Introduction to Algebraic Topology (Part – II)
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Lecture – 2
Attaching Cells

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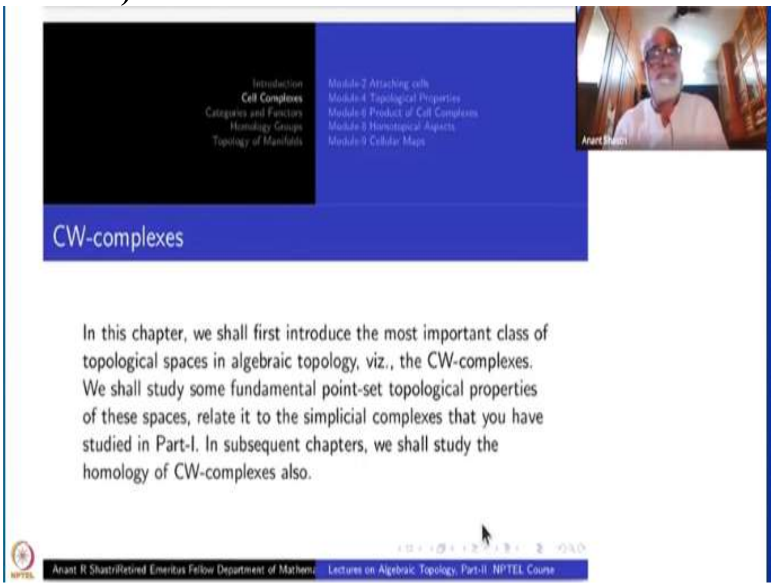

Today we begin the subject the CW-complexes. In this chapter, we shall first introduce the most important class of topological spaces for algebraic topology, namely, CW-complexes. We shall study some fundamental point-set- topological properties of these spaces, and relate it to the simplicial complexes that you have studied in part I. And also in subsequent chapters we will study its relation with fundamental groups and various other things such as homology and so on as and when time permits.

CW-complexes are built up out of nowhere perhaps, by a sequence of operations called attaching cells. The concept of attaching cells itself was studied in part I but I will recall it so as to refresh your memory.

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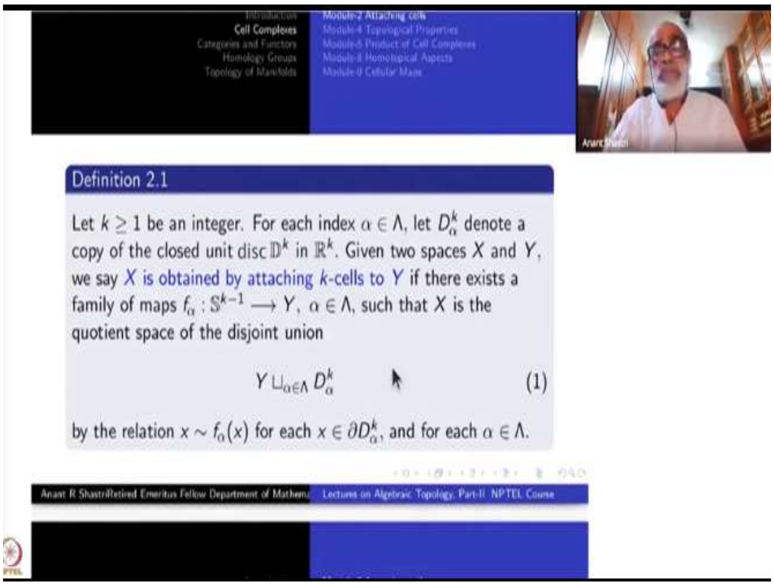

So, fix a positive integer k and indexing set Λ for each index α, choose a copy of the unit disc in $\mathbb{R}^k$. So, all these copies are supposed to be disjoint now, they are copies of the same disc $\mathbb{D}^k$. Now, suppose X and Y are topological spaces, we say X is obtained from Y by attaching k-cells from this indexing set if there exists... I repeat, given 2 topological spaces X and Y, we say X is obtained by attaching k-cells to Y, if there exists a family of functions f_α from the boundary of the disc $\mathbb{D}^k$, which is $\mathbb{S}^{k-1}$, the sphere, to Y, for each α, you must have a function such that X is the quotient space of the disjoint union of Y and all the copies of the disc $\mathbb{D}^k$ by the relation:

x is equivalent to $f_\alpha(x)$ whenever x is in the boundary of $\mathbb{D}^k_\alpha$. Remember boundary $\mathbb{D}^k_\alpha$ is $\mathbb{S}^{k-1}$ and then there is a map f_α; X is a quotient space.

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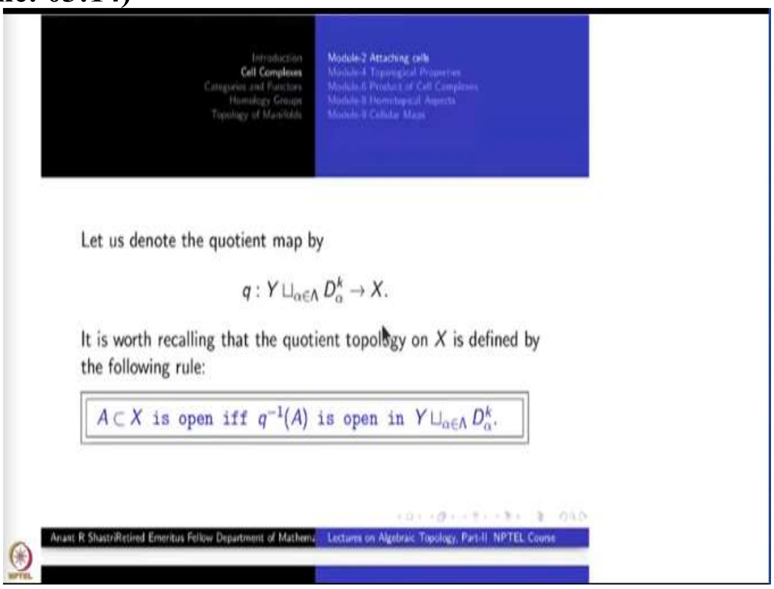

Let us denote the quotient map itself by q. Now remember, what is the meaning of the quotient space, the topology on X? What is it? A subset of X is open if and only if its inverse image under q is open in the disjoint union of these things. The disjoint union of these spaces is given the disjoint topology, we have to remember that.

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The maps f_α are called attaching maps for k-cells. Let us restrict the quotient map q here to each $\mathbb{D}^k_\alpha$ and call them ϕ_α. These ϕ_α's are called characteristic maps of the cells. They are homeomorphisms in the interior of $\mathbb{D}^k$ and they are equal to f_α on the boundary. The image of $\mathbb{D}^k_\alpha$ is a compact subspace of X because $\mathbb{D}^k_\alpha$ is compact and f_α and ϕ_α are continuous. So, we call them closed k-cells in this pair (X, Y).

