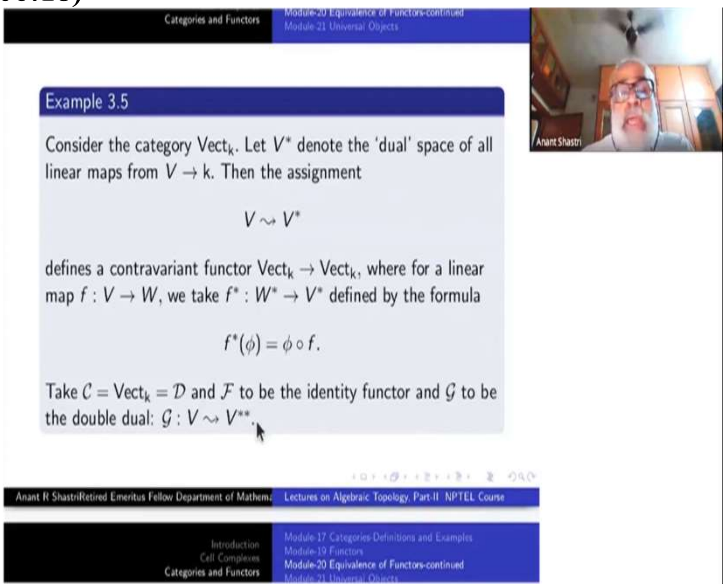

Now, here is the second example, I wanted to explain, which is easier among the two of them. It is about vector spaces that you learn in linear algebra. So, there is not much to explain here. So, let us consider the category $Vect_k$, where k is a field. We have defined this with the family of all vector spaces over k as objects and k-linear maps as morphisms. Given a vector space, define V^* as the dual space of linear map from V to k. This is just a notation. I hope this is standard notation. The assignment V to V^* itself a contravariant functor. It is not just an assignment it is a contravariant functor from $Vect_k$ to itself. For example, you also know that given a linear map f from V to W, what is the corresponding associated linear map

f^* from W^* to V^*, so that the association V to V^* and f to f^* defines the contravariant functor $*$.

So, V will be V^*, W will become W^*, the arrow will be the other way around and that is some f^* which must be linear map. We have to tell what is this f^*? It should have the property that $(g \circ f)^*$ is $f^* \circ g^*$ and Id^* must be identity of the corresponding stars.

These properties must be there. So, first of all I am going to define what is f^*. f^* is nothing but pre-compositor, f^* of $\phi = \phi \circ f$. Now, let us put $\mathcal{C} = Vect_k = \mathcal{D}$, both categories are $Vect_k$, capital $\mathcal{F}$ to be the identity functor and $\mathcal{G}$ to be the double dual functor. What is that? You are composing $*$ with itself $*$ is contravariant functor and therefore $\mathcal{G}$ will be a covariant functor.

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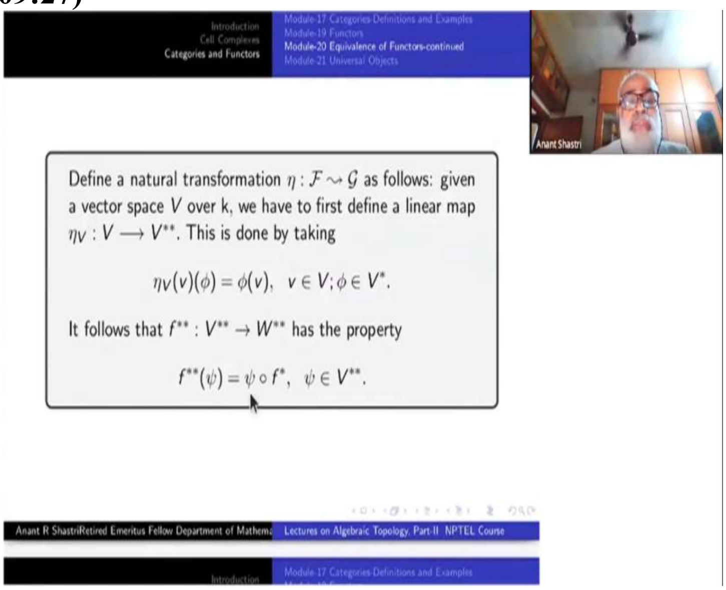

So, V leads $\mathcal{G}(V) = V^{**}$ a double star is a contravariant functor from the category $Vect_k$ to itself. Now, define a natural transformation. First I want to say there is a natural transformation from the identity functor to this double star. Namely, η from $\mathcal{F}$ to $\mathcal{G}$ as follows.

