Introduction to Algebraic Topology Part – II
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Lecture – 17
Categories Definitions and Examples

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So, this chapter is a quick introduction to the language of categories and functors. What we are going to do is just a limited introduction. Interested reader may look into the books I refered at the end of the lecture notes. The book by Adamek is freely downloadable and it is quite a readable book. The topic as I told is the language of modern mathematics, takes some time to mast it. So, I would not say that, right now you will become a master of whatever I want to introduce here, though, that itself is very minimal. But because I am going to use this language again and again, certainly by the end of this course, I hope you will all know how to use this language. Studying categories and functors for its own sake is not at all done here. You must take note of that. It is not like we are going to write poetry in this language, we are merely trying to learn the market language or know how to do day-to-day business. That is all.

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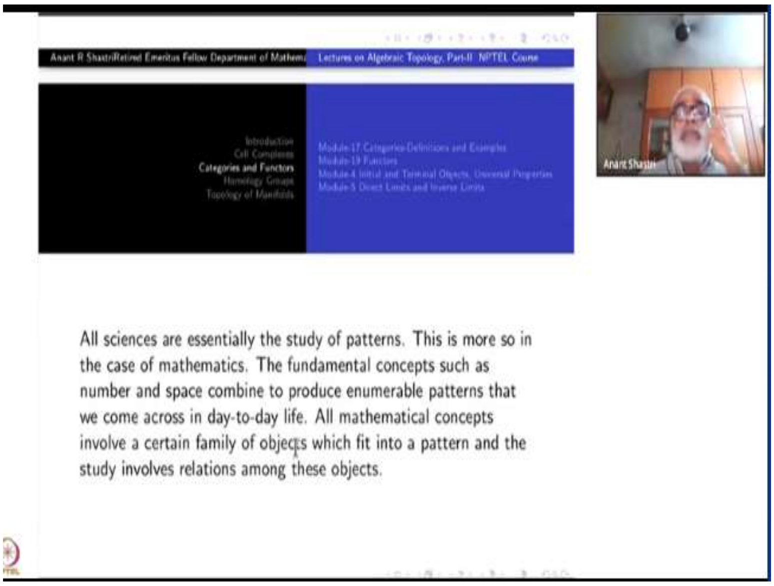

All sciences are essentially study of patterns. This is more so in mathematics. The fundamental concepts such as number and space combine to produce enumerable patterns that we come across in day-to-day life. All mathematics concepts involve a certain family of objects which fit into a pattern and then the study involves relations among those objects.

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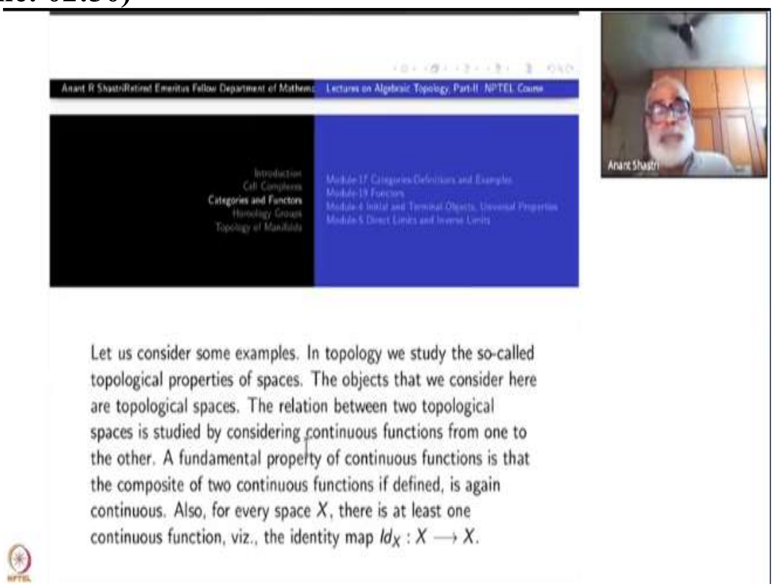

Let us consider some examples. In topology, we study the so called topological properties of topological spaces. The objects that we consider here are topological spaces with the relations among them being what are called continuous functions. The fundamental property of continuous functions is that the composite of two continuous functions if defined, is a continuous function. And identity map from any space to itself is also continuous. Okay?

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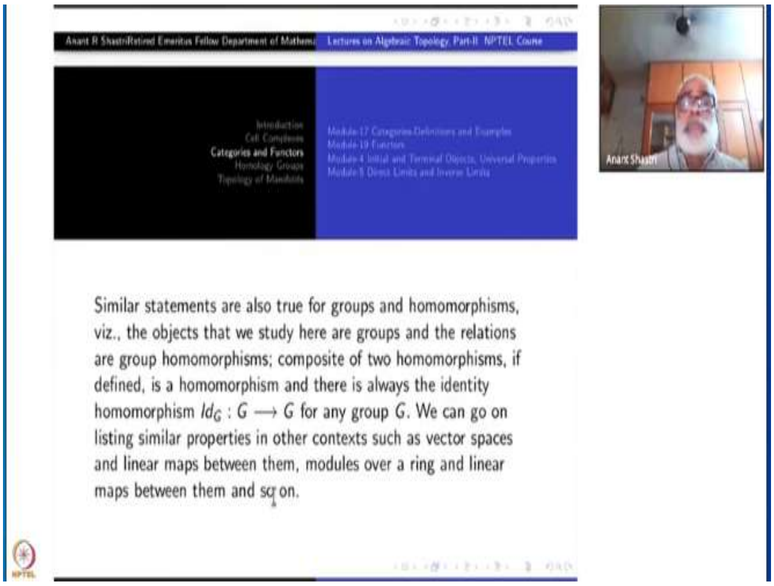

