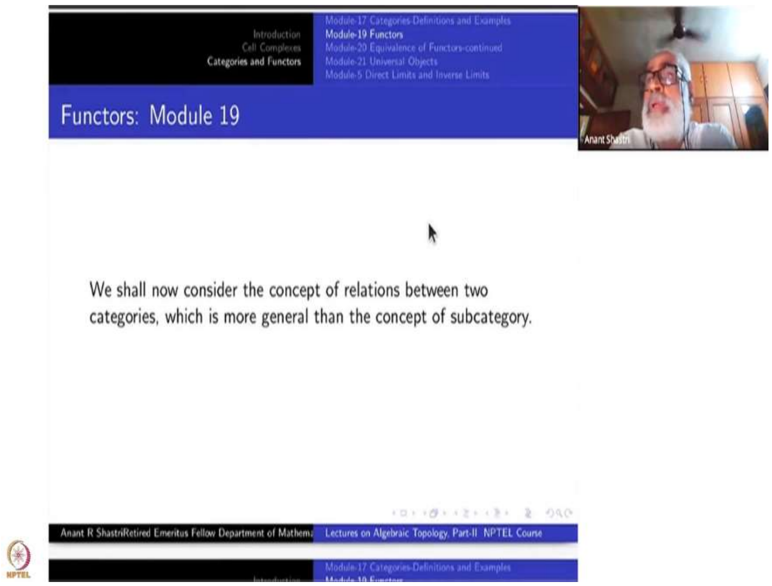
Introduction to Algebraic Topology Part – II

Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology – Bombay

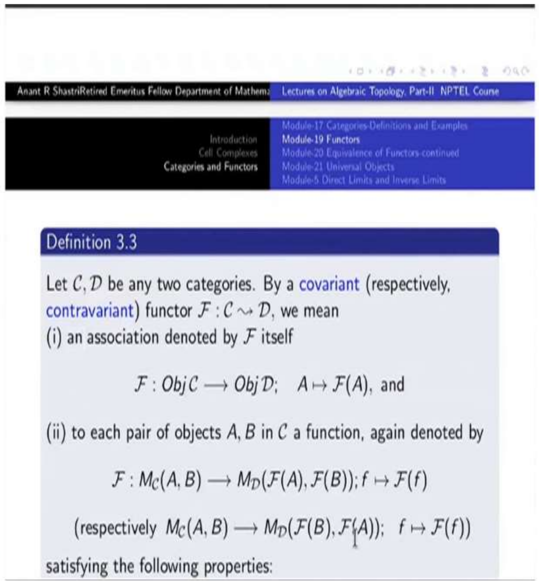
Lecture – 19
Functors

(Refer Slide Time: 00:11)



Having introduced categories and lots of examples now we want to study relations between them. One simple relation we have already introduce, namely, subcategory. When is one category a subcategory of the other? It is just like the inclusion maps between sets. But then we want to have more maps. So, now let us have little more generalization of this concept of subcategory, okay? That leads us to functors.

(Refer Slide Time: 00:50)



So, $\mathcal{C}$ and $\mathcal{D}$ be any two categories. There are two types of functors, one called covariant another one called contravariant, okay? Classically, this are the names used. There is no way to change them, though many people have objections for these names, It does not matter but the names stand. So, the people who object for these names, they want to say that this `contravariant' should be called simply `variant' and other one is covariant and there is no need to bringing `contra'. it is the variant that is more fundamental concept. This argument is similar to the practice where we have algebra to co-algebra, or finite tocofinite, dimension and co-dimension etc., where the prefix `co-' indicates a definite meaning. So, the contravariance should actually be called `variance' and the covariance is then co- of the variance.

However, what actually happened hystorically is that the covariance was studied in beginning and then there attention went to this other concept which is dual to covariance. Obviously they would like to call it co-covariance and hence the name contravaince was introduced and it has stuck. This is just an unfortunate historical background, okay?

So, let us see what is the difference between these two concepts. First let us look at the a definition of a covariant functor.

