

Introduction to Algebraic Topology Part – II
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology – Bombay

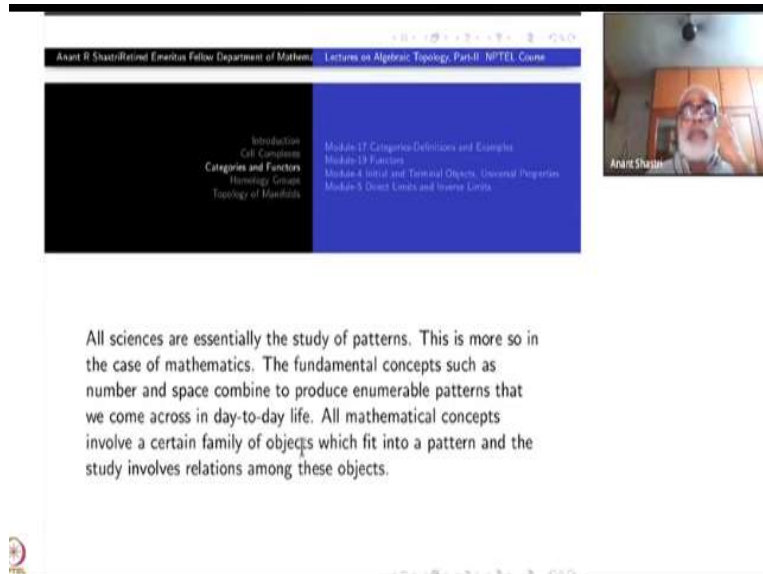
Lecture – 17
Categories Definitions and Examples

(Refer Slide Time: 00:11)



So, this chapter is a quick introduction to the language of categories and functors. What we are going to do is just a limited introduction. Interested reader may look into the books I referred at the end of the lecture notes. The book by Adamek is freely downloadable and it is quite a readable book. The topic as I told is the language of modern mathematics, takes some time to master it. So, I would not say that, right now you will become a master of whatever I want to introduce here, though, that itself is very minimal. But because I am going to use this language again and again, certainly by the end of this course, I hope you will all know how to use this language. Studying categories and functors for its own sake is not at all done here. You must take note of that. It is not like we are going to write poetry in this language, we are merely trying to learn the market language or know how to do day-to-day business. That is all.

(Refer Slide Time: 01:54)



NPTEL