Gien a vector space V over k, we have to define a linear map η_V from V to V^{**}. Because, $\mathcal{F}(V) = V$ and $\mathcal{G}(V) = V^{**}$. That means for each vector $v \in V, \eta_V(v)$ must be a linear map from V^* to k. So, we take $\eta_V(v)$ operating on a linear map ϕ from V to k to be equal to $\phi(v)$. I am sure that you have seen these things before.

So, we can verify the linearity part etc. very easily. v is in capital V, ϕ is in V^* and V^* is the space of all linear maps from V to k. Similary, it follows that f^{**} from V^{**} to V^{**} has the property that $f^{**}(\psi) = \psi \circ f^*$. f^{**} is nothing but $\mathcal{G}(f)$ in our notation G. Just verify this. So, double star being $*$ repeated twice.

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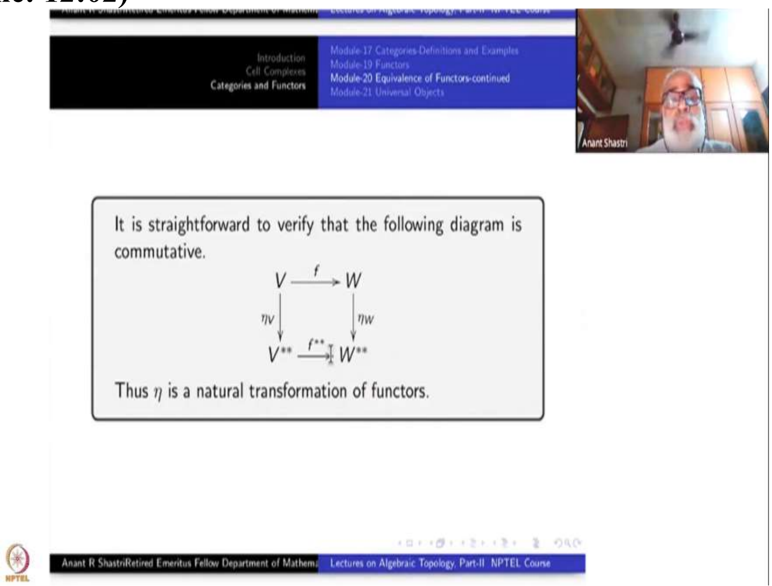

So, it is straightforward to verify that the following diagram is commutative: V to W, f here, $\mathcal{G}(V)$ to $\mathcal{G}(W)$, we have $\mathcal{G}(f)$ which is nothing but f^{**}. And vertical arrows are η_V and η_W respectively. You are verify that the diagram is commutative, in the definition, remember this one. So, this $\mathcal{F}(f) = f$ is identity here and $\mathcal{G}(f)$ is f. We have to be verify that this is commutative. That is straight forward. For each $v \in V$ we have to see that $f^{**} \circ \eta_V(v)$ is equal to η_W operating on $f(v)$. So, take any $\alpha \in W^*$ and verify that the two sides operating on α produce the same of k.

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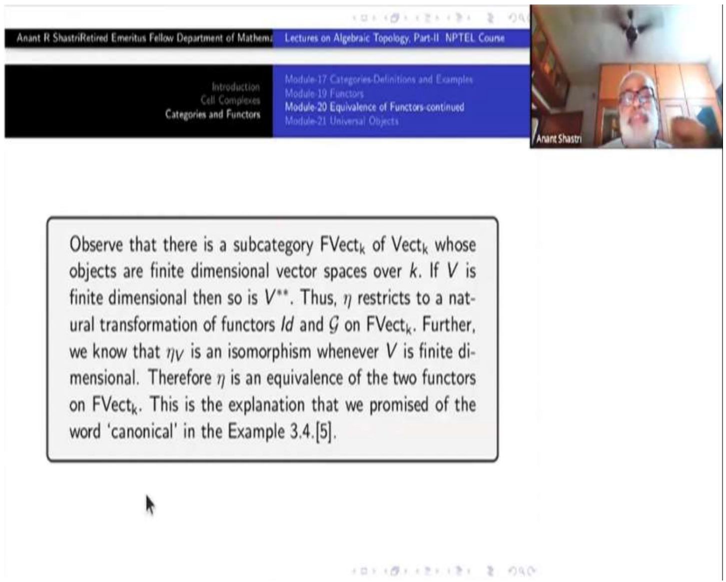