Notice that why am I writing this ordered pair $(X, Y).Y$ is the given space and X is obtained by this operation. The image of each of these $\mathbb{D}^k_\alpha$'s will be denoted by corresponding e^k_α and they are called closed k-cells of X.

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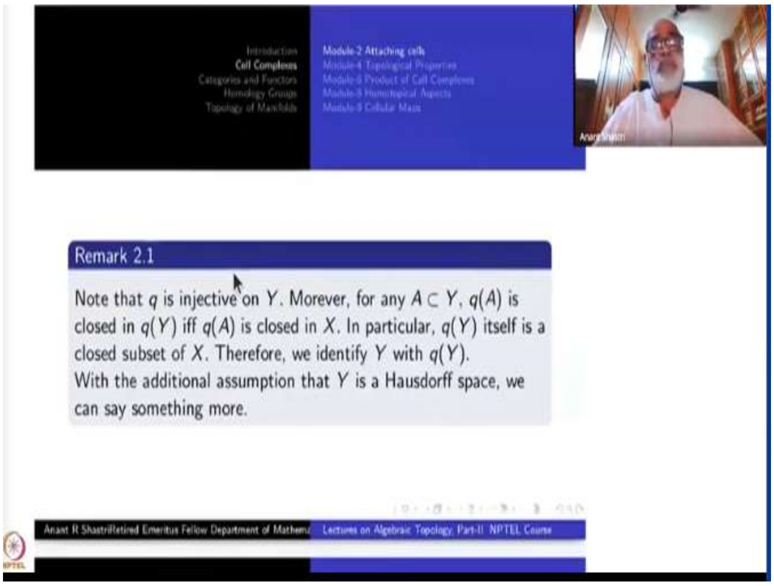

Note that q is an injective map on Y; there is no identification within Y, therefore we can use the same notation Y for the image of Y under q.

For any subset A of Y, $q(A)$ can be identified with A and A is a closed subset of Y iff $q(A)$ closed in X. That is by the definition; if A is closed subset in Y, then its inverse image under q is the disjoint union of A and $f_\alpha^{-1}(A)$ for all $\alpha . A$ being closed, f_α being continuous, $f_\alpha^{-1}(A)$ are all closed subsets of $\mathbb{S}^{k-1}$. Therefore they will closed subset of $\mathbb{D}_\alpha^k$.

So, with this identification, it is not just set-theoretic identification, it is a topological identification.

We shall now on assume that Y is a Hausdroff space. It follows that Y is a (closed) subspace of X and that is why we can use this notation (X, Y), the usually notation for pairs of topological spaces. Whenever the notation (A, B) is used, B is a subspace of A. That is the standard notation for pairs of topological spaces.

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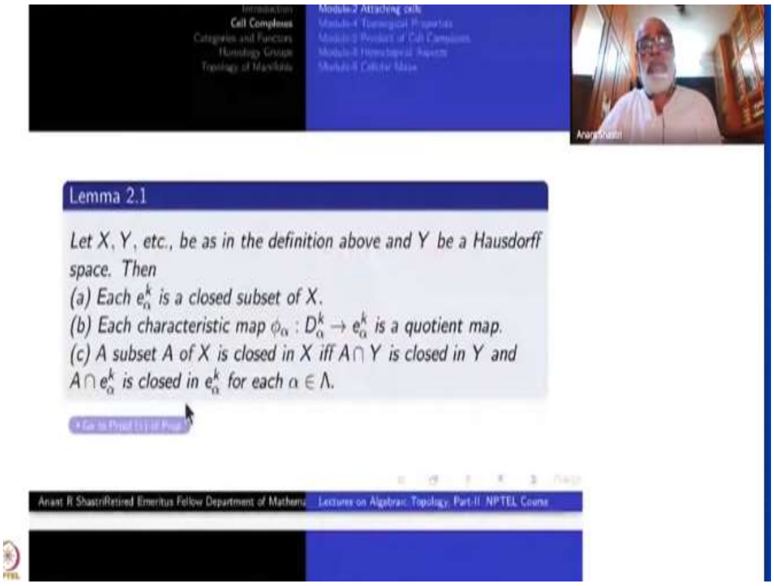

So, here is a basic lemma out of this quotient space topology. All these things are consequences of just taking the quotient space topology on the disjoint union of Y and $\mathbb{D}^k_\alpha$. So, first one is:

(a) Each e^k_α is a closed subset of X. So, that is where we have to assume that Y is Hausdroff space, e^k_α being an image of a compact space $\mathbb{D}^k$, is a compact subset of X. The boundary of $\mathbb{D}^k_\alpha$ is a subset of Y that will be closed. So, that will helped to see that e^k_α itself is closed X.

(b) Each characteristic map is supposed to be a quotient map.

(c) A subset A of X is closed in X if and only if $A \cap Y$ is closed in Y and $A \cap e^\alpha_k$ closed in e^k_α of X. So, this statement (c) is exactly a copy of the characterization of the quotient topology here. Here the characterization is in the top space here, on the mother space, whereas this part (c) here transfers the whole thing inside the quotients X itself. You do not have to go above. Nothing new but you have to prove this one.

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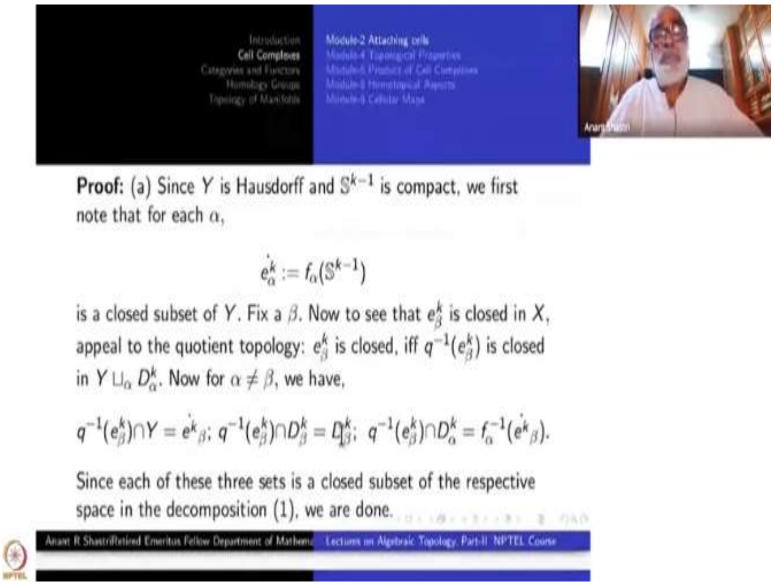

What is the proof? Proof is using again and again Hausdroffness of Y and this the quotient map. I mean what is the definition of an open subset in X or closed subset in X? So, first of all, Y is Hausdroff and $\mathbb{S}^{k-1}$ is compact for each alpha $f_\alpha(\mathbb{S}^{k-1})$ which we denote by e^k_α, denoting the boundary of f_α; all the interior points are taken away from this cell. This is a closed subset Y.