Similar statements can be done about groups. Suppose you are studying groups, then what are the functions from one group to another group, that we are going to study? Answer is homomorphisms. Once again, composite of two homomorphisms if defined, is again a homomorphism and identity map from any group to itself is also a homomorphism, Okay? We can go on listing such examples like vector spaces and linear map, modules over a ring and the module homomorphism which are also called linear maps, and so on. Okay?

So, what we shall do is give a strictly rigorous definition of categories and then re-examine some of these examples, in the light of this definition which extract certain basic properties that we are trying to explain.

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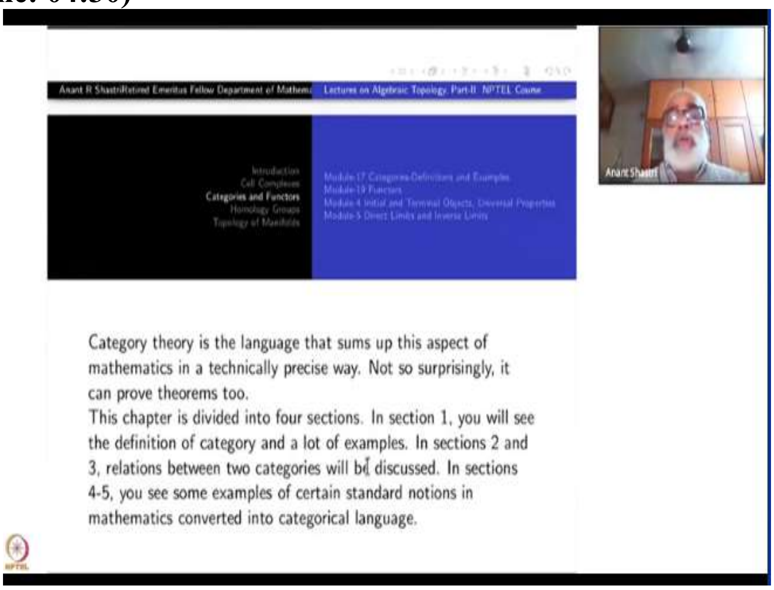

Category theory is a language that sums up this aspect of mathematics in a technically precise way. Not so surprisingly, it can help you prove theorems also. This chapter is divided into four sections. First section, you will see definition of a category and sufficiently many examples. In second and third, we will study relations between them under the name of functors. In 4th and 5th, we see more examples of certain standard notions which can be expressed, which have been expressed, which have been studied without the categorical language, but when you put them in the categorical language, how nice they become, okay? So, with this background, let us begin with the abstracts.

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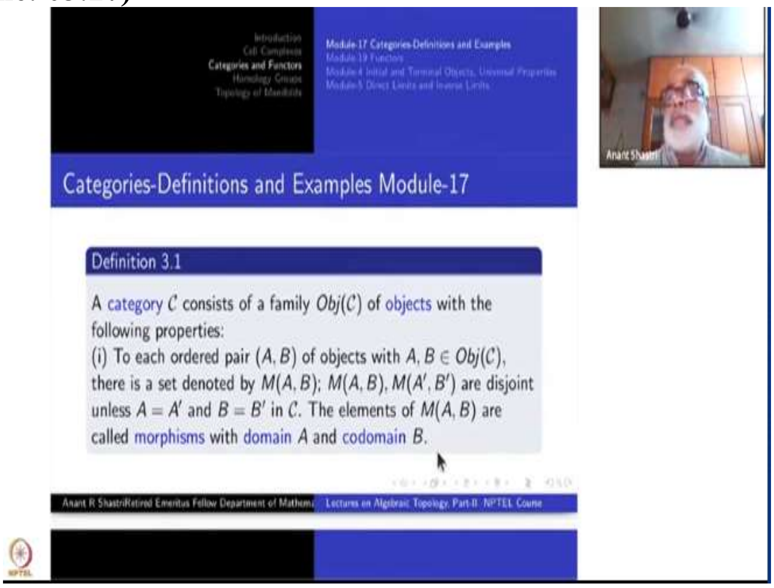

A category $\mathcal{C}$ consists of, (I can say that it an ordered quadruple of certain things but, I do not want to use that kind of language), it consists of a family $\mathrm{Obj}(\mathcal{C})$ (read it as objects $\mathcal{C}$C), members of this family are called objects, okay, satisfying certain properties. That will come in the definition. The whole definition has to be understood together, before you understand individual stuffs here. So, what are the properties? One by one I am going to list them.

