Introduction to Algebraic Topology (Part-II)
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Lecture – 18
More Examples

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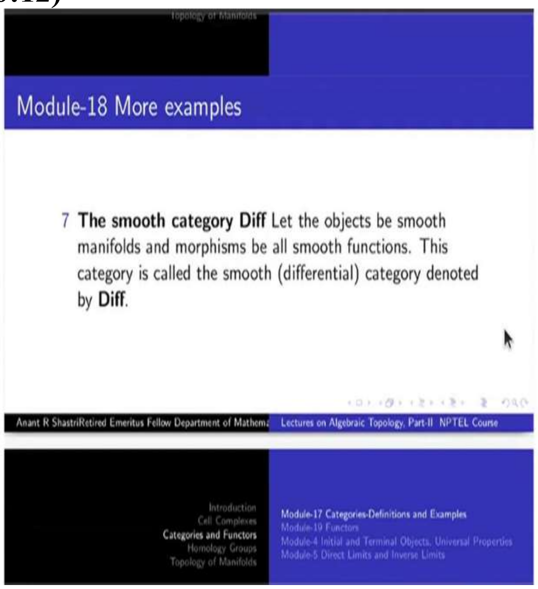

We should now continue to give some more examples, familiar examples of categories. We studied the topological category, the homotopy category and the simplicial category and CW category. The more important category of topological spaces is the smooth category denoted by Diff by me, but other people may denote differently, notation is not all that uniformized. So, what are the objects of this category? They are smooth manifolds and morphisms are smooth functions.

So, this category is called the smooth category and is denoted by **Diff**. Some of you who have not studied any differential topology, this may not be quite familiar. In any case, we are not going to do anything in this category right now, in this course. However, I would like to introduce a closely related category to this one which is actually somewhat larger than this category **Diff**, in some sense and you will be comfortable with it.

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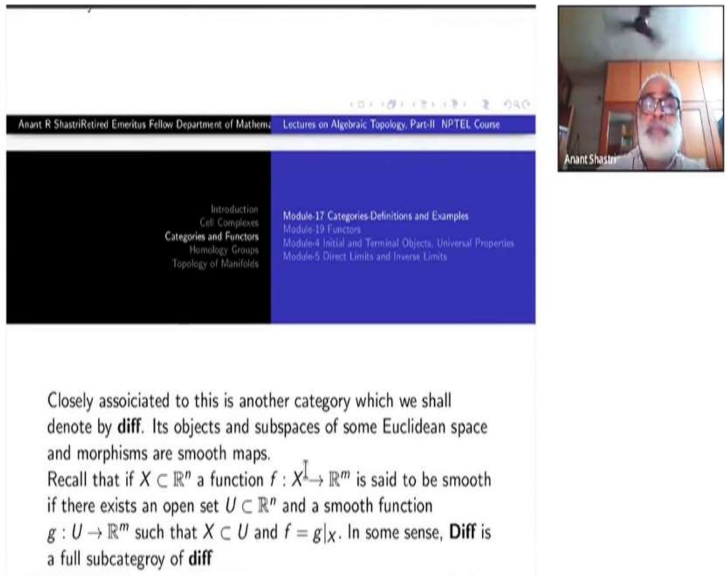

So, that is denoted by small `**diff**'. It is actually larger than **Diff** in some sense. The objects of this **diff** are subspaces of some Euclidean space. There is no other condition. All subspaces Euclidean spaces are allowed here. And what are the morphisms from one object to another object? They are all smooth functions. Now, this word `smooth functions' on arbitrary subsets of Euclidean space you may not be familiar to you. So I am recalling it for those who know it and for those who do not know this, they will learn.

A function from X to $\mathbb{R}^m$, where X is a subspace of some $\mathbb{R}^n$ is said to be smooth if there exists an open subset U inside $\mathbb{R}^n$, (not X is a subset of $\mathbb{R}^n$) and a smooth function g from U to $\mathbb{R}^m$ which extends f. X is subset of U and U is open and g is a function on U to $\mathbb{R}^m$, g restricted to X must be f. Why I am making such a definition. Because you all know the meaning of first-differentiability, twice-differentiability, smoothness etc., of functions defined on open subsets of $\mathbb{R}^n$. But you may not know what is the meaning of differentiability of functions on arbitrary subsets. This is the meaning. With this definition, **diff** category becomes very familiar to you from your calculus courses. So, this is one of the important categories.