Now, fix a β. To see that e^k_β is closed for every β, I am fixing one β, is that clear? All that you have to do is to go back to the mother space and check the criterion, namely, $q^{-1}(e^k_\beta)$ must be closed in the disjoint union of Y and all the $\mathbb{D}^k_\alpha$. $q^{-1}(e^k_\beta)$ intersection Y is nothing but e^k_β which is a closed subset of Y. $q^{-1}(e^k_\beta \cap \mathbb{D}^k_\beta)$ is the whole of $\mathbb{D}^k_\beta$, since q is the identity map here. And finally, its intersection with $\mathbb{D}^k_\alpha$ is equal to to f_α inverse of $f_\beta(\mathbb{S}^{k-1})$ and hence is closed in $\mathbb{D}^k_\alpha$, for $\beta \neq \alpha$.

So, this proves (a).

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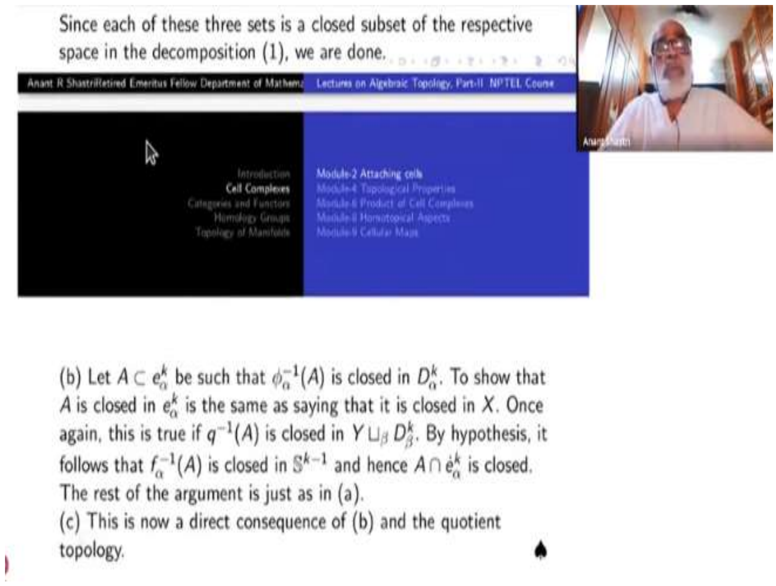

Once (a) is proved, to prove (b) i.e., the quotient map q restricted to each $\mathbb{D}^k_\alpha$ onto its image is a quoteint map, we will take a subset A of e^k_α such that its inverse image under ϕ_α is closed in $\mathbb{D}^k_\alpha$. We have to show that A is closed in e^k_α, which is the same as proving that it is closed in X because e^k_α is closed in X.

So, once again, this means that we have to show that $q^{-1}(A)$ is itself closed in here in the disjoint union.

By the hypotheses it follows that $f_\alpha^{-1}(A)$ is closed $\mathbb{S}^{k-1}_\alpha$, because you have started with the hypothesis that $\phi_\alpha^{-1}(A)$ is closed in $\mathbb{D}^k_\alpha$. Intersect it with the boundary, it will be closed in there. So, once you have this the rest of the argument will be as in (a).

Now, (c) is now a direct consequence of (a).

What is it? A subset A of X is closed in X if and only if $A \cap Y$ is closed in Y and $A \cap e^\alpha_k$ is closed. 'Only if' part is obvious. To see the if part, we have to see that $q^{-1}(A)$ is closed. $q^{-1}(A \cap Y)$ is $A \cap Y$. So, that part is fine. $A \cap e^k_\alpha$ is closed in e^k_α and $q^{-1}(A \cap \mathbb{D}^k_\alpha)$ is nothing but $\phi_\alpha^{-1}(A \cap e^k_\alpha)$ and hence is closed. So that will complete the proof that A itself is a closed subset of X.

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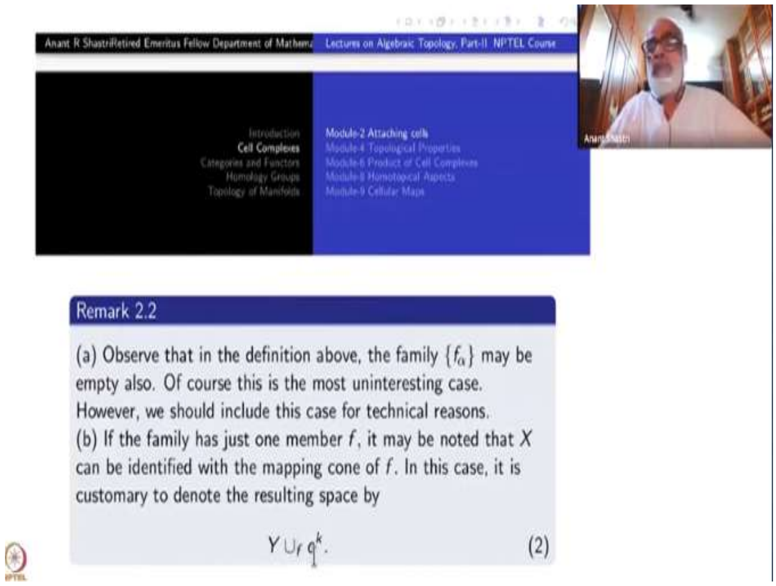

In the definition of the attaching cells, we could have taken the indexing set to be empty also, just for logical completeness. We should include this case. We should not assume that Λ is non empty. Of course what you get is the space Y itself, i.e., when you do not attach anything. Then X will be Y itself. So, the family Λ could have been empty.

If the family has just one member $\{f\}$, then X is nothing but the mapping cone of f. f from $\mathbb{S}^{k-1}$ to Y is a function and then you are filling up this $\mathbb{S}^{k-1}$ with a cell, with $\mathbb{D}^k$ inside the mapping cylinder of f, that gives you the cone. So, this would have been obvious if you know the definition of mapping cylinder and mapping cone. I am not recalling that here. Anyway, that is a special case. In that special case, we have this simple notation also $Y \cup_f e^k$. Since there is only one cell, there is no need for writing indexes α.