A functor $\mathcal{C}$ to $\mathcal{D}$ is just like a function one set another or morphism form one object to another. But we will not write just an ordinary arrow, a simple arrow which used to denote a morphism.

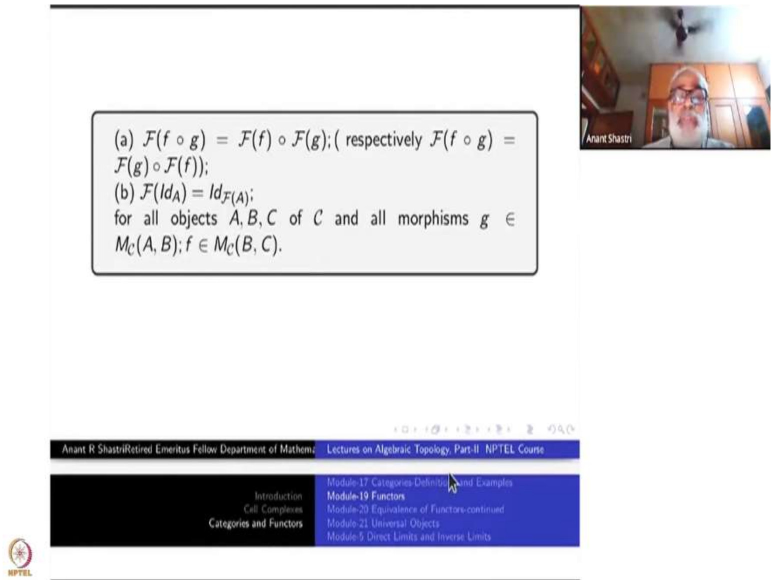
morphism like a function that will not be indicated. So, we have to have some other notation. So a twisted arrow is used, okay? That will be the notation for a functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$. By this, we mean

i) an association denoted by $\mathcal{F}$ itself again, okay?
(several times, you will have to use this notation) $\mathcal{F}$ is an association from objects of $\mathcal{C}$ to objects of $\mathcal{D}$. Both $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Obj}(\mathcal{D})$ are some classes or some families, and so $\mathcal{F}$ is an association. If these families were sets then we could have called $\mathcal{F}$ a function. You cannot call it a function just because the domain and co domain are not necessarily sets okay? So, we use the association which you express as follows: for each object A in $\mathcal{C}$, the image will be written as $\mathcal{F}(A)$, under the association $\mathcal{F}$. Okay? So that much we do just like function theoretic notation.

(ii) for each pair of objects, A, B in $\mathcal{C}$, you have objects $\mathcal{F}(A)$ and $\mathcal{F}(B)$ in $\mathcal{D}$, right? On the other hand, you have the sets $M_{\mathcal{C}}(A, B)$, $M_{\mathcal{D}}(F(A), F(B))$, $f \mapsto \mathcal{F}(f)$. So, these morphisms to morphisms, now these are sets and we have a function for which we write $\mathcal{F}$ again or a little more elaborately $\mathcal{F}_{A,B}$ from $M_{\mathcal{C}}(A, B)$ to $M_{\mathcal{D}}(F(A), F(B))$, we are writing the same $\mathcal{F}$, each morphism here is taken to a morphism there. So the image of f under $\mathcal{F}$ is called $\mathcal{F}(f)$ where $\mathcal{F}$: from $M_{\mathcal{C}}(A, B)$ to $M_{\mathcal{D}}(F(A), F(B))$.

(This is a difference here for contravariant functor. What is happening is that the arrows are going the other way, viz., $\mathcal{F}$ from $M_{\mathcal{C}}(A, B)$ to $M_{\mathcal{D}}(\mathcal{F}(B), \mathcal{F}(A))$, i.e., instead of from $\mathcal{F}(A)$ to $\mathcal{F}(B)$, we have $\mathcal{F}(f)$ from $\mathcal{F}(B)$ to $\mathcal{F}(A)$. Okay? That also is written as $\mathcal{F}(f)$, but domain and codomain are interchanged in contravarient, okay? That is property (ii) okay? Property (i) is the same for both covariant and contravariant functors; here is (ii) for contravariance, f and $\mathcal{F}(f)$ work in opposite direction.