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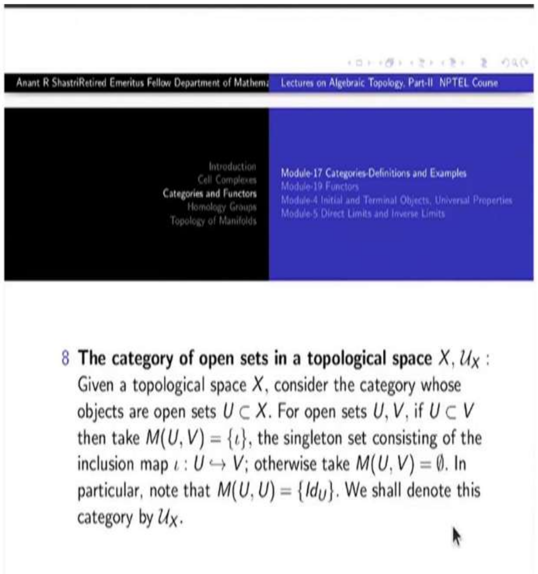

Now, the category $\mathcal{U}_X$ of open sets in a single topological space X. This is a wonderful category. But this is again not going to be used in this course, but this will be useful when you do cohomology theory, sheaf theory and so on. What is $\mathrm{Obj}(\mathcal{U}_X)$? For this category the objects are nothing but all open subsets of X. If you denote the space by $(X, \mathcal{T})$ then $\mathcal{T}$=$\mathrm{Obj}(\mathcal{U}_X)$. So, object sets are open subsets, each object is an open subset of X. What is a morphism from U to V ? There are two cases.

Whenever U is contained inside V, take $M(U,V)$= the singleton set consisting of the inclusion map. If U it is not contained V, take it to be the emptyset. Note that $M(U,U)$ is the singleton containing Id_U and that serves as a 2-sided identity. So, this category is quite useful in the study of sheaf theory.

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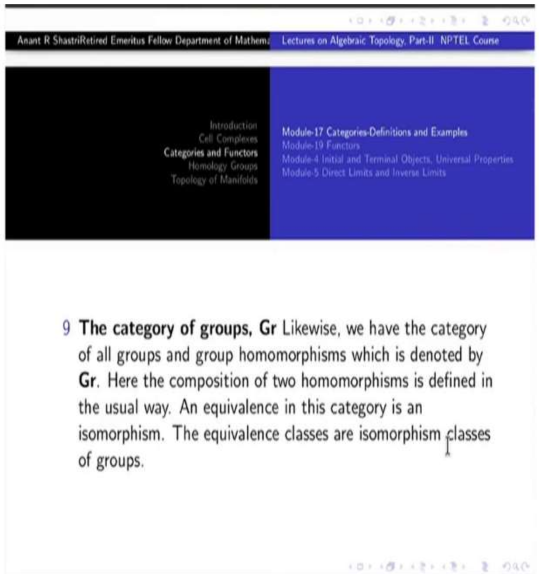

Now, I come to examples from algebra. The first example is the category of groups **Gr**. All groups are taken as objects, morphisms from one group to another group are homomorphism and composition is just like set theoretic composition. Does that convince you that it is a category? Associative law is there and identity law is there and that makes it a category. What is an equivalence here? Isomorphism of groups. What are equivalence of objects? Isomorphism classes of groups.