(i) For each ordered pair (A, B) of objects, i.e., A, B are members of this family or a class, (take note that we are careful not use the word `set' this family, there is this slight difference here, $\mathrm{Obj}(\mathcal{C})$ is a collection or you can say it a class but do not use the word set. However, what makes sense is the membership relation between objects), there is assigned a set (this time it is a set) denoted by $M(A, B)$ is a set with the property that the two sets $M(A, B)$, $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$. This must be true for all members A, A', B, B' of $\mathrm{Obj}(\mathcal{C})$. These are all axioms, all are part of the definition, each word each comma here is a part of the definition that we are making, okay? The elements of $M(A, B)$ are called morphisms in $\mathcal{C}$, with domain A and codmain B. Okay?

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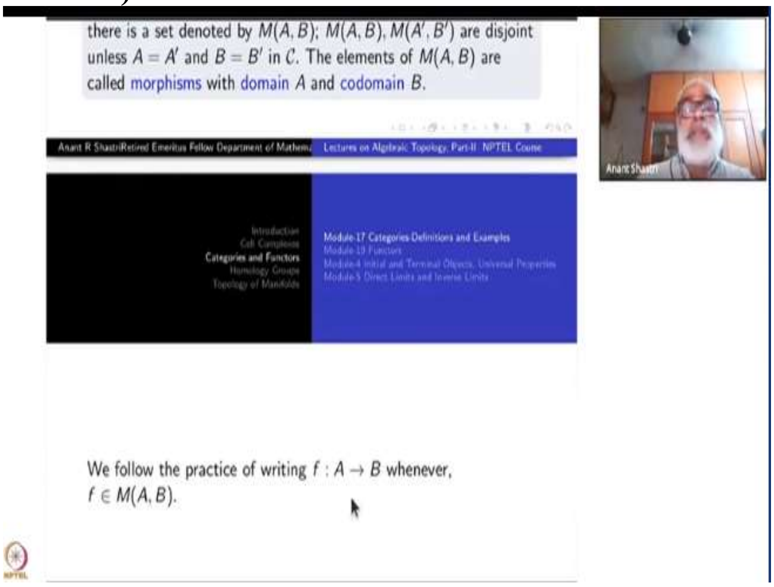

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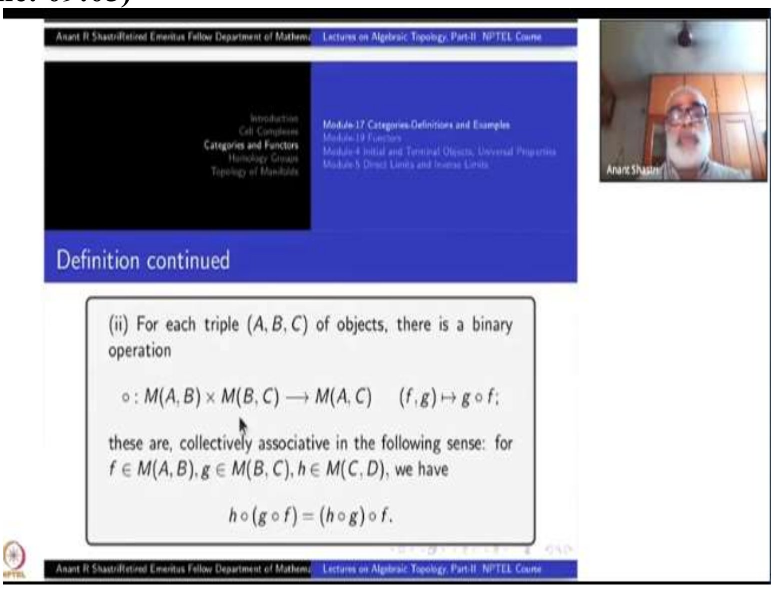

(ii) If you take a triple of objects, A, B, C, then you have three sets $M(A,B), M(B,C)$, and $M(A,C)$. There is a binary operation from $M(A,B) \times M(B,C)$ to $M(A,C)$. Now these three are all sets and so binary operation makes sense. So I write it as (f, g) goes to $g \circ f$. Okay? Similar to the standard practice of writing compositons of functions. These binary operations are for each triple there is one okay? So, if you change those triples the operations also change but we are using the same notation circ. They are collectively associative. What is the meaning of `collectively associative'? In the following sense namely, given f from A to B, g from B to C, h from C to D, then we have $g \circ f$, right, we also have $h \circ g$, these two things make sense.

What we want is $g \circ f$ is follows by h to get an element of $M(A, D)$ that element should be the same f followed by $h \circ g$. That is the brackets can be interchanged like this, just like associativity of compostions of functions. Collectively associative means whenever the compositions are defined then we have the associative law. The compositions may not be defined unless the arrows match, just like the case of sets and functions. Okay?