(Refer Slide Time: 05:28)



(iii) Now third condition is about the composites $\mathcal{F}$ of the composites is composites of the $\mathcal{F}$, both for covatiance and contravariance, but you are taking them in the opposite directions, the correct one which makes sense. So, $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for covariance and $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for contravariance.

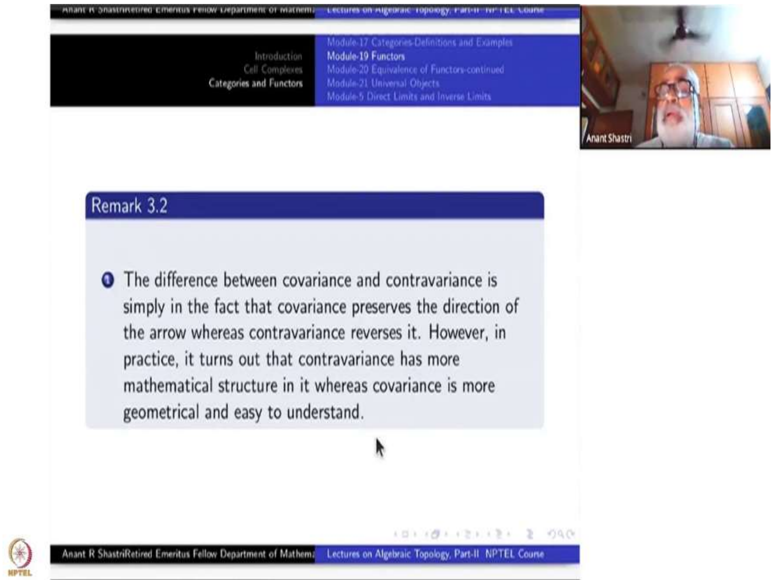
Finally there is one more condition which is very important: $\mathcal{F}$ of identity is identity of the corresponding object: $\mathcal{F}(Id_A) = Id_{\mathcal{F}(A)}$.

This must be true for whenever A, B, C etc are objects in $\mathcal{C}$ and for all morphisms f and g wherever they are inside the category $\mathcal{C}$, okay? So, basic thing is that there is an assignment to each object in $\mathcal{C}$ an object in the other category $\mathcal{D}$ and to each morphism in $\mathcal{C}$, there is a morphism in the other category v. This assignment must respect the composition law, and the identity morphisms must go to Identity morphisms.

This is just like a homomorphism of groups; identity goes to identity, compositions should go to composition that is the property of homorphisms. So that has been generalized here okay? So such a thing is called a functor. If the arrows are reversed, you say it is a contravariant functor. Still it is a functor, covariant or contravariant. What you have to know and keep trach of is whether the arrows are going in the same direction or are reversed, okay?

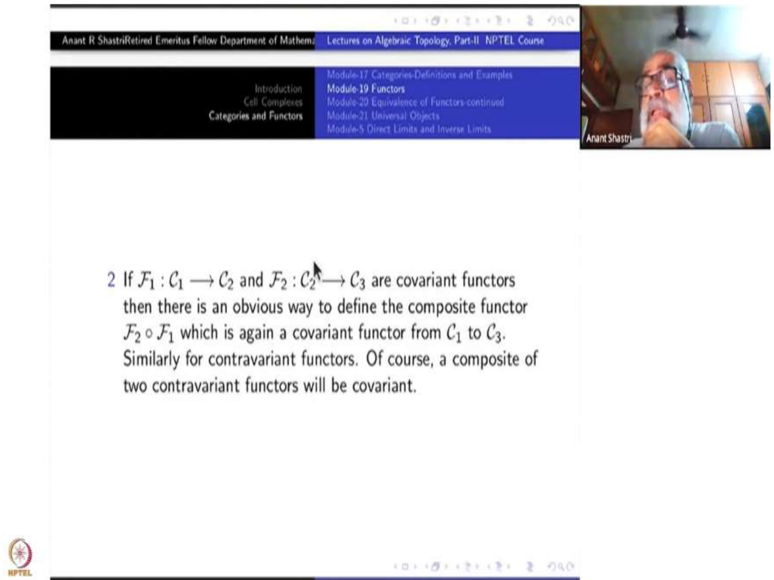
The point is that we are already familiar with plenty of examples. This definition has come much, much later in our education. Now we are just adjusting our vocabulary to the new definition okay, which is actually an ideal thing. So, let us examine what are all known things for us which fit into this definition, okay?