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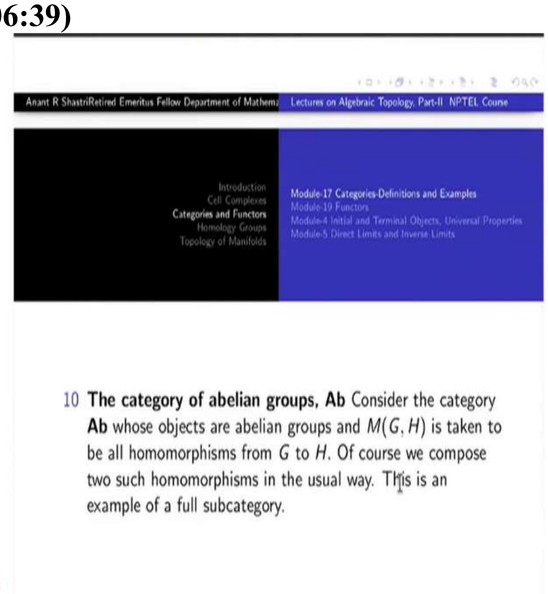

There is a very nice subcategory **Ab** here. Namely, you only take abelian groups as objects, the rest of the things are same as in **Gr**. Namely, if you have two abeliean groups, what is the set of all morphisms, same as in **Gr**, viz, all homomorphisms. No special conditions on homomorphisms between abelian groups. Every homomorphism is fine. And the same identity map is the 2-sided identity here.

So, that is a category which is a subcategory of category of all groups **Gr** and it is a full subcategory, because whenever you have $M_{\mathbf{Ab}}(G, H) = M_{\mathbf{Gr}}(A, B)$, where A, B are abelian groups.

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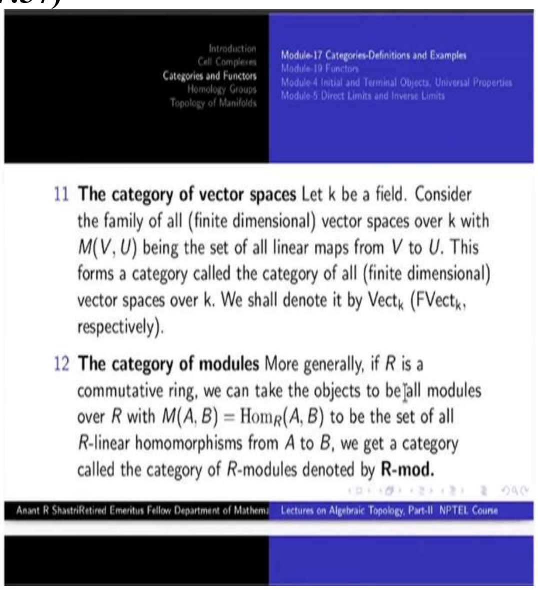

Similarly, another important example which is very familiar to you is $Vect_k$. Take a field k, fix it. Then take all vector spaces over k as the family of objects. What are morphisms? Vector space linear maps. Linear maps from one vector space to another vector space over k. The field has to be fixed to make it a category then if you take the composition of linear maps it will be again a linear map. Identity map is a 2-sided identity. All that is fine.

Similarly, instead of k being a field, if you take $k = R$ to be a commutative ring and objects are what are called modules over R, instead of vector spaces. This is another important category which we will be using all the time in this course. So, what are morphisms between two modules A and B? They are the so called R-linear maps. The ring R is fixed here, you cannot take morphisms from a module over R to a module over R', where R and R' are at different rings. (That is an entirely different concept, which needs deeper algebra to be handled but which may not lead to any category.)

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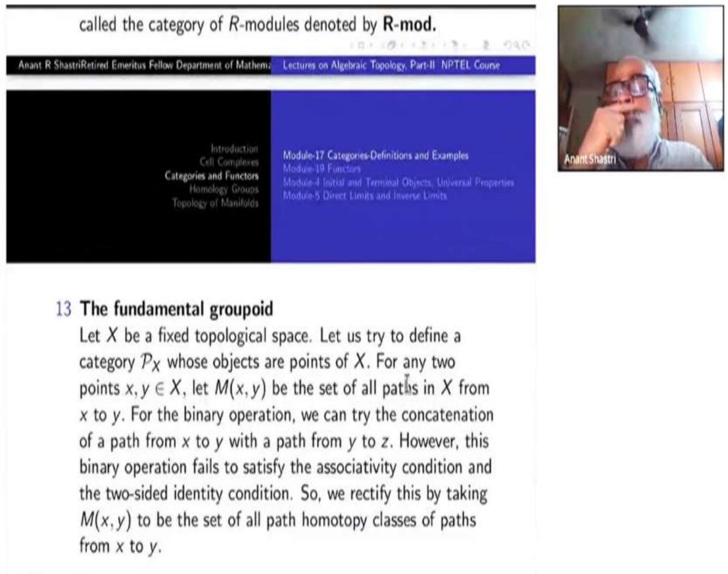