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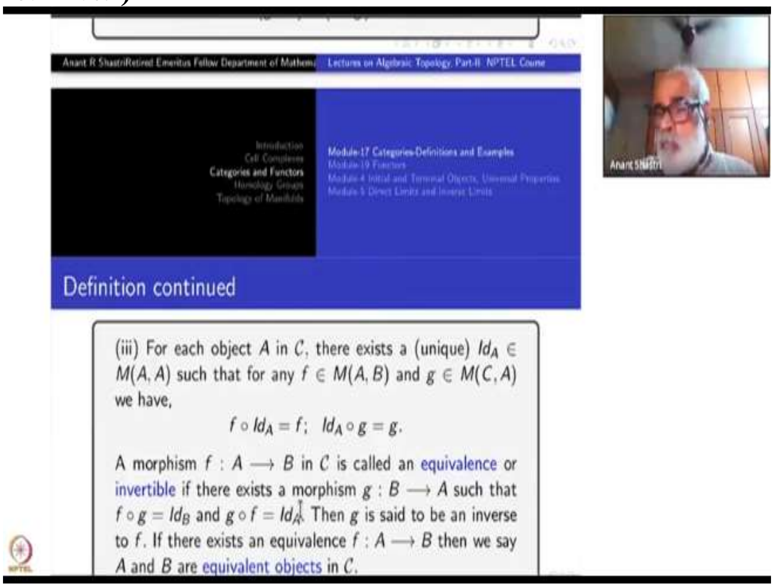

(iii) The third condition is that for each object $A \in \mathcal{C}$, there exists a unique Id_A (note this notation) inside $M(A, A)$, the set of morphisms from A to A, there is a unique element which has the property that it is a 2-sided identity; for all f from A to B and for all g from C to A, you have $f \circ Id_A$ (this makes sense) is equal to f and $Id_A \circ g$ is equal to g. Thus Id_A is a 2-sided identity for the binary operations okay? This completes the definition of a category.

A morphism f from A to B is called an equivalence in the category $\mathcal{C}$, if there exists a morphism g from B to A such that $g \circ f$ is equal to Id_A and $f \circ g$ is equal to Id_B. Then g is said to be the inverse of f and f is said to be the inverse of g. In fact, it is an elementary algebra to verify that the identity elements in $M(A, A)$ as well as inverse of morphism it exist are unique. There may not be any inverse of a given morphism in general. Morphisms with inverse are also called invertible morphisms.

So, if there is an equivalence f from A to B, then we say A and B are equivalent objects in $\mathrm{Obj}(\mathcal{C})$. Okay? That is the end of some definitions. So, we have defined a category? Objects in it, morphisms, equivalences and equivalent objects. Okay?

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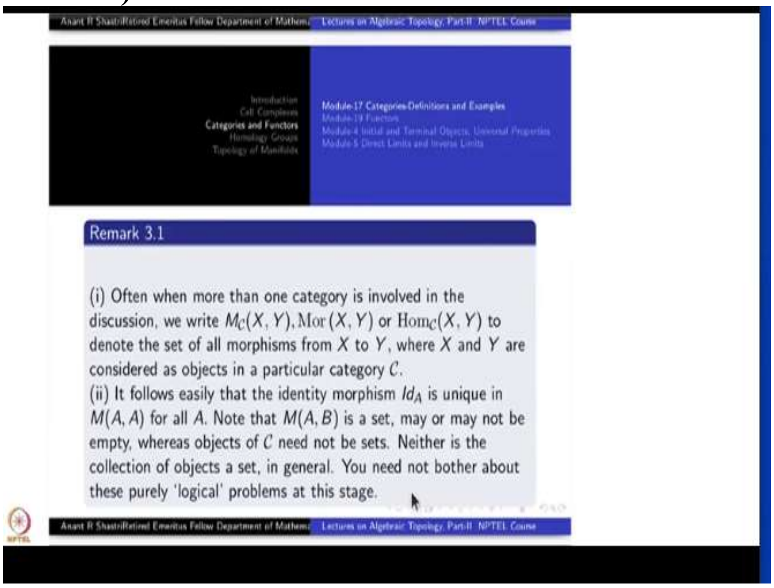

Now, I want to make a few elementary remarks. Often when more than one category is involved at a time in the discussion, for object $X, Y \in \mathcal{C}$, instead of just the simple notation $M(X,Y)$, we write $M_{\mathcal{C}}(X,Y)$ to indicate which category we are working in. Also the notation $Mor(X,Y)$ (or $Mor_{\mathcal{C}}(X,Y)$) is used in place of $M(X,Y)$ (respectively, $M_{\mathcal{C}}(X,Y)$).

Suppose, I am having two different categories $\mathcal{C}$ and $\mathcal{D}$ at hand. It may happen that objects under discussion are objects in both of them. Therefore, if I just use $M(X,Y)$, you do not know whether I am talking about morphism is it is you category $\mathcal{C}$ or category $\mathcal{D}$, right? So, that is why in that case, we use the elaborate notation. When you know that the discussion is going on in single particular category, then we us the simpler notation as usual is a practice. But this kind of thing should not be done when you are dealing with a computer for instance. Computers would not understand unless you are strictly following the notation that you have fed into it. By the way, computer scientists use the category theory very much okay?