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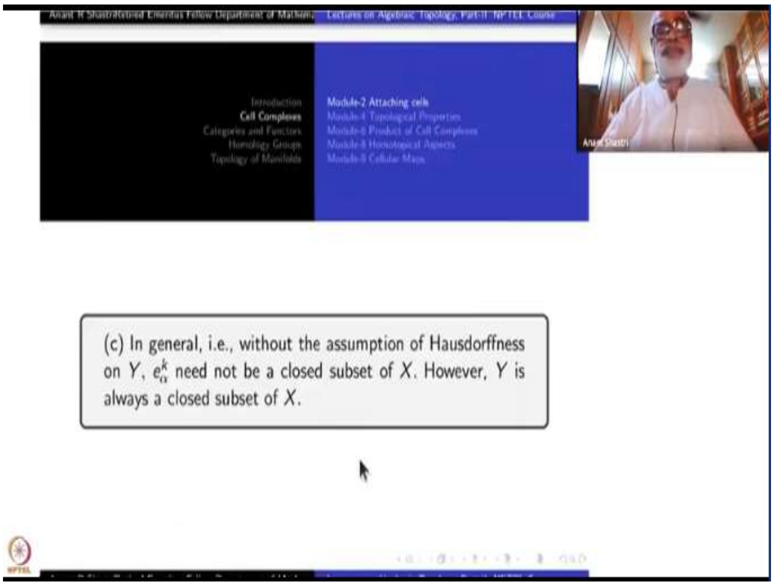

In general, without the assumption of Hausdorffness, e^k_α may not be closed in X. Very simple examples can be given. There is no problem there. However, even if you do not assume Y is Hausdorff, Y will always be a closed subset of X in the attaching process.

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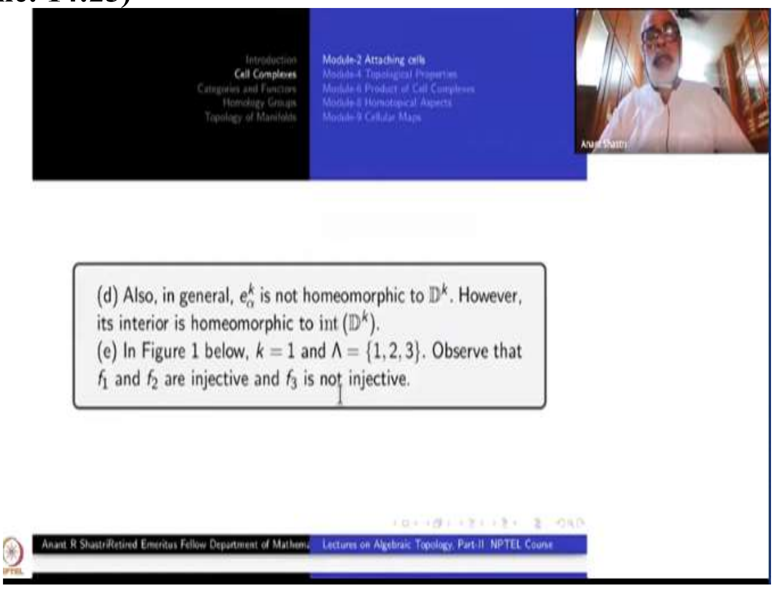

Also, unless on the boundary, namely f_α, the attaching map is 1 to 1, you will not have e^k_α homoeomorphic $\mathbb{D}^k$.

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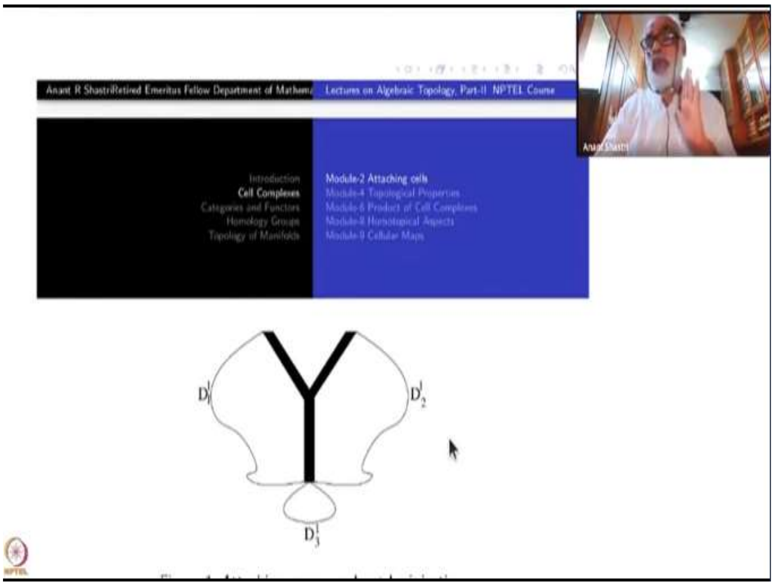

So, here in the figure, I have given you an example of attaching 1-cells, i.e., for $k = 1$. The pictures can be always 1 or 2 dimension anyway. So, this is your Y, which looks like Y anyway. $\mathbb{D}^1$ is the interval $[-1, 1]$. In the first cell, -1 has gone here and 1 has gone here. $f_1(-1)$ is this point and it is identified here, $f_1(1)$ is this point and is identified here, that is the meaning of this identification space.

The rest of the interval could be just anything homeomorphic to open interval here. Similarly another open interval here another cell. Remember interior of the cell, interior of the cell, interior of this cell they are already on the boundary what happens that is left to f_α, for boundary should always be mapped somewhere inside Y. In this D_3^1, the third cell both the points of the boundary have gone to a single point, viz., it corresponds to f_3 being a 1-point map.

Now you see that D_1^3 is not homeomorphic to $\mathbb{D}^1$ at all, these things are homeomorphic this is not. So, this is what happens if you draw a picture. Suppose I draw a line like this that is not considered that cannot be attaching map that is not part of attachment. On the other hand I can just take a point here just a point here that is allowed then it will be attaching a 0-cell. It is not attaching 1-cell. So, just now we have defined k equal to a constant and $k \geq 1$. So, let me define what is the meanings of attaching a 0-cell also.

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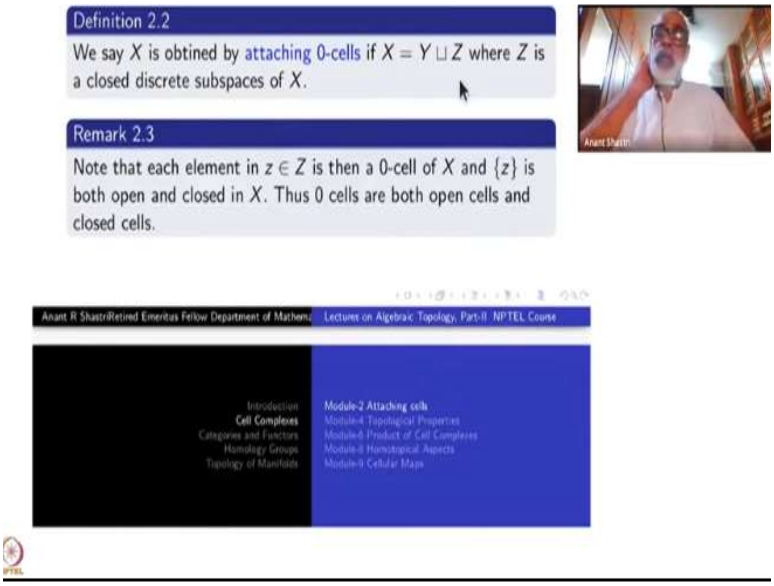

So, we say X is obtained by attaching a 0-cell (I will define $k \geq 1$ here k-cells, so I want to include 0-cell also what is the meaning of attaching a 0-cell?) Namely, if X is a disjoint union of Y and Z where Z is a closed discrete subspace of X, discrete and closed subspace of X. Y itself will be closed in X. Infact both Y and Z will be closed in X by this definition, because X is a disjoint union of the two topological spaces. Y is a given space, Z is a set of points the topology on those points is discrete.