Next comes a groupoid category $\mathcal{P}_X$. It is an algebraic category, however, it depends upon a topological space X. Later on, we will conceptualize this also. But right now, it is just an example of a special kind of category. What is this category? Fix a topological space X. Then what are the objects? Objects are elements of the underlying set X. So $\mathrm{Obj}(\mathcal{P}_X)$ is is nothing but X itself; elements of X are objects. What are morphisms?

$M(x, y)$? What is it when x and y are elements of a set X? Let us try taking $M(x, y)$ to be the set of all paths in X from x to y. A path ω is a continuous function defined on the closed interval $[0, 1]$, we must have $\omega(0) = x$ and $\omega(1) = y$. The collection of all such paths will be taken of $M(x, y)$. They do not look like functions, they are certainly not functions with domain x and codomain y. How I am going to define the binary operation? Intuitively, I am trying the concatenation of paths as the binary operation, whenever they are defined--- if the first path is from x to y and another path is from y to z, then their concatenation is defined, which you may call composition of paths, viz. first you trace the first path and then the second path.

Now, the only problem here is with associativity as well as identity. You the concatenation of paths does not satisfy these laws. Therefore, we have to modify the definition of morphisms, namely, instead of taking all paths, what you do is to take homotopy classes of paths, path-homotopy classes. Recall that path-homotopy is what a homotopy which keep the endpoints fixed. Take each class as one single member of $M(x, y)$. Then you know that concatenation is associative as well as has 2-sided identities. The constant paths play the role of 2-sided identities.

For each $x \in X$, the constant path c_x will be a 2-sided identity, left identity for paths starting at x and right identity for paths ending at x. That will completes the definition of this category, denoted by this $\mathcal{P}_X$. It clearly depends upon the topological space X. The letter P corresponds to paths. This category is called the fundamental groupoid category.

So, the constant map at x, that will be 2-sided and at P for path is fixed which end there as well as path is start from there that will complete a definition of this category. This category is denoted by this $\mathcal{P}_X$ depends upon x this P corresponding to whatever paths and this is called the fundamental groupoid category. Why it is called groupoid because every morphism is invertible you know that if you take a path from x to y, there is a the tracing the path from the other way around y to x.

Why it is called groupoid? Because every morphism is invertible. You know that if you take a path from x to y, there is a path obtained by tracing this path the other way round from y to x. So, $\tau(t) = \omega(1-t)$. That defines a homotopy inverse for the path ω. Therefore, every morphism is invertible. So, such categories are called groupoids in a more general setup. $\mathcal{P}_X$ is one such.

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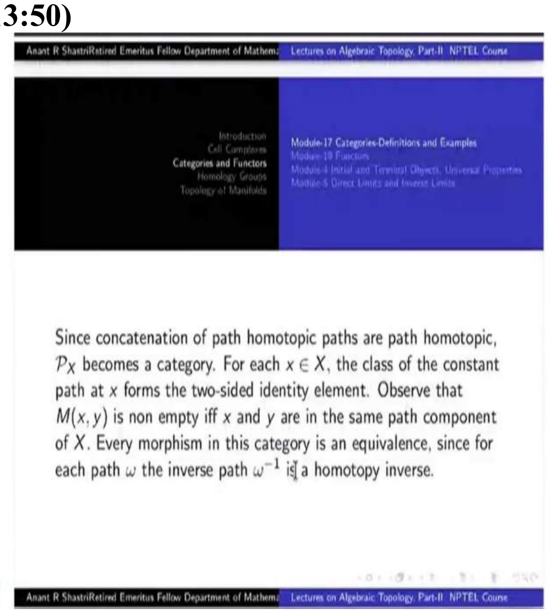

So, concatenation of path homotopy classes, with the class of constant paths as two sided identities, and every morphism is invertibile, these are the things in $\mathcal{P}_X$.