(Refer Slide Time: 08:09)



So, I already told you the difference between covariance and contravariance is simply the fact that covariance preserves the direction whereas the contravariance reverse it. However in practice it turns out that contravariance has more mathematical structure in it whereas covariance is more geometrical and easy to understand. And obviously, historically study of contravariance was carried out much much later because covariance was easy to understand.

(Refer Slide Time: 08:52)

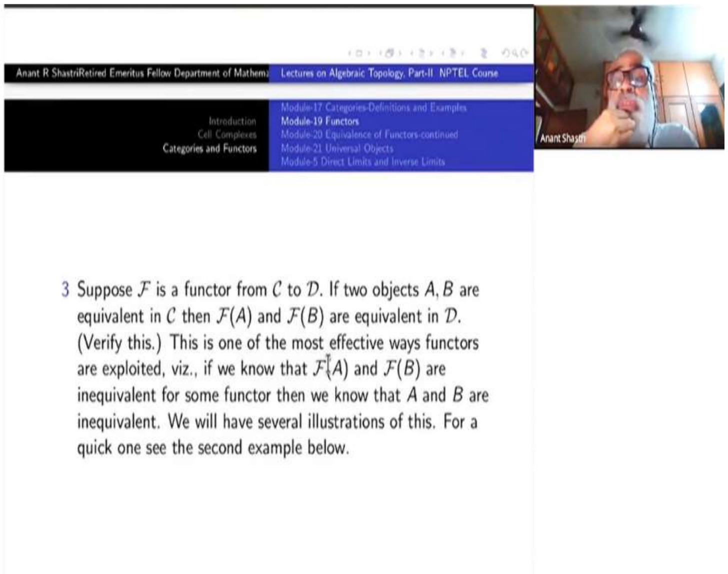


The second point is the following: Suppose $\mathcal{F}_1$ from $\mathcal{C}_1$ to $\mathcal{C}_2$, and $\mathcal{F}_2$ from $\mathcal{C}_2$ to $\mathcal{C}_3$ are both covariant functors. Then there is an obvious way to define a composite of these functors, viz., take a take an object A in $\mathcal{C}_1$ and associate directly an object in $\mathcal{C}_3$, how? by taking $\mathcal{F}_2(\mathcal{F}_1(A))$. Likewise, if you have a morphism f from A to B you will take $\mathcal{F}_2(\mathcal{F}_1(f))$ as the morphism from $\mathcal{F}_2\mathcal{F}_1(A)$ to $\mathcal{F}_2\mathcal{F}_1(B)$. okay?

So, it is obvious that composition defined is this way makes sense. And if both of them are covariant, then the composition is covariant from $\mathcal{C}_1$ to $\mathcal{C}_3$. Similarly for contravariant functors However, if you compose two contravariant functors, it will become a covariant functor. Now this is one of the reasons why covariance has to be studied first before contravariant, okay?

So, covariance is easy in that sense but it is mandatory for you to study that before studying contravariance because if we compose two contravariant functors you will get a covariant functor okay? Composite of a contravariant functor and a covariant or a comopsite of covariant functor and contravariant one will be contravariant. So, contravariant functors are similar to anti-homomorphism in group theory and anti holomophic functions in complex analysis, okay? So, there are examples of these phenomena all over mathematics, alright?