So, it follows easily that identity morphism is unique in $M(A,A)$ for all A. Try to write down a proof yourself. Note that $M(A,B)$ is a set but it may be empty also. Clearly, $M(A,A)$ is non empty, because Id_A is there.

The collection of all objects in C need not be sets, Okay? We will see some examples later on. We have already remarked that the family $\mathrm{Obj}(C)$ need not be a set. You need not bother about these purely logical problems at this stage. That is the key word here. Because if you get into this logical problem at this stage, you will never learn the category theory. You may bother about it only when you are talking about foundational mathematics, like logic, machine learning etc., The logicians are more interested in this aspect of category okay? Unless, we have some problems we should not bother about that aspect. Right now you learn the language.

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We have also remarked that if a morhism f is invertible then its inverse is unique. therefore, I can use the notation f^{-1} for the inverse of f, Okay? The equivalence of any two objects in $\mathrm{Obj}(C)$ defines an equivalence relation, namely, it is reflexive, transitive and symmetric. The central problems in mathematics is to determine the equivalence classes in any category that you have chosen.

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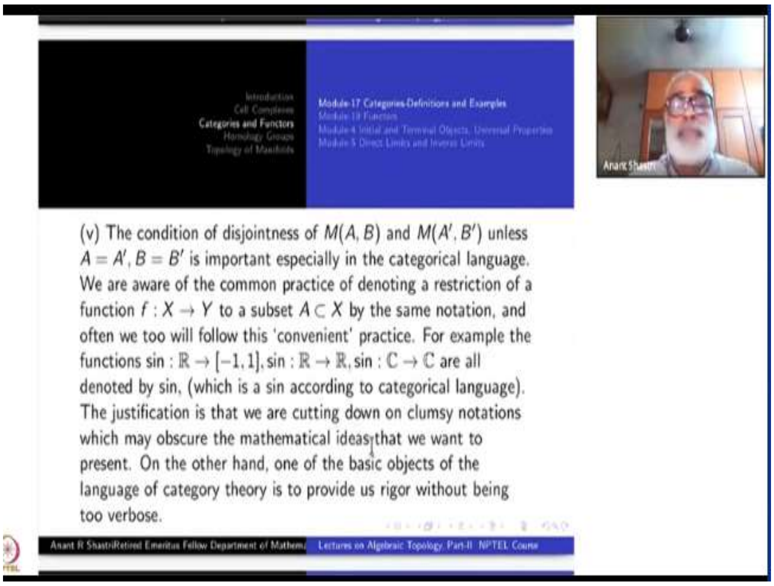

One more remark. The condition of disjointedness. Remember that I told you $M(A, B)$ and $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$. There is no overlap among them. That is very important here right now, you may not understand the importance of this right now, because in standard mathematics such as in calculus and so on, right from the ancient days of Euler, Gauss and so on, we are not doing this, and we have to keep that tradition and cannot give it up, okay? For example, I will give an example. You are writing `sin' for the sine function on an open interval. The interval could be the whole of $\mathbb{R}$ or just $[-1.1]$, okay? You can treat the same function as from $\mathbb{R}$ to $\mathbb{R}$ also, even though its image will be only in $[-1.1]$. Moreover, there is also a complex valued sin function defined all over the complex numbers and we use the same notation `sin' for it okay? In all these examples of the function denoted by `sin', you take a function, you restrict it to a subset of the domain and still use the same notation. This practice occurs all the time in mathematics, right? According to categorical language this is not allowed. The moment domain or codomain are different, morphisms have to be treated as different. They are different elements.

That is what we have chosen in the definition of a category. However, I told you that cutting down clumsy notations is a must in all mathematics that we do, okay? But, for example, this is not at all done in computer science, by computers, okay? On the other hand, one of the basic objects of language of category theory is to provide us `rigor without being too verbose'. However, to achieve this one, right in the beginning, we need to be very very verbose. Right? The reward will come later.

Insisting on having $M(A, B)$ and $M(A', B')$ disjoint all the time and so on, in the beginning seems to you to be very, very verbose. You will see the reward coming in a little later. Just tell one single statement in the categorical language, it will mean thousands of theorems. That is the kind of achievement which category theory has done. All right.