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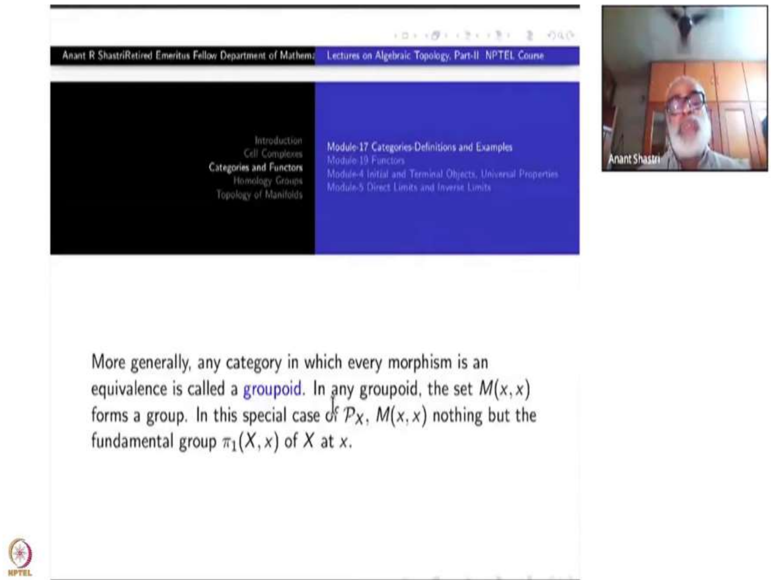

More generally, any category in which every morphism is invertible is called a groupoid. That is why we call $\mathcal{P}_X$ also a groupoid. As a particular case in this category, look at $M(x,x)$. What are the elements? Path-homotopy classes of loops at x. They form a group, namely, $\pi_1(X,x)$. this is the notation. This is called the fundamental group of X at x which we have studied thoroughly in part I. So, more generally, you have fundamental groupoid.

$M(x,y)$ is empty if x and y are in different path components of X. You should observe this also. $M(x,y)$ is nonempty means x and y are in the same path component. So, there is no assumption the space X itself. X may be path connected or may not be connected, but groupoid is well defined.

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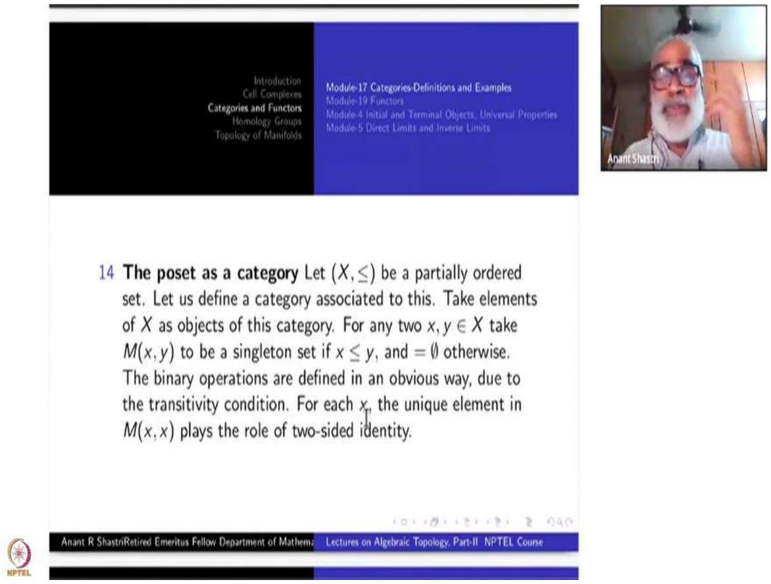

Now, I come to some other kind of examples, slightly from a different area, viz., posets. This is another very important example. You start with a partially ordered set. Associated to that you can define a category and associated to that category you can define a partial order. This way partially ordered sets can be converted into category and you can apply the theory of categories to derive theorems in partially ordered sets, and so on.

So, how do we do this? Let us define a category associated to this $(X, \leq)$, that is the partially ordered set. What are the objects of this new category? They are just elements of X. For any two elements x and y in X, take $M(x, y)$ to be a singleton element if and only if $x \leq y$, otherwise take $M(x, y)$ to be empty.