This definition is completely justified because first of all, you have to say what the meaning of a 0-cell or $\mathbb{D}^0$.$\mathbb{D}^0$ or 0-cell. In the zero dimensional vector space there is only one vector, which is the 0 vector. So, 0-cell is just a singleton. It has no geometric properties, this singleton. What is then its boundary? Boundary is empty. So, there are no attaching maps there. Therefore, no identification takes place. So, this is a logical justification for defining the operation of attaching 0-cells. If you have difficulties in this logic you can just take this as the definition.

Elements that belong to Z are then the 0-cells of X. The space Y is a black box. There is no name for it. You call it the base space if you like. When you attach 0-cells, they come from a disjoint set indexed by a family. By the very definition, each point in Z is closed as well as open in the whole space. Thus a 0-cell is both an open cell as well as a closed cell.

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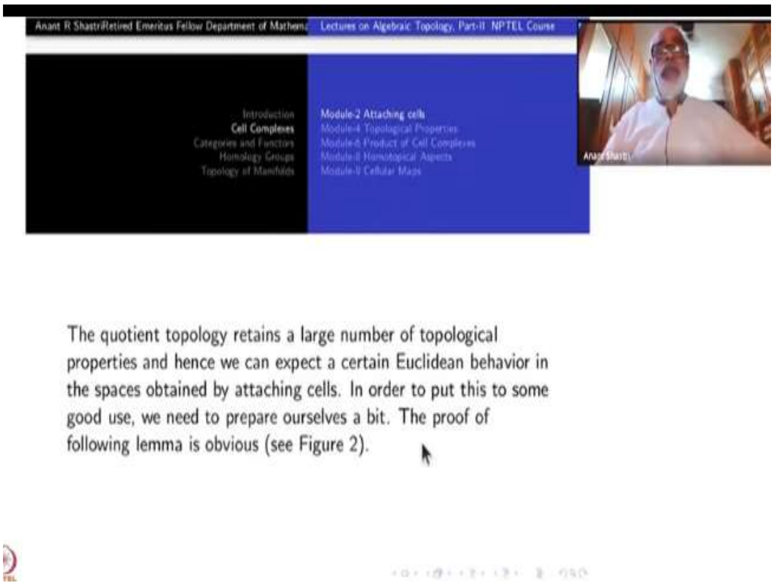

The quotient topology retains a number of properties of the base space. There are quite a few topological properties such as Hausdroffness, which in general, do not go to the quotient space. You have studied quotient space thoroughly last time. So, what happens is that away from Y, you can expect a few properties of the Euclidean space inside X, I mean $X \setminus Y$ has a number of topological properties inherent from the Euclidean spaces, because we have attached copies of $\mathbb{D}^k$.

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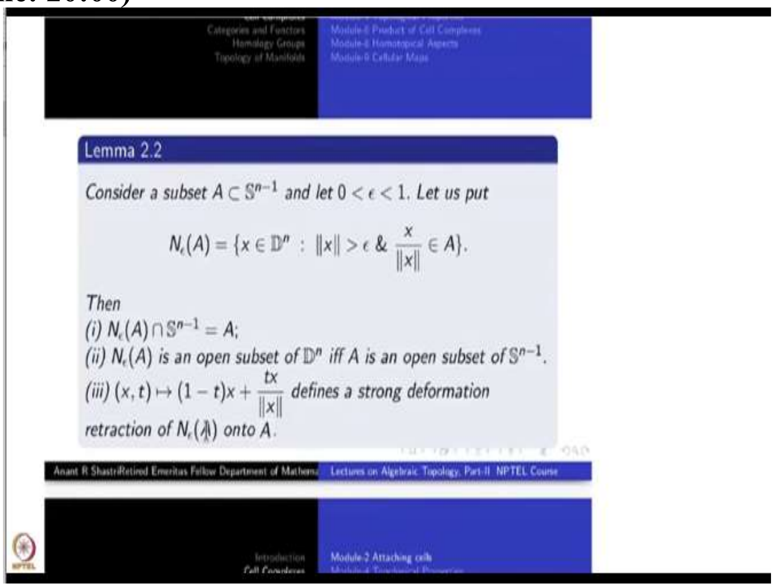

The first attempt here is to check what kind of things may happen inside $\mathbb{D}^n$ and $\mathbb{S}^{n-1}$---- some elementary observations. So, I have put it as a lemma which is the starting point of our topological study here.

Take a subset A contained in $\mathbb{S}^{n-1}$. Fix an ϵ strictly between 0 and 1. Let us put $N_\epsilon(A)$ equal to the set of all $x \in \mathbb{D}^n$ such that $\|x\| > 1 - \epsilon$ and such that when you divide it $\|x\|$, namely, the corresponding unit vector is inside A.

Remember A is a subset of $\mathbb{S}^{n-1}$. So, it has to be unit vector. Take all such elements x such that $x/\|x\|$ is in A.

How does an element of $N_\epsilon(A)$ look like? It is just like taking a vector $v \in A$ and multiplying it by some number r between 0 and 1. I can also say that $N_\epsilon(A)$ consists of a point in A and then a small line segment going towards 0, but 0 is strictly avoided because ϵ is less than 1. All those points you take. Since ϵ is bigger than 0, we can take r to be one as well. Therefore, A is contained in $N_\epsilon(A)$

Infact, $N_\epsilon(A)$ intersection $\mathbb{S}^{n-1}$ will be exactly equal to A.A is contained in the intersection is clear but intersection $\mathbb{S}^{n-1}$ is precisely equal to A because $x/\|x\|$ is equal to x itself when $\|x\| = 1$.

$N_\epsilon(A)$ is an open subset of A if and if only A is open in $\mathbb{S}^{n-1}$. That is also clear. If $N_\epsilon(A)$ is an open subset, its intersection with the sphere is an open subset. That is clear.