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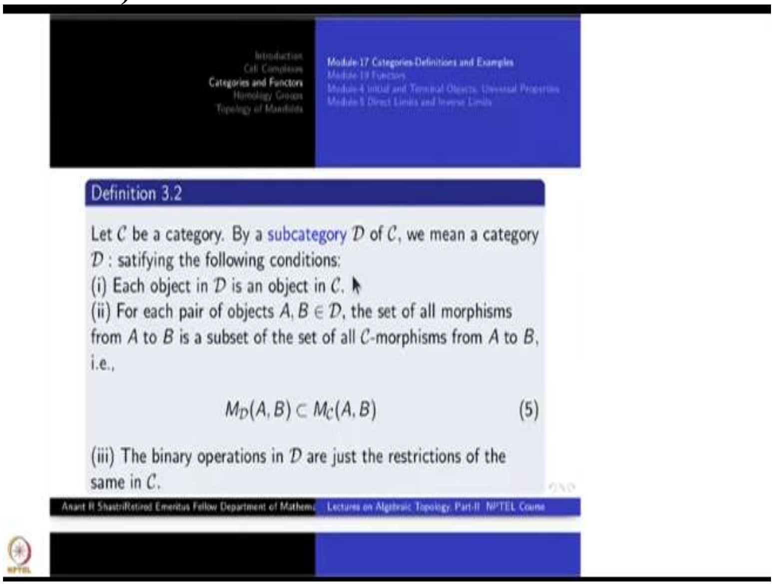

Let $\mathcal{C}$ be a category. Let us see what is the meaning of a subcategory that is what I am introducing right now. Okay? By a subcategory of $\mathcal{C}$, we mean a category $\mathcal{D}$ which satisfies the following conditions.

(i) So, $\mathcal{D}$ is going to be a category by itself. But what is this relation between $\mathrm{Obj}(\mathcal{D})$ and $\mathrm{Obj}(\mathcal{C})$? Each object of $\mathcal{D}$ is an object in $\mathcal{C}$ also.

(ii) For each pair of objects A, B inside $\mathcal{D}$, the set of morphisms in $\mathcal{D}$ from A to B is a subset of the set of all morphisms in $\mathcal{C}$ from A to B, i.e., $M_{\mathcal{D}}(A, B)$ is a subset of $M_{\mathcal{C}}(A, B)$.

The third condition is: the binary operation inside $\mathcal{D}$ are the restriction of the corresponding binary operations inside $\mathcal{C}$. This condition is similar to the definition of a subgroup, a subring, vector subspace etc. Okay?

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Further if equality holds in this one, namely, for A and B in $\mathrm{Obj}(\mathcal{D})$, $M_{\mathcal{D}}(A,B) = M_{\mathcal{C}}(A,B)$. , if this happens for all pairs of A and B, whenever A and B are inside $\mathcal{D}$, then we call $\mathcal{D}$ a full subcategory of $\mathcal{C}$.

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So, we will soon have examples of this no problem. Right now we will give examples of categories. As you might have guessed, the most important and easiest category, which you may call the mother of many, many categories, mother category, (or daddy category) which is denoted by **Ens**, (**Ens** is a short for `ensemble' a French word, which means `sets').

Take the collection of all sets, that is the Obj(**Ens**). for this category. All objects are sets. Okay? What are the morphisms? Morphisms are the usual set functions. The binary operation is the ususal compostion of functions.

So, verification of the axioms is totally easy because the axioms have been modelled on what is happening in the sets okay? So, you see that many other examples that we are going to discuss, they are all in some sense, some subcategories of this category **Ens** or some slight modifications. That is why I call **Ens** the mother category.

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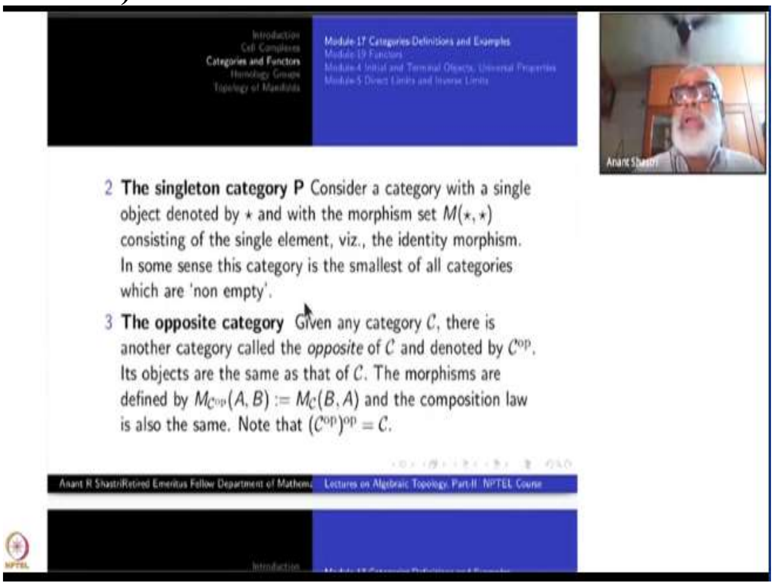

So, the next example I will give you a very simple one. Consider the singleton category **P**, a category with a single object. How to do denote this single object? Put just a $\star$. There is only this object. It has no structure, nothing. but it is the only object in **P**, okay? Then I have to define morphisms $M_P(\star,\star)$. So, take that also equal to the singleton set. You have to take at least one element there because the axiom says that $M(A,A)$ has to be nonempty. And that element is also determined viz., the identity morphism.