(This is similar to what we had in the category $\mathcal{U}_X$. For a given in topological space X. we had the category $\mathcal{U}_X$. It is similar to this one I can see. It is actually more general than that. $\mathcal{U}_X$ can be obtained as a special case of this one.)

So, $M(x, y)$ is a singleton if $x \leq y$, otherwise it is empty. The binary operations are defined in an obvious way because of the transitivity of the partial order relation-- if $x \leq y$, and $y \leq z$, then you know $x \leq z$. That is the transitivity of the partial order. Since all the sets $M(x, y), M(y, z)$ and $M(x, z)$ are singletons, the is exactly one function from $M(x, y) \times M(y, z)$ to $M(x, z)$ and we take that to be the composition. Finally, $M(x, x)$ has only one element there, which clearly is a 2-sided identity for this composition. This is the category associated to a given posets, a partially ordered set.

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So, what I want to say is these steps can be reversed to some extent. Start with a category $\mathcal{C}$ whose family of objects is a set X. That condition is a must. (In general, $\mathrm{Obj}(\mathcal{C})$ may not be a set.) Next suppose that for each pair of objects x, y, the set of morphisms $M(x, y)$ is either empty or a singleton. If it is empty, do not do anything. If it is a singleton define $x \leq y$. Further assume that if $M(x, y)$ is nonempty then $M(y, x)$ is empty. Then you get verify that the above order is a partial order on X and the associated catagory as above is the category $\mathcal{C}$ that you started with.

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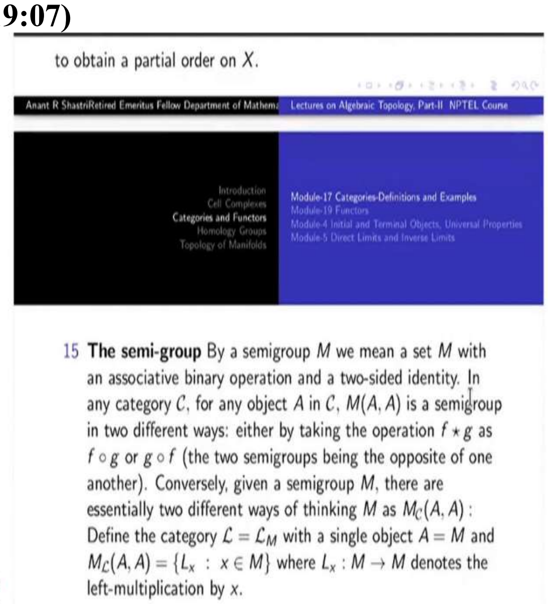

So, now let us look at another example from algebra. Recall that by a semi-group, we mean a set with an associative binary operation, and a 2-sided identity. Like the set of non-negative integers with addition, for example. That is a simple example of a semigroup. In other words, as compared to a group, inverse may not exist in a semigroup. In particular, every group is also a semi group. A semi group may not be a group because inverses may not exist. But inverse, if it exists, it is unique. I don't have to say that separately. So, all groups are semi-groups also. The collection of semi-groups is a larger category than the category of groups, that is all. The semi groups are very important in function theory and so on.

So, in any category $\mathcal{C}$, for any object A in $\mathcal{C}$, $M(A, A)$ is a semigroup in two different ways. What are these ways? By taking the binary operation $f \star g$ in two defferent ways, viz, either equal to $f \circ g$ or equal to $g \circ f$. Both of them will give you a semi-group structure on $M(A, A)$.

So, conversely, suppose you are given a semigroup M. There are essentially two different ways of thinking this as $M_{\mathcal{C}}(A, A)$, where $\mathcal{C}$ is a category. What is this category? The category $\mathcal{C}$ has only A as an object. $M_{\mathcal{C}}(A, A)$ is the semi-group M. But unfortunately, you do not know whether you are going to take the left multiplication as the binary operational or the right multiplication. So, there is that much of ambiguity. So, actually both of them can be taken so that we get two different categories, one is the opposite of the other.