If A is open in the sphere, why is $N_\epsilon(A)$ open? Because it is first of all a subset of $\mathbb{D}^n \setminus \{0\}$ which is open in $\mathbb{D}^n$ and $\mathbb{D}^n \setminus \{0\}$ is homemorphic to $\mathbb{S}^{n-1} \times (0, 1]$, and under this homeomorphism $N_\epsilon(A)$ corresponds to $A \times (1 - \epsilon, 1]$; clearly $A \times (1 - \epsilon, 1]$ is open in $\mathbb{S}^{n-1} \times (0, 1]$.

The last thing which is very important is that: you look at this map which joins x and $x/\|x\|$, along the line segment $(1 - t)x + tx/\|x\|$. If $t = 0$, this is x; if $t = 1$, it is $x/\|x\|$. Thus it defines a strong deformation retract of $N_\epsilon(A)$ onto A.

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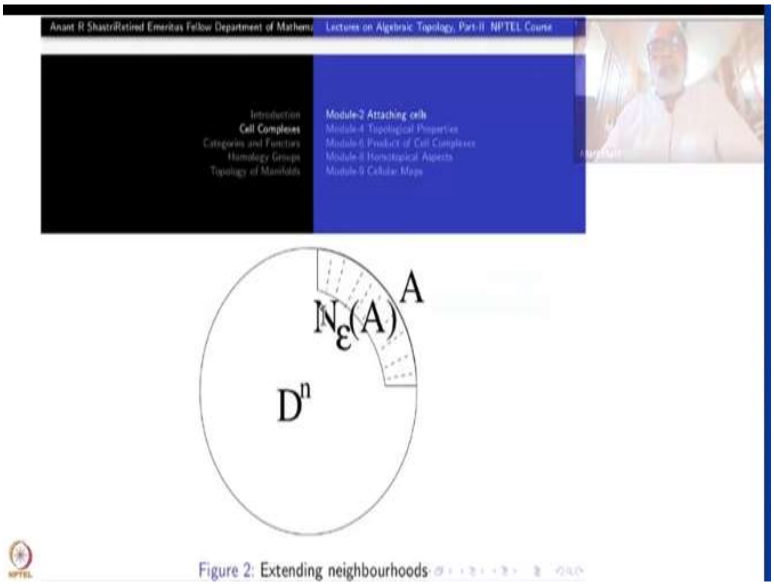

That is the picture. If you allow epsilon to be zero also then there may not be any deformation because you do not know where to send the zero vector. Since we have assumed ϵ to be positive, $N_\epsilon(A)$ does not contain the zero vector of $\mathbb{D}^n$. This is important. As ϵ varies the collection $N_\epsilon(A)$ forms a fundamental system of neighbourhoods of A in $\mathbb{D}^n$.

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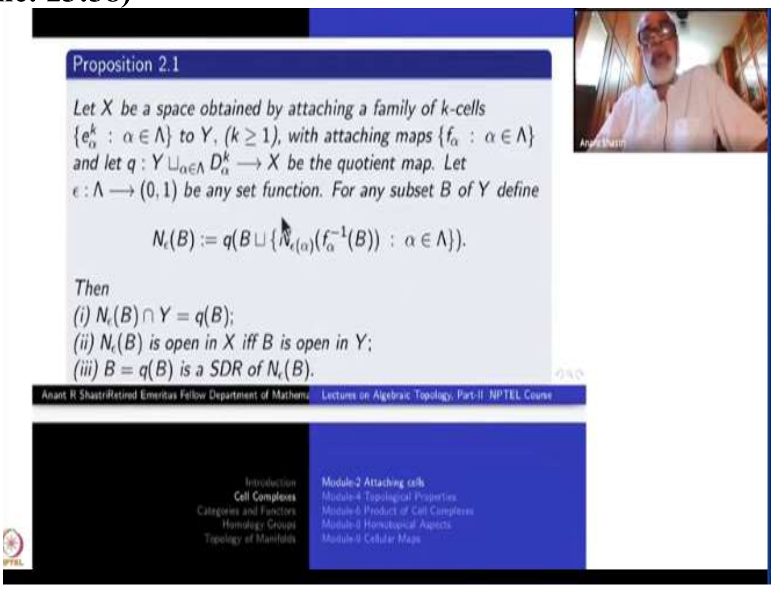

So, from this result, now, we want to extend this one to the attaching cells from Y to X. So, what do we do? Take a family f_α's and suppose that we have attached them to Y just like in the definition above. Now start with a subset B of Y and then for each α look at $f_\alpha^{-1}(B)$ that is a subset of $\mathbb{S}^{k-1}$ the boundary of $\mathbb{D}^k$, take this as A in the previous lemma. Choose some ϵ between 0 and 1 and form the neighbourhood N_ϵ of that.

Let us say that B is open in Y. Then that these are open in $\mathbb{D}^k_\alpha$ (not otherwise). Take the disjoint union over all α and include B also. Now take the image under q and call it $N_\epsilon(B).N_\epsilon(A)$ in the earlier was defined only for this picture. Now, I have defined it for the general case when X is obtained from Y by attaching k-cells.

The first thing is to see that $N_\epsilon(B) \cap Y$ is nothing but $q(B)$. $N_\epsilon(B)$ is open in X if and only if B is open in Y.

These claims are straightforward from previous lemma. if B is open in Y implies $f_\alpha^{-1}(B)$ is open in $\mathbb{S}^{k-1}$, which im turn implies that N_ϵ if it is open $\mathbb{D}^k_\alpha$ for each α. Therefore the disjoint union of all these together with B is open and finally under q. This is the full inverse image $N_\epsilon(B)$.

So, that follows from directly from previous lemma and quotient topology definition. B which is equal to $q(B)$. We can B or $q(B)$, because we have identified these things they are subsets of Y. The strong deformation retracts for each α patch up to define a strong deformation $N_\epsilon(B)$ onto B.

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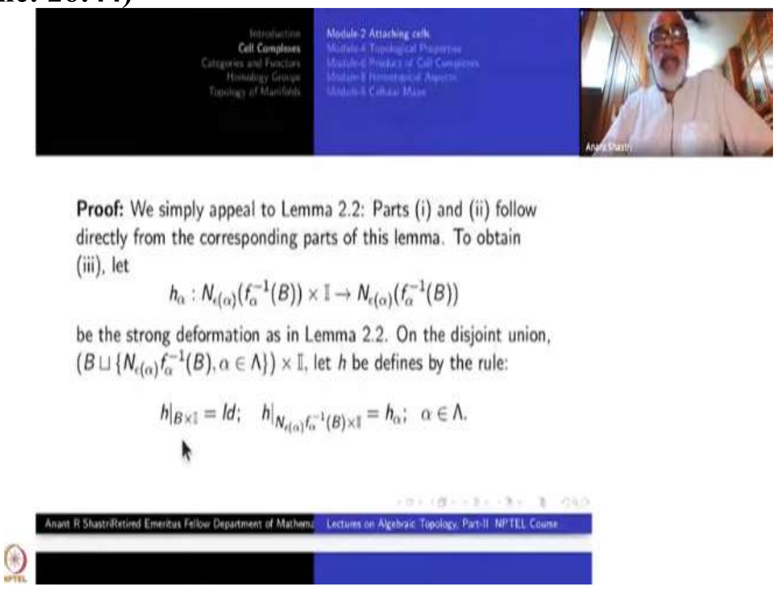

What you get is h is shift to $B \times I$ put it this identity and in the rest of these things put this is h_α.