So, this is true in the category of all vector spaces. So, far I have not assumed anything about finite dimensionality. Now, I come to finite dimensions. So, there is a subcategory $FVect_k$ consisting of of finite dimensional vector spaces which is a full subcategory of $Vect_k$. Full means what? All morphisms in $Vect_k$ are allowed, in this smaller category also. Only the domains and codomains are finite dimensional.

If V is of finite dimension, then we know that V^* and hence V^{**} is also finite dimensional. So, the two functors star and double star namely $\mathcal{G}$, are from $FVect_k$ to $FVect_k$, from the subcategory to subcategory, they become functors. Further, elementary linear algebra, we know that η_V is an isomorphism whenever V is finite dimensional. So, this is a linear algebra that you have learnt.

I am not going to prove that. I am going to explain the word `canonical' attached to this isomorphism you know. So, what you might not have bothered about is that such a diagram is commutative. No matter what vector spaces and what linear maps you take. The corresponding diagrams are commutative. You do not have to worry about these transformations, these isomorphisms the same isomorphism will work for all V, W and f. That is the beauty of this statement. So, we know that η_V is an isomorphism whenever V finite dimensional. Therefore η is an equivalence of the two functors, identity functor and the double dual functor. It is not just for one vector space that you have got an isomorphism. For all vector spaces together, in a such a compatible way, that is the meaning of the canonical isomorphism.

So, this was just a technical or just a verbose explanation of technically very precise statement, namely, there is a natural transformation which is an equivalence. So, category theory has achieved that.

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So, our next example is the word canonical occuring in the first isomorphism theorem in group theory. This example is slightly subtler, more difficult than the vector space case wherein the necessary stuffs were ready buitl-in for us. Here I have to do some circus. Nevertheless, I appreciate this great thing. It must be due to Emmy Noether. These are so called Noether isomorphism theorems right? But, the first isomorphism theorem perhaps is not named after.

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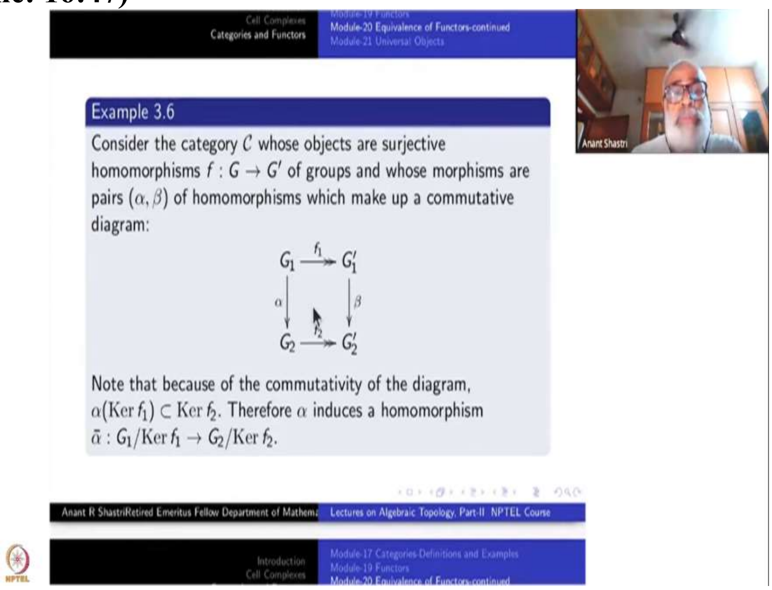

Consider the category $\mathcal{C}$ whose objects are surjective homomorphisms from G to G' of groups. What are the objects? They are not groups. I am making up another category whose objects are surjective morphisms from one group to another. These are the objects. You understand, how can you make it a category? I am going to do that. I am going to do several constructions here you will see and this is only beginning.