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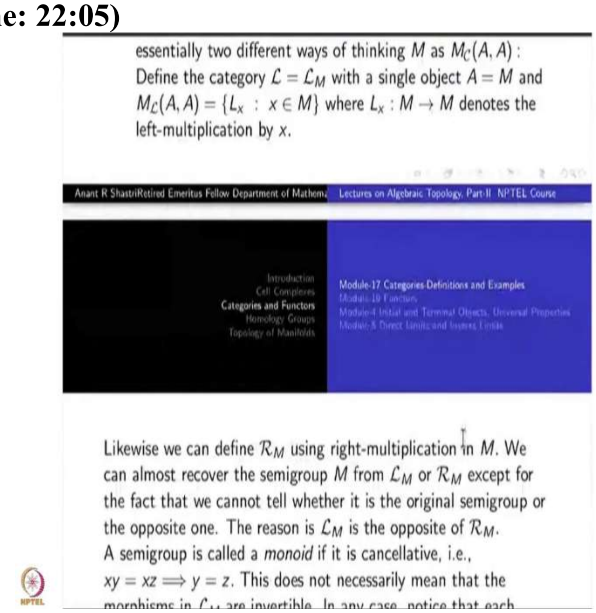

Sometimes in a semi-group you may not inverses but a weaker condition which is quite useful. A semigroup is called a monoid (these are some words wich may differ from author to author, you may not pay much attention to them) if it is cancelative, i.e., $xy = xz$ implies $y = z$. A group is of course, automatically, a monoid. Look at the semigroup of positive integers with the operation of multiplication. It is cancelative. Similarly, the set of non negative integers with the operation of addition.

For your school days, you know that if $5 + 10$ is equal is to something equal to $5 +$ some other number, then that other number must be equal to 10, even without knowing the operation of subtraction. This is built in a child's understanding. This is nothing but the cancellation law. It is easier to understand this cutting down operation, That cutting down operation later on becomes the subtraction and then goes on to define neagive integers. So, cancellation law can be understood without having invertibility.

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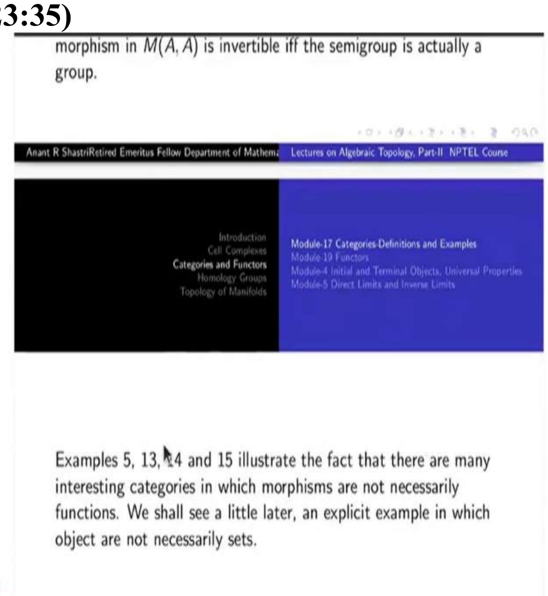

So, the example $5, 13, 14, 15$ illustrate the fact that there are many interesting categories in which morphisms are not necessarily functions. The first such category is example 5. The homotopic category. A homotopy class becomes a morphism. At least here objects are sets.

What is example 13? Let us see. What was example 13? The fundamental groupoid; here morphisms are path- homotopy classes of paths. They are not even functions from x to y of homotpy class of class of functions. They are all represented by functions from the same domain viz., the closed interval $[0,1]$.

In example 14, you see that a morphism is just a binary operation $x \leq y$ that will be a morphism. So, this kind of abstractness has to sink inside your mind slowly. All these are concrete examples in category theory. There are also some categories called abstract categories. You can imagine how abstract they are. Things that we have considered so far are very concrete examples. There are so called abstract categories which we are not going to deal with in this course.

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So, since I already introduced what is the meaning of a subcategory, let us have a quick look at some more examples of subcategories. Now, you have already studied one such example, namely, the **Ab** category of abelian groups is a subcategory of the category of groups. Not only that. It is actually a full subcategory.