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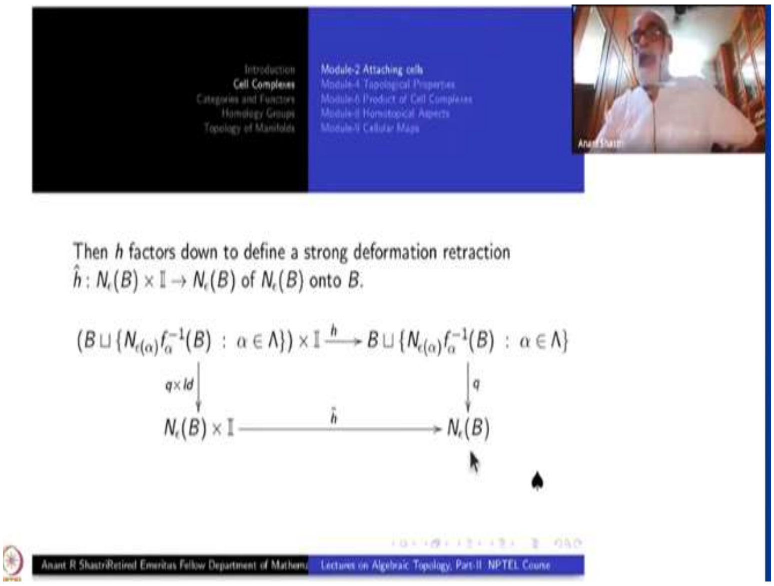

So, this patch is up to define a big homotopic from this entire space to be space, q of that cross I have taken identity, we will come back to h_α. So, this will give you a homotopic from $N_\epsilon(B) \times \mathbb{I}$ to $N_\epsilon(B)$ homotopic of the identity map with the retraction there on to B everything will go inside B. So, this property recently we are going to use it again and again.

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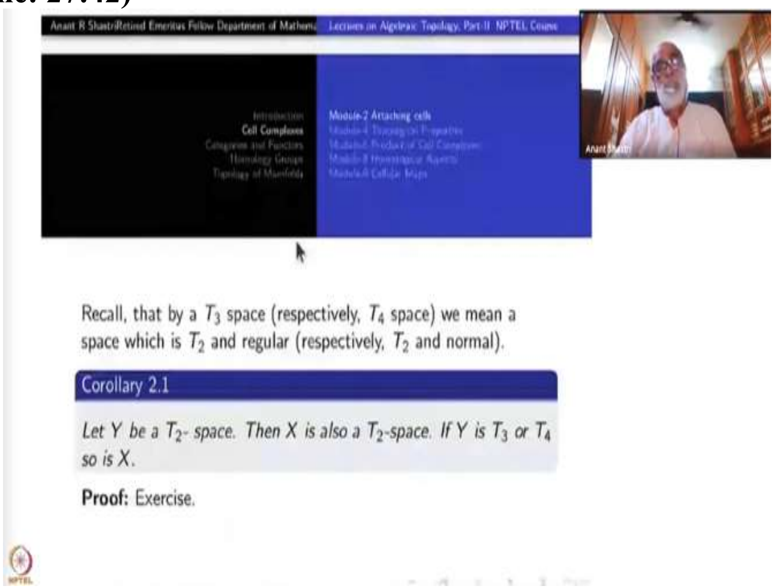

Here is a corollary. Recall that this notation T_3 space means what? A Hausdorff space T_2 plus regular similarly T_4 means a Hausdorff space which is normal. Now, this is a simple exercise. Maybe, you should take some time to verify each of them, one by one. X is obtained from Y by attaching k-cells where k is any non negative integer. If Y is Hausdorff, T_3 or T_4 then so is X.

This is left to you as an exercise which you should be able to do by yourself. Do it so that you get familiar with what is going on with this lemma.

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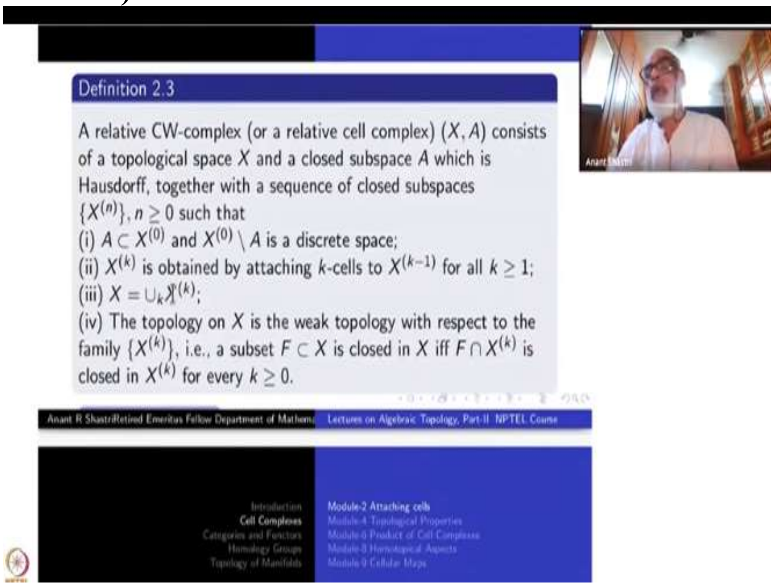

Now, let me give you the definition of CW-complex, a relative CW-complex. We will define slightly more general things, just like attaching maps from Y to X.

Before proceeding further, by the way, in the attaching process, note that we can take Y to be the emptyset also. Can we really? If Y is the emptyset and $k \geq 1$. What are the maps? There is no map. So, you will never get to attach any cells. But k can also be 0. You can attach 0-cells. So, taking Y equal to emptyset is allowed in which can you must remember that you can attach on 0-cells) and of course, you can always take the family of attaching maps itself to be empty as well. That is also permitted.

So, let come to the definition of a relative CW-complex.

Some people like to start with a space A and say that X is obtained by a sequence of processes or operations namely each operation is attaching k-cells where k ranges from $0, 1, 2, 3$ and so on. Instead I shall give a slightly more elaborate definition here.