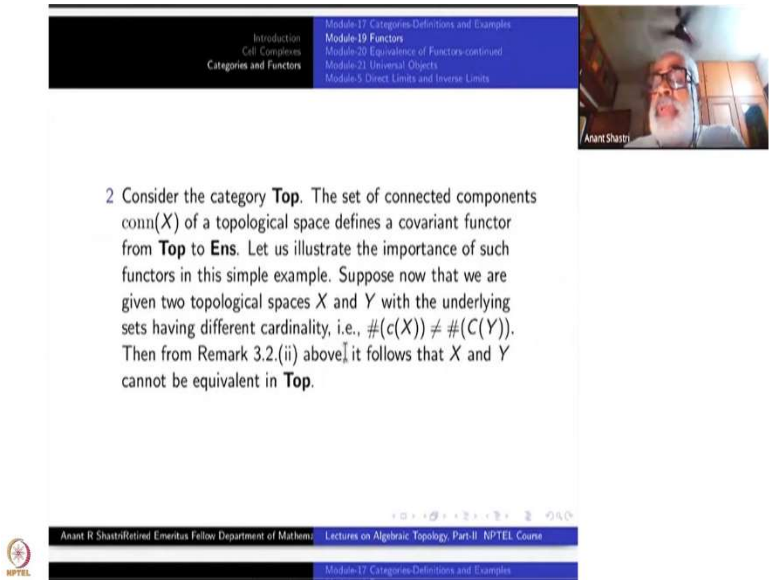
(Refer Slide Time: 11:12)



So, now suppose $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$. If two objects A, B are equivalent in $\mathcal{C}$(remember what is an equivalent in $\mathcal{C}$? There is a morphism from A to B and another from B to A which are inverses of each other; if there is a such a morphism then the two objects are equivalent) then $\mathcal{F}(A)$ and $\mathcal{F}(B)$ will be automatically equivalent in $\mathcal{D}$. Namely, if f from A to B is an equivalence with g as its inverse, then $\mathcal{F}(f)$ will be an equivalence $\mathcal{F}(A)$ to $\mathcal{F}(B)$ with $\mathcal{F}(g)$ being its inverse.

So, this is one of the most effective way functors are exploited, namely, if you know $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are not equivalent, then A and B are not equivalent. okay? So, this contrapositive of the statement will be used again and again in practice okay, to derive lots of results. You will have several such illustrations okay? So, already one of them we have discussed right in the beginning in part I. I can repeat it here, okay? First let us do that example and then come back.

(Refer Slide Time: 12:49)