Similarly, if you take the category **Diff**. What are the objects? They are manifolds? So, before being manifolds, they are actually topological spaces and a smooth function is automatically a continuous function. Therefore, **Diff** can be thought of as a subcategory of topological category of all the topological spaces and continuous maps. But this is not a full subcategory because all continuous maps may not be smooth. Therefore, it is strictly smaller, not as a full subcategory. So, we have examples as well counter examples also to understand clearly what going on.

Now I come to the category **diff**. And I have told you that this is, in some sense larger than **Diff**. Why? because there is a theorem is differential topology which says that any smooth manifold can be embedded inside some large $\mathbb{R}^n$. When you say `embedded' it will become diffeomorphic to a subset of $\mathbb{R}^n$, for some large n, depending upon the manifold. This is a theorem in differential topology. It tells you that every manifold can be thought of as subspace of some $\mathbb{R}^n$. Therefore, you can think of objects in **Diff** as objects in **diff**.

What are morphisms? They are smooth functions in either case. This way you can think of **Diff** as a full subcategory of **diff**. But inside **diff**, there are more objects which may not manifods, such as the union of x-axis and y-axis in $\mathbb{R}^2$. But you can define (and we have defined) what is the meaning of smooth functions on it. Similarly, if you take a square in $\mathbb{R}^2$, that is not a smooth manifold because it has corners.

Similarly you can take triangle. It is not a smooth manifold. But that are all smooth objects as members of **diff.** You know what is the meaning of smooth functions on them. So, there are many more objects in **diff** which are not smooth manifolds. Smooth manifolds form a small part of objects in **diff**. So, in that sense this capital **Diff** is a subcategory of small **diff**.

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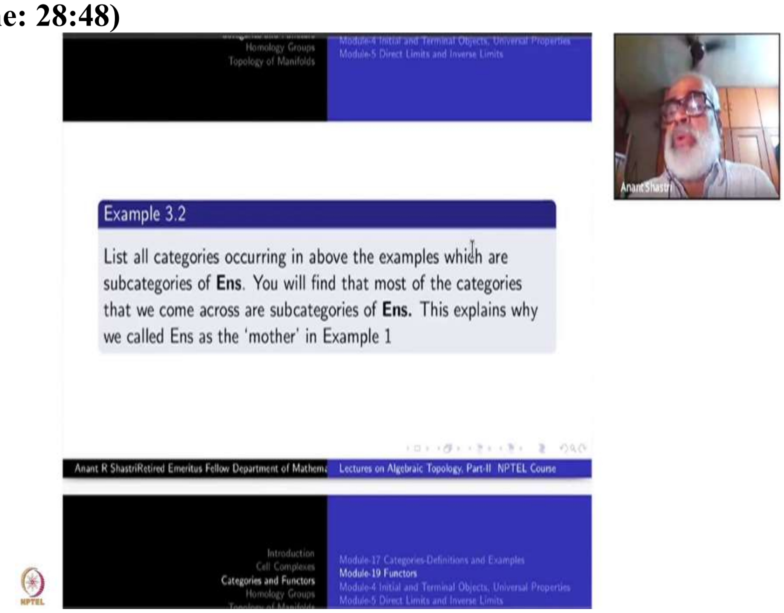

Now, this is an open kind of exercise for you while studying these two models attaching go back and just keep seeing what are the examples of what category is an example can be thought of as a subcategory of the category of sets you will receive many of them are you will see some of them are not and that is why I told this **Ens** is a mother category for many of them. This is what you have to do, then we know which one are which one are not.

Many of these categories structural categories. They are introduced starting with **Ens**. Namely, you start with a set X and then you put extra structure on it, such as a topology, put a binary operation which makes it a group, may put a two or more operations which makes them into a vector space and so on. These are what extra structures. These are all to begin with, they said, even if you put extra structure. Still, the objects are sets.

What are morphisms? Again they are functions. So all structural categories are subcategories of **Ens**. So to understand these comments, you know, understanding takes you more time. You have reread whatever has been done so far. Read again and again, a couple of times. Be sure of what is going on. Thank you. That is all for today.