A relative CW complex (X, A) consists of a Hausdorff topological space and a closed subspace A, (Hausdorffness is a part of the definition some people do not assume it but I am going to assume this), together with a sequence of closed subspaces $X^{(n)}, n \geq 0$ satisfying the following conditions:

(i) A is a subspace of $X^{(0)}$ and $X^{(0)} \setminus A$ is a discrete space. (Note that this precisely means that $X^{(0)}$ is obtained by A by attaching 0-cells. Instead of saying that I have just said $X^{(0)} \setminus A$ in discrete space A is a closed subset.)
(ii) $X^{(k)}$ is obtained by attaching k-cells to $X^{(k-1)}$ for all $k \geq 1$.(So, what is $X^{(1)}$? It is obtained by attaching 1-cells to $X^{(0)}$. Remember that $X^{(0)}$ could have been just equal to A because all that you need is A is contained in $X^{(0)}$ and the complement is a discrete space which may or may not be empty. So, do these for each k, then what is $X^{(k)}$?)
(iii) X is just the union of all these spaces $X^{(k)}, k \geq 0$. (Remember by the very definition here $X^{(k)}$ will be a subset of $X^{(k+1)}$ and so on each of them closed in the next one. So, your sequence of closed subspace A contains an $X^{(0)}$ contains $X^{(1)}$ and so on take the union that is your X.)
(iv) The last condition is about the topology on X. This topology is called the weak topology or what is also known as the coherent topology or... all these names are of no use unless you know what is the meaning. So, meaning is precisely this. A subset of X is closed if and only if its intersection with each $X^{(k)}$ is closed in $X^{(k+1)}$. Remember each $X^{(k)}$ is a well defined topological space being obtained from $X^{(k-1)}$ by attaching k-cells.

So, the topology on each $X^{(k)}$ is well defined as soon as you know what is the space A and you know what are all the attaching maps, inductively. There is no ambiguity there. But what is the topology on X which an infinite union that has to be defined and this condition (iv) for here takes care of that. It says the topology on X is coherent with respect to each $X^{(k)}$. Automatically with this definition each $X^{(k)}$ will be a closed subset of the whole space X itself and not just closed inside $X^{(k+1)}$. All these are easy consequences of this condition (iv).

A very interesting special case I told you is that when A is empty. Then $X_0 \setminus A$ cannot be empty, cannot be mean what? If it were empty then inductively each $X^{(k)}$ will be empty because $X^{(k)}$ is obtained by attaching k-cells to an empty set $X^{(k-1)}$. Therefore, X itself will be empty.

So, we conclude that if X is nonempty then X_0 is non empty.

Likewise, in between you may not attach any cells. For example there may not be any 1-cells at all or you may not attach 2-cells at all you can directly attach 3-cells. It is possible that, in between, some $X^{(k)}$ may be equal to $X^{(k-1)}$. So, all these things are allowed.

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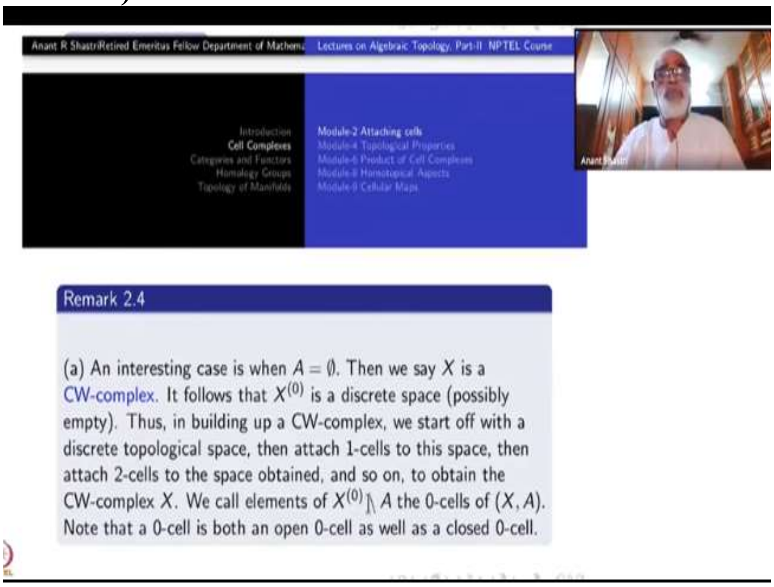

As told before, one of the interesting case is A is empty. Then we do not write (X, A) at all; we just write X and that is called a CW-complex. So, let us re-examine this special case. A non empty CW-complex starts with some non empty discrete space $X^{(0)}$ (So, there will be a set of points you can call them as vertices just like in the case of a simplicial complex, points of $X^{(0)}$ are called 0-cells or vertices. So, it is a discrete space then. Then $X^{(1)}$ is contained by attaching some 1-cells to $X^{(0)}$ and then we attach 2-cells and so on. Each time there is no condition on the number of cells, there may be just one 1-cell, there may be 50,000 or more 2-cell and so on. The only thing that is needed is if you take a k cell, the attaching map of that k -cell must be a map from $\mathbb{S}^{k-1}$ into $X^{(k-1)}$. So, these $X^{(k)}$ are called the k^{th}-skeleton of X. It is just a name and useful name.

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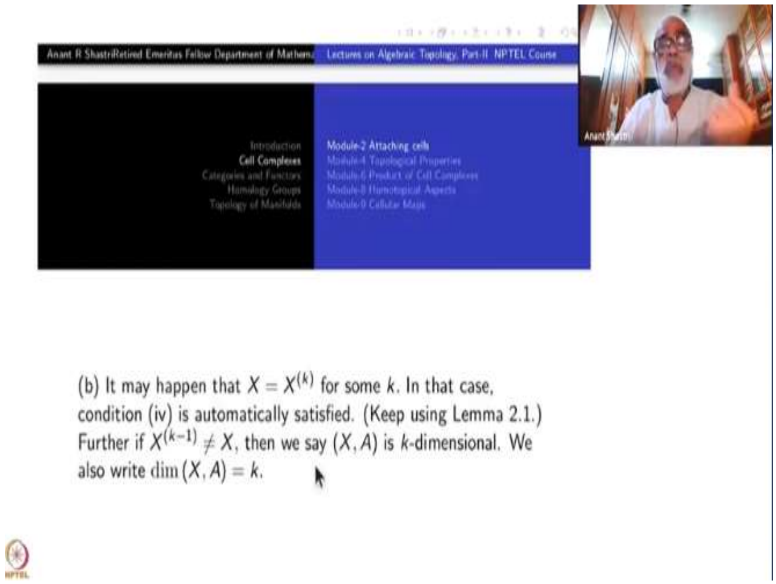

So, that is what it may happen; after certain stage, there are no cells, no cells have been attached, i.e., $X = X^{(k)}$. If that happens, we call this $X = X^{(k)}$ of dimension $\leq k$. Suppose there is at least one k-cell and no n-cell for any $n > k$, then we say dimension of $X = k$.

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These are just definitions. So open cells, characteristic maps, attaching maps all these things make sense in the case of(relative) CW-complexes also all the terms, you have learnt in the case of attaching k-cells. as you maps. That is the just the gist of the definition. We shall take up this study next time. Thank you.