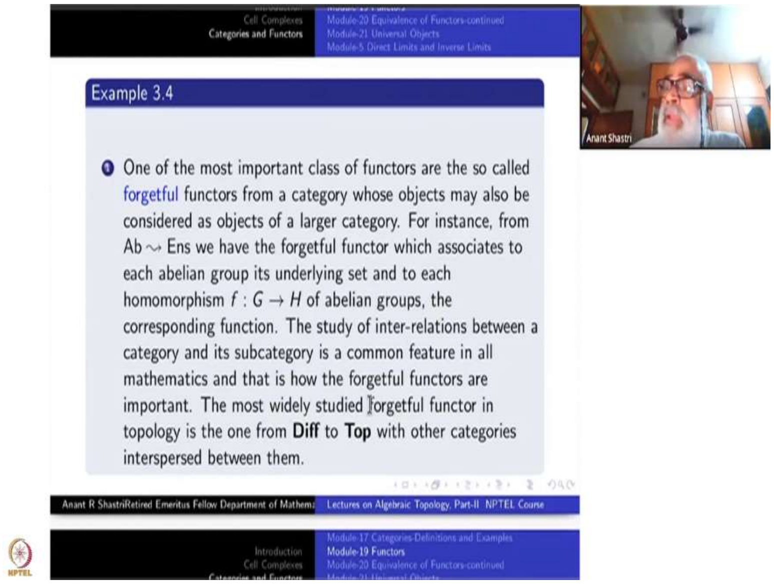


Namely, from the category **Top** to **Ens**, consider the functor, the set of path connected components of a space $\mathrm{conn}(X)$. If you have a continuous function f from A to B, then automatically it induces a set theoretic function $\mathrm{conn}(f)$ from path connected components of A to path connected components of B. So, one can easily verify that this association is a covariant functor. Under the identity map, the corresponding path connected components will go into the same path connected components and so the induced function is also identity map of the path connected components, okay. So, the collection of connected component is a set, if two sets are equivalent, means that just that their cardinalities are equal.

Suppose now the cardinalities are different okay? Then you can conclude that the original space is X and Y are not homeomophic with each other, (in fact, they are not even homotopy type of each other) because homeomorphism implies the set of path components are in bijection with each other. So, this is the very simplest way how a functor can be useful. This is used several times in ordinary topology.

Like look at some thing you may have proved earlier. Suppose you have the union of two axes. Why it is not a manifold of dimension one? Many other such instances, such as the topological classification of alphabets as subspaces of $\mathbb{R}^2$, okay?

(Refer Slide Time: 15:35)



So, let me come back here to one of the most important class of functors, the forgetful functors. What is the meaning of this forgetful functors? It is a very strange name here okay? Start with a category whose objects may also be considered as objects of a larger category okay? For instance, from the category **Ab** of abelian groups to the category o **Ens** of sets. What you have doing? An abelian group is a set together with a particular structure on it. Only looking at the underlying set you get this functor.

Similarly, you can take a vector space and forget the vector space structure and only look at the underlying set. That gives another `forgetful' functor from $Veck_k$ to **Ens**. Like this you can have several such instances okay? You can start with an abelian group and forget that it is abelian but retain the group structure, then you will a forgetful functor into the category of **Gr**.

So that functor is called a forgetful functor. The whole idea that often we start with a sucategory and pass to the larger category but keep the binary operation the same. Only certain conditions, certain extra structures in the smaller category are ignored. That is the meaning of these forgetful functors okay. So, study of interrelation between categories and subcategories is a common feature in all mathematics.

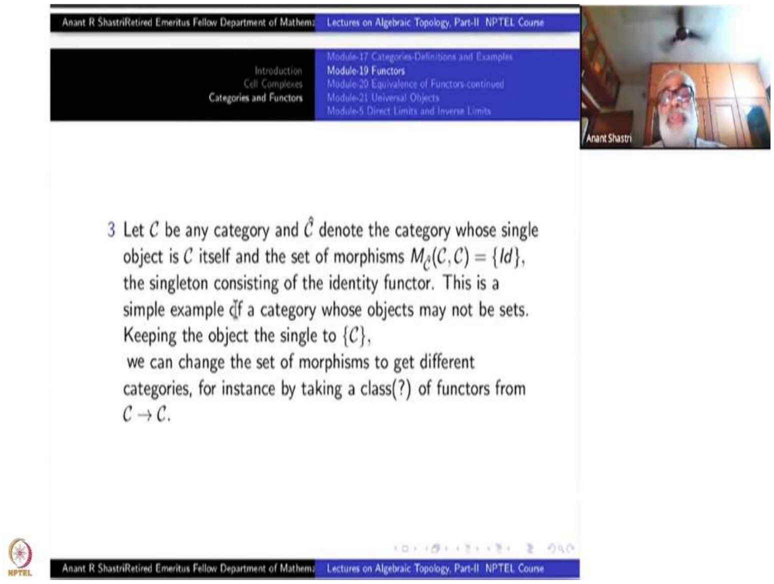
One of the most widely studied forgetful functor in topology is the one from **Diff** to **Top**. Begin with a manifold with its differential structure but you can just consider it as topological space. This is a very, very important forgetful functor. As topological spaces if two maifold are inequivalent, i.e., non homeomorphic, then there is no chance that as differentiable manifold they will be diffeomorphism to each other right?