So objects of $\mathcal{C}$ are epimorphisms. You may denote them by the triples such as (G, f, G'). So what are morphisms from (G_1, f_1, G_1') to (G_2, f_2, G_2')? It is a pair (α, β) of homomorphisms such that the following diagram is commutative. All of them are group homomorphisms the top thing is one single object. The bottom thing is another single object, a morphism is a pair of homomorphisms not arbitrarily, but such that the diagram is commutative. This kind of is going to occur several times here. We will see that commutative diagrams are the essence of the whole thing.

Note that because we have a commutativity diagram, if you take α of the kernel of f_1 will be contained in the kernel of f_2. What is the kernel of f_1, all those $g \in G_1$ which go to identity in G_1'. Therefore, f_2 of α of g which is equal to β of f_1 of g is the identity element of G_2' This just means $\alpha(g)$ is in the kernel of f_2.

So, this next statement is also a theorem that you have studied in group theory: So, for each pair (α, β) as above, there is homomorphism $\bar{\alpha}$ from quotient to the quotient. This $\bar{\alpha}$ will depend upon both (α, β) as such, remember that, but we are not using such an elaborate notation. Usually people just write $\bar{\alpha}$ because it is given from $G_1/Ker(f_1)$ to $G_2/Ker(f_2)$.

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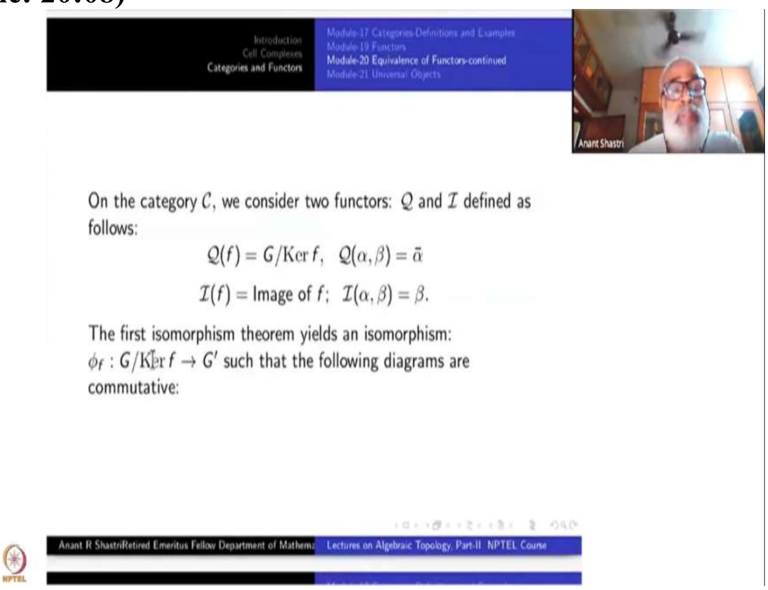

On the category $\mathcal{C}$ that we started with, we have two functors into **Gr** now. What is the first one? I am writing it as $\mathcal{Q}$ and the second one I am writing it as $\mathcal{I}$, defined as follows: For $f = (G_f, G'), Q(f) = Q(G_f, G')$ is equal to $G/Ker(f)$. This makes sense. And $\mathcal{Q}(\alpha, \beta)$ must be a homomorphism from $\mathcal{Q}(f)$ to $\mathcal{Q}(g)$. So take it to be equal to $\bar{\alpha}$. Define $\mathcal{I}$ of f equal to is the image of ff which is nothing but G'. And $\mathcal{I}(\alpha, \beta) = \beta$.

Note that in the definition of $\mathcal{Q}$, there is no mention of β and in the definition of $\mathcal{I}$ there is no mention of α. Easy to check that both are covariant functors. The first isomorphism theorem tells you that there is an isomorphism ϕ_f, which depends on f of course, from $G/Ker(f)$ to G'. The canonicalness corresponds to the assertion that the following diagrams are commutative.