So, the composition is also well defined by the 2-sided identiy axiom. There is no other choice okay. So, this is, in some sense the smallest category. Maybe one can make a category where the family of objects is empty. I am not going to discuss it further, but that is also allowed okay? Empty objects. Other than that, this is **P** is a nice category. Sometimes category theorist write this as 1 and the empty category as 0 okay? I am not going to do all that, but I am aware of such things okay?

The opposite category of a given category. Opposite category means that there is already a category $\mathcal{C}$, I am going to construct a category, call $\mathcal{C}^{op}$, the opposite of $\mathcal{C}$. Objects of opposite category are the same as the objects of the original category $\mathcal{C}$, but whenever A and B are members of this category $\mathcal{C}$, $M_{\mathcal{C}^{op}}(A, B)$ is taken to be $M_{\mathcal{C}}(B, A)$, okay? And the binary operation law is also the same except that you write it in the reverse order:

So, you can easily verify that left identity in $\mathcal{C}$ becomes right identity in $\mathcal{C}^{op}$ and vice versa. So, the 2-sided identities are there, no problem, okay? Associativity is also no problem. So, these are all easy to verify. Not only that, if you take $(\mathcal{C}^{op})^{op}$, i.e., the opposite of the opposite category of $\mathcal{C}$, then you get the category $\mathcal{C}$ back. This idea just looks like a totally useless thing. But proof of some of very deep results in category theory use this idea, which goes under the name `duality principles'. We would not have time for discussing that one. But, because of the importance of this idea in general category, I have just introduced it here, a very simple idea of constructing in the opposite category. Similarly, the singleton category is also very important in some sense, though we may not have much use of them.

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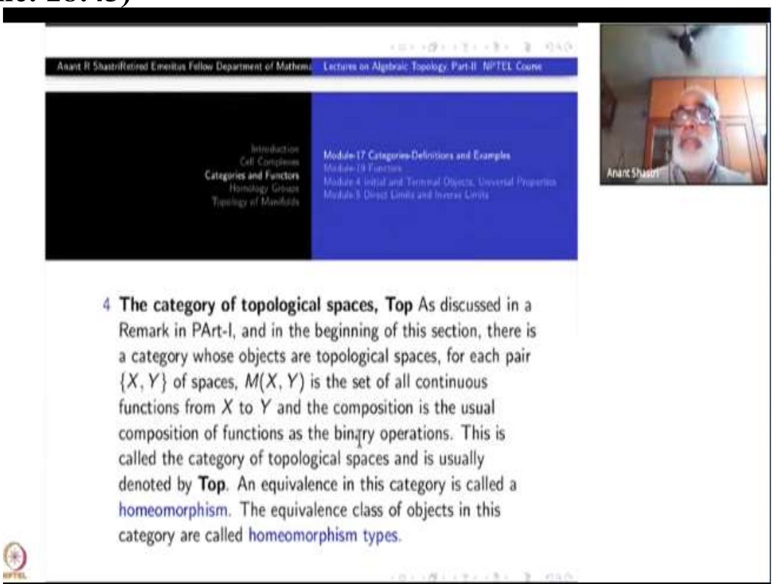

The category of topological spaces is one that is what we are interested in. What are the objects? All topological spaces. That collection by the way, is not a set okay? The collection of all topological spaces is not a set but it is a class. What are morphisms? Continuous functions. Okay? Binary operations are compositions of functions. What are the equivalences? Homomorphisms. What are equivalence classes of objects? They are homeomorphism types, okay? So, this is the first example we started with, even before giving the definition. So, now we can see that it is a good example.

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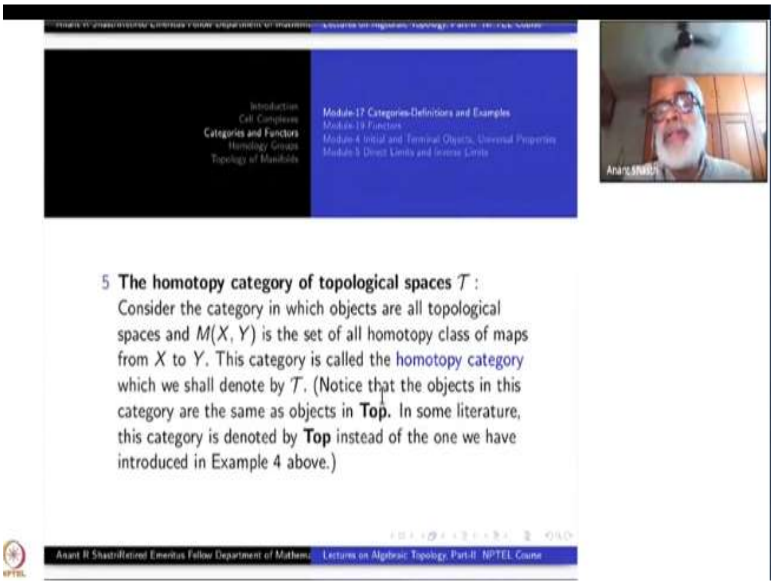

In the next example, you will start seeing the power of category theory --- how one can just change a little bit and express a lot of ideas. Consider the category in which objects are all topological spaces but morphisms from X to Y, where X and Y are topological spaces, are not continuous maps, but the homotopy classes of continuous functions. A homotopy class of a continous function from X to Y is treated as a morphism from X to Y, okay? This category is called the homotopy category.