So, first you try to understand them as topological spaces by forgetting the differential structure. So, whenever you are in trouble, you may use extra structure. For instance, given a smooth function between two smooth manifolds, while studying some property of a C^1 function, if you have to use higher order derivatives, then only you appeal to the smoothness of the manifolds and consider functions which are twice differentiable etc.

Often, the concept of forgetful functors is used by us without even being aware of it. Giving a name to it and pinpointing that this is what is happening makes the concepts much more clear and much more powerful. That is all. It has its advantage okay?

Long, long back, I learned this as a student of chess. I used to play chess reasonably well, without knowing any of chess theories from any books. One of my friend used to beat me easily. One day when my chess friend explained me certain terms and names to certain types of moves, such as a pin, a fork and a discovered attack etc. I said to myself, Oh! after all I am doing all this and there is nothing new. But after that realization, my score with him became far better than what it was. So that made me read some chess books and chess theory also okay? So that is the story of what happens when you have better knowledge of the weapon, the tool that you are using, whether you are using 2-sided weapon or not or how to hold it properly so that the strokes will be more powerful.

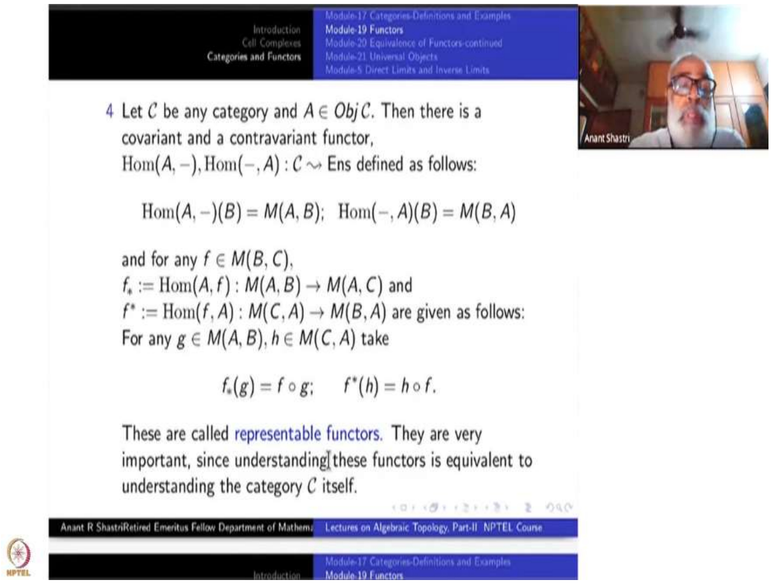
(Refer Slide Time: 20:30)



So, in this way, not only forgetful functors but many of these categorical notions will help you to make you a better mathematician, if not teach you any new mathematics as such okay? So, let us go ahead. Consider the category **Top**. I have already told you this example. As you know cardinality of the underlying set and cardinality of the connected components etc, all these things are some kind of functors okay? There are several of them.

Because the rest of them composition etc obviously defined and what is the objects set singleton what is the element of this one that is not a set possibly because we know category is set of all sets is not a set now that becomes a one single object, the objects themselves may not be sets. So, this is an example for that once you have this you can make many more examples of going to just to illustrate that objects of a category may not be sets.

(Refer Slide Time: 22:52)



Now let $\mathcal{C}$ be any category. I want to construct another category out of this, namely, let $\hat{\mathcal{C}}$ denote the category whose single object is $\mathcal{C}$ itself okay, the singleton $\{\mathcal{C}\}$ is the $\mathrm{Obj}(\hat{\mathcal{C}})$. There is only one object. I have to define morphisms. Take $M_{\hat{\mathcal{C}}}$ from $\mathcal{C}$ to $\mathcal{C}$ to be the singleton set consisting of the identity functor. So, this is an example of a category in which objects may not be sets.