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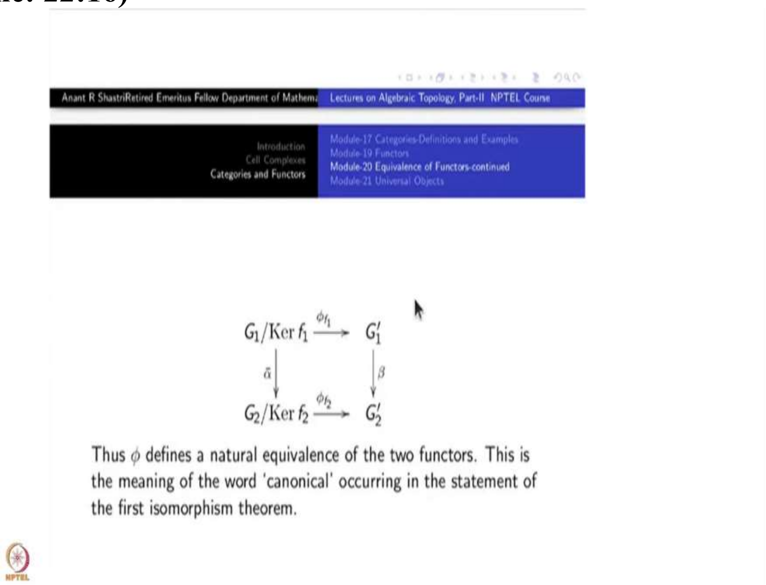

$G_1/Ker(f_1)$ to G_1, we have the isomorphism ϕ_{f_1}. Similarly, $G_2/Ker(f_2)$ to G_2', we have the isomorphism ϕ_{f_2}, making this diagram commutative. Now, look at the first vertical arrow the functor $\mathcal{Q}$ and the second one is the functor $\mathcal{I}$. And ϕ is a natural transformation from one to another which is an equivalence. I hope this explains the word `canonical' used in the ordinary group theory theorem, that you have studied.

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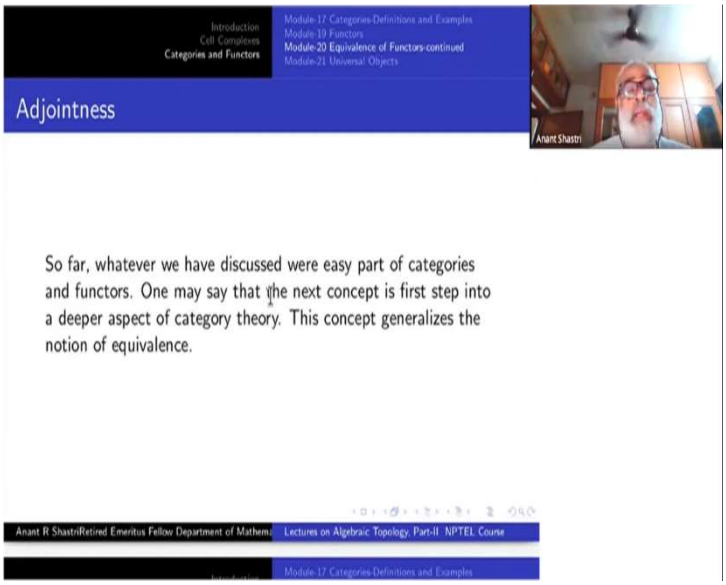

Let me now introduce one of the first deep results in category theory. So far, whatever we have done were all easy. This is the first deep step you are taking in category theory. We are not going to do anything further. Just a little bit of this one, and then we stop there. Later on, we will do something else.

I would say that adjointness is the starting point of serious category theory. So, let us just make a small beginning here. And then interested reader can pick up more from elsewhere, form source such as I have given you the reference of that book. That is a good book you can read from. Of course there are many other books also.

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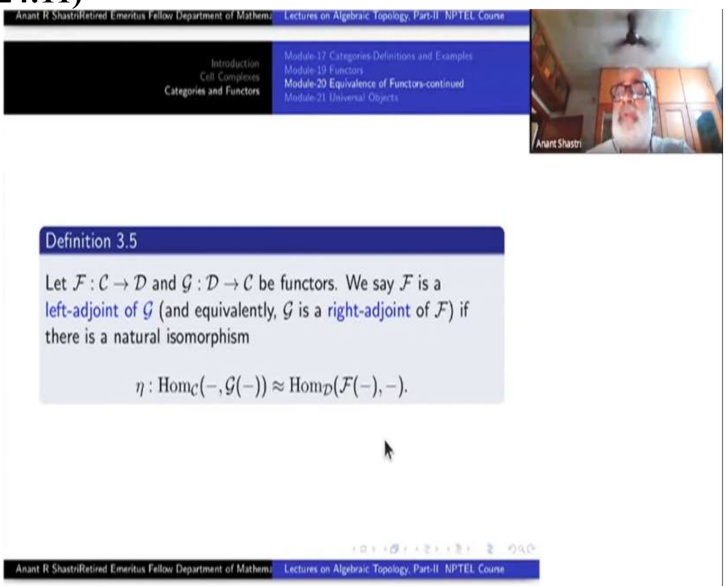

So, join us once again, I am going to define two things simultaneously, left adjointness and right adjointness. So, you can have a very vague picture of it by starting with a homomorphism and then a left inverse and a right inverse for it. It is similar to that. Having said that, the similarity ends there. This is much more subtle and much more stronger statement than that. So, let $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$, $\mathcal{G}$ from $\mathcal{D}$ to $\mathcal{C}$ be two functors. (Now, once again, I am taking only the case of covariant functors here. For contravariant functors there is exactly the same kind of definition and same kind of result, got by simply reversing the arrows. So, I am not going to do that. So, here they are both covariant functors.)

We say $\mathcal{F}$ is a left adjoint to $\mathcal{G}$ and (at the same time) $\mathcal{G}$ is right adjoint to $\mathcal{F}$(just like for functions, when α is left inverse to β, then β is the right inverse to α, $\mathcal{F}$ left adjoint to $\mathcal{G}$ will imply and implies by $\mathcal{G}$ is a right adjoint to $\mathcal{F}$) if there is a natural equivalence of the these two functors.

Both are actually bi-functors from the product category $\mathcal{C} \times \mathcal{D}$ (which I have not explicitly defined so far) to **Ens**. Note that both functors have two slots, to be filled up with objects from $\mathcal{C}$ and $\mathcal{D}$ respectively. Actual definition of this product category etc is time consuming and not necessary to understand what is going on, except when I have to do all this rigorously and systematically.)

So, what is $Hom_{\mathcal{C}}(-,-)$. Starting with an ordered pair (C, D) of objects in $\mathcal{C}$ and $\mathcal{D}$ respectively, since $\mathcal{G}(D)$ is an object in $\mathcal{C}$, $M_{\mathcal{C}}(C, \mathcal{G}(D)) = Hom_{\mathcal{C}}(C, \mathcal{G}(D))$ makes sense as an object in **Ens**. Similarly, $Hom_{\mathcal{D}}(\mathcal{G}(C), D)$ also makes sense. So, this $\eta_{C,D}$ must be a bijection from the first set to the second.

Now, when I say natural transformation, automatically, there are many other things built in here. See, first of all, a bijection for every C and D will be ready that itself a lot of data, but it is when I say it is a natural isomorphism, what does it mean? Let me recall the definition.

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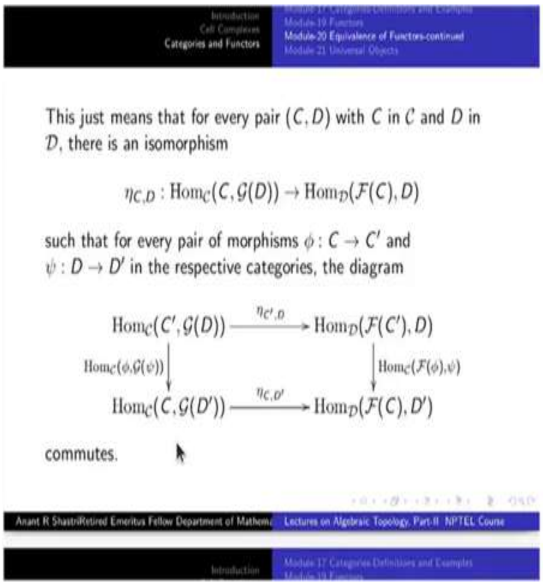

That just means that for every pair (C, D), with C in $\mathcal{C}$ and D in $\mathcal{D}$, $\eta_{C,D}$ is an isomorphism which depends on both C and D, $\eta_{C,D}$ from $Hom_{\mathcal{C}}(C, \mathcal{G}(D))$ to $Hom_{\mathcal{D}}(\mathcal{F}(C), D)$. Note that C occurs on the left slot and D occurs on the right. That is the way to remember.

You could have η itself in the other direction here, no problem. Indeed that would be η inverse no problem. I have not yet finished the description yet. These are isomorphisms such that whenever you have a morphism ϕ from C to C' in $\mathcal{C}$, and a morphism ψ from D to D' in $\mathcal{D}$, you must have a commutative diagram as shown. Once we have, ϕ from C to C' here and $\mathcal{G}(\psi)$ from $\mathcal{G}(D)$ to $\mathcal{G}(D')$, there $Hom_{\mathcal{C}}(\phi, G(\psi))$ is a double functor which assign α to $G(\psi) \circ \alpha \circ \phi$. Similarly the other functor $Hom_{\mathcal{D}}(F(\phi), \psi)$ is defined. (There is a typo here in the slide.)

So, this entire diagram must commute. The most difficult thing to obtain here is this natural transformation like this. It has so much of data built in this one. So, unravelling the definition is of the major difficulty in understanding this. You have to practice it at this level itself. Then you will see that you have been given so much of data in one single statement.

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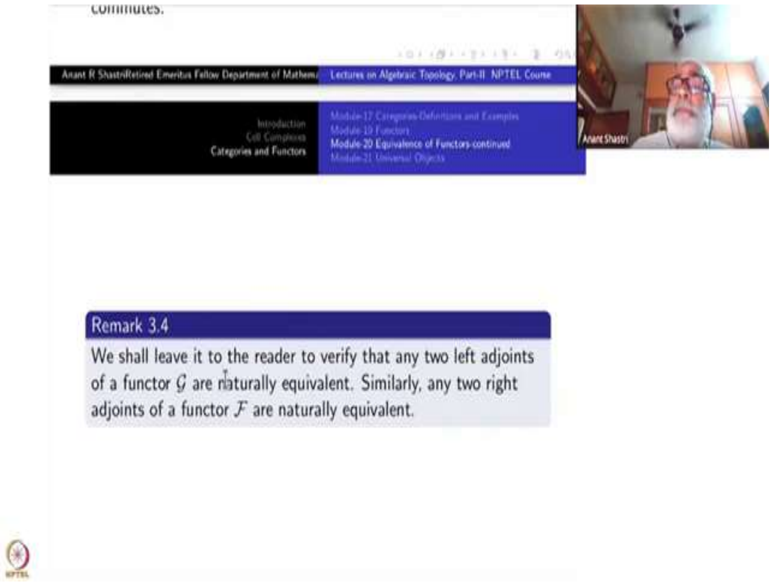

So, I have put it here some remarks. We shall leave it to the reader to verify that any two left adjoint functors of $\mathcal{G}$ are naturally equivalent. This is not a very difficult exercise. But the difficulty is that you may not be knowing what to do with this kind of statement. You may do very well in computing even very difficult ones. You are familiar with that kind of mathematics where you have to show left hand side equal to right hand side. This kind of kind of mathematics is new to you. That may be one of the reasons why you have difficulty with it.

Secondly, even if I explain all this, it will remain almost as difficult as it was. But maybe slightly less that is all, because you have not spent much time on this one. And that is the reason, I am leaving it as exercise. Find out yourself, unravel these hypotheses and definitions, take some simpler special cases, and see what it gives. If in early attempts you cannot get something, come back again and read again, maybe after some time.

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I can give a number of exercises to you later on as assignment to you and so on. Right now, this is a remark which not even a well-defined exercise. But I am imploring you to try to work out this one. That two left adjoints for the same functor will be naturally equivalent to each other. Same for right adjoints. Try to prove that. First you must think clearly what is to be done. Then it will be easier. Easier for me to explain as well.

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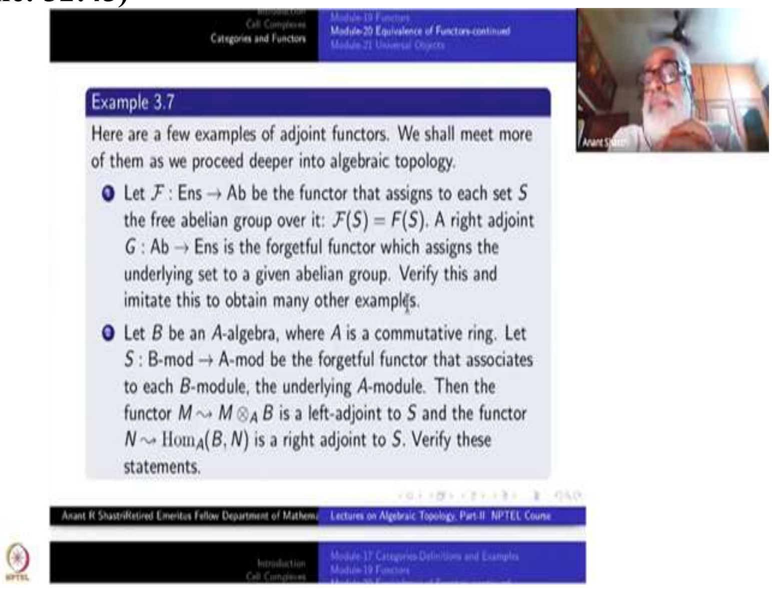

Right now, let us give two examples of these functors and then stop. Examples of adjoints. The first one is the functor $\mathcal{F}$ from sets to abelian groups, **Ens** to **Ab**, that assigns to each set S, the free abelian group $\mathcal{F}(S)$ over S. So what is the definition? (Note the notation $\mathcal{F}(S)$ is used usually for free groups, but I am taking free abelian group here. So, please ignore the notation in the slide). I am assigning for each set S, the free abelian group over it. Now a adjoint $\mathcal{G}$ from **Ab** from **Ens** is got by just taking the forgetful functor, namely, take a group and forget its group structure and look at underlying set. That is a forgetful functor. So I want to say that $\mathcal{G}$ is a right adjoint to $\mathcal{F}$. That is all. Similarly, you can consider the functor which assigns free group and think of a right adjoint to it.

The second one is a little subtler, that is all. But here you have to know a little more commutative algebra of tensor products and so on. Let A be a commutative ring and B be an algebra over A. (Like a polynomial algebra with integer coefficients or a tensor algebra and so on, it is like vector space with a compatible, commutaitve multiplicative structure also). Let A-mod (respectively, B-mod) denote the category of modules over A (respectively, over B). Consider the forgetful functor $\mathcal{S}$ from B-mod to A-mod which assigns to each B-module the underlying A-module. A left adjoint to this is the functor M goes to $M \otimes_A B$.

Similarly, a right adjoint to $\mathcal{S}$ is the functor N goes to $Hom_A(B, N)$. B is an algebra so B is also a ring. So, you can take the modules over that, that is the category B-mod. You can think of B as an A-module so $Hom_A(N, B)$ makes sense.

So, a typical example is: B=complex numbers form an algebra over the A=reals. You can take a complex vector space and treat is a real vector space. That will give you a forgetful functor. So, this is just an example but not exactly the same. Here B-mod to A-mod, we have forgetful functor that associates to each B-module, the underlying A-module, because B is somewhat a larger ring than A. Since scalar multiplication by elements of B make sense and so scalar multiplication by elements of A also makes sense. That is the meaning of this and that is all. So, that is one functor. Then you consider the functor M going to $M \otimes B$ over A.

Now, here B is thought of as a module over A, as a left-module and M is taken to be a right module. This is possible because A and B are commutative. $M \otimes B$ becomes a right B-module. It is a left adjoint this functor $\mathcal{S}$. And the functor N going to $hom_A(B, N)$ going means what? All A-homomorphisms from B to N. This is a right adjoint to $\mathcal{S}$.

But there are two of them here. Of course, the tensor product itself is left adjoint to $\mathcal{S}$. So, $\mathcal{S}$ itself is a right adjoint. But of couerse a right adjoint to S will be something else. So, for this examples, you will need to understand meaning of tensor product, algebra and so on. So, it is only for those people who know enough algebra. But the first about abelian groups all of you must be able to verify the details.

Next time we will define we will study some general topics of what are called as universal constructions. Such as the construction of free abelian groups. We shall do that kind of things in a categorical language. Thank you.