Recall an elementary property of homotopy that if f is homotopic to g and $g \circ h$ and $g \circ h'$ are defined, then $f \circ h$ is homotopic to $f \circ h'$. We have verified that in part I. Such properties are necessary to verify that this homotopy category makes sense, viz., we can define the binary operations on homotopy classes to be the homotopy classes of composition of representative functions.

What is an equivalence here? A homotopy equivalence. What are the equivalence classes of objects? Homotopy equivalent spaces?

Suddenly, it has a totally different meaning okay? And this is the category in which algebraic topology is all the time interested in, okay?

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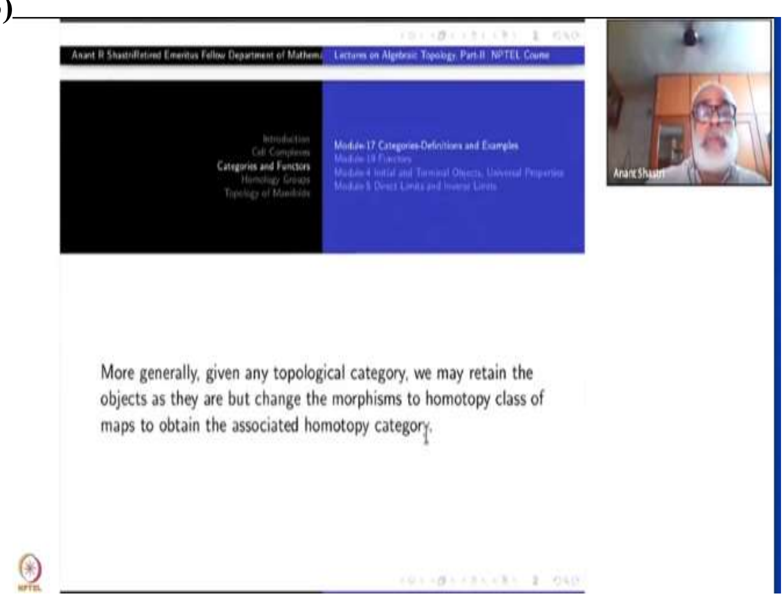

So, such a change can happen in different ways. For example, suppose you start with the category of topological spaces, **Top**. Change the family of objects slightly. There are various ways of doing it, like, you can take objects to be simplicial complexes, or CW complexes, (I am going to discuss them below in detail,) okay? Or just take all Hausdorff spaces as objects. But remain all continuous functions as morphisms. Then you may change them to homotopy classes, okay and so on. So, there is a lot of scope in this language. Okay?

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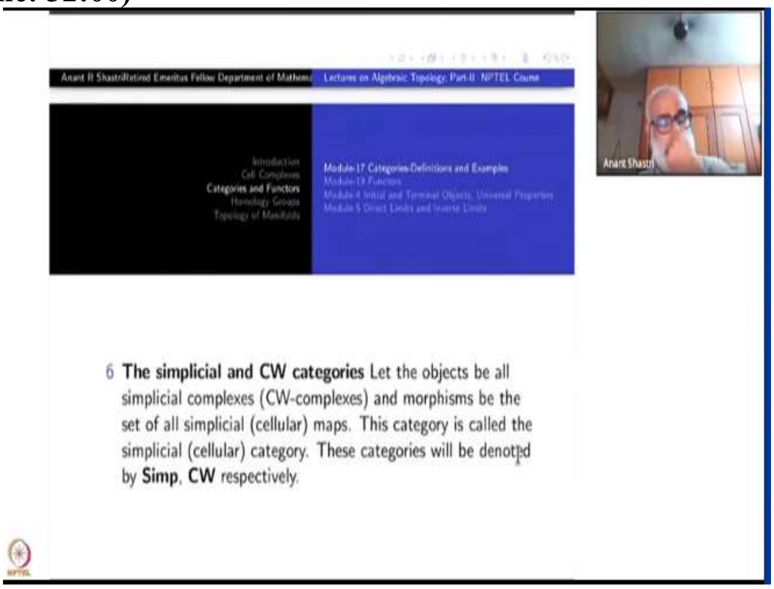

So, now these two categories are of importance. The simplicial category was essentially introduced in part one, though we did not have this terminology of category then. Now we will just verify all those definitions terminology. Similarly, the CW-category was introduced just a couple of days back. Let go through them again.

What are the objects in CW? Topological spaces which have an extra structure namely a CW-complex structure. Remember that topological space itself is a set with an extra structure. Now we have one more extra structure, namely the CW-structure. What are the morphisms? cellular maps between CW compelxes. Composite of cellular maps is cellular, and identity map is always cellular. This is what we need to verify but that is already done. Similarly the simplicial category in which morphisms are simplicial maps. Okay So we will discuss more examples next time. Thank you.