I mean I could I give many other examples, more complicated examples of categories in which objects need not be sets. However, I waited for this particular one so that I can tell you something else. Now we know what is the meaning of a functor, and we have made a functor as a morphism. A morphism occurs within a category from one object to another object, whereas a functor is from one category to another category. However, here is an example wherein a functor has become a morphism of some other category.

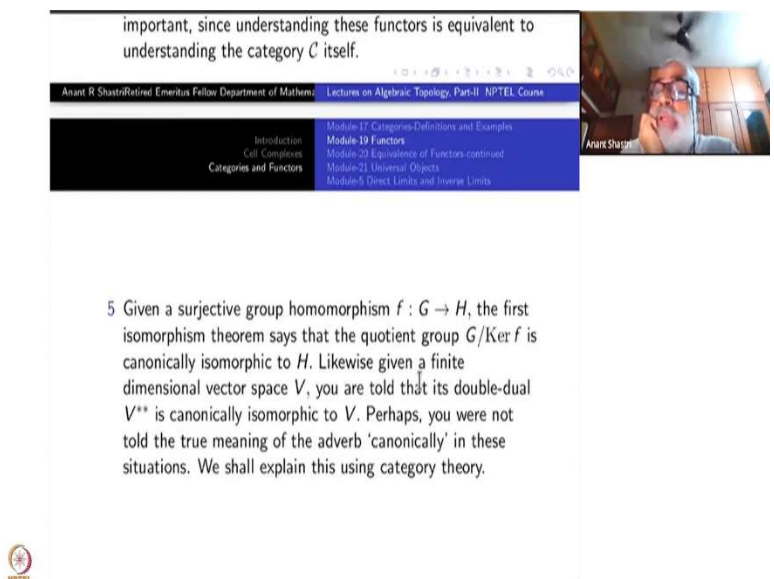
Now I am going to define two functors here one is covariant and contravariant. These are not for fun. These examples are very, very important. But the way I introduce them looks like I am simply cooking something. Start with a category $\mathcal{C}$ and fix an object A in it. Vary the objects in the second slot here. For each object B of $\mathcal{C}$, you consider $Hom(A,B) = M_{\mathcal{C}}(A,B)$. (Or you can do the other way round, namely take the assignment A to $Hom(B,A) = M_{\mathcal{C}}(B,A)$. There are two of them.)

So, this will be an assignment from the category $\mathcal{C}$ to the category of sets **Ens** because you know that the morphisms always form a set, okay? So, this definition of $Hom(A, -)$ operating upon B is $M_{\mathcal{C}}(A, B)$. So, this dash is the second slot on the right. So, B occupies that slot. (Here, B comes on the left slot, $Hom_{\mathcal{C}}(B, A)$ okay?)

Now having defined the association on the objects I have to define the association of morphisms. Take any morphism f from B to C in $\mathcal{C}$, define $f_* = Hom(A, f)$ from $Hom(A, B)$ to $Hom(A, C)$ (and $f^* = Hom(f, A)$ from $Hom(B, A)$ to $Hom(C, A)$ as follows: For any g in $Hom(A, B)$ put $f_*(g) = f \circ g$ (respectively for any $h \in Hom(B, A)$, put $f^*(h) = h \circ f$).

You will see easliy that both of them are functors, first one is covariant functor and the second one contravariant okay? So, these are called representable functors of the category $\mathcal{C}$. I have cooked up these functors right? In fact, from the singe category, so many functors viz., for each object A there are two functors one covariant and another contravariant. The whole idea is that if you know these two functors for all object A in $\mathcal{C}$, then you know the category $\mathcal{C}$ perfectly. In other words, these two functors will bring out various properties present inside $\mathcal{C}$. So knowing these two representable functors is the key to understand a given category okay?

(Refer Slide Time: 27:17)



So, this is where I will stop today. Next time I have to explain the canonicalness of the assignment G to $G/Ker(f)$ in group theory and the assignment V to V^{**} in vector spaces. So, this is where I will stop today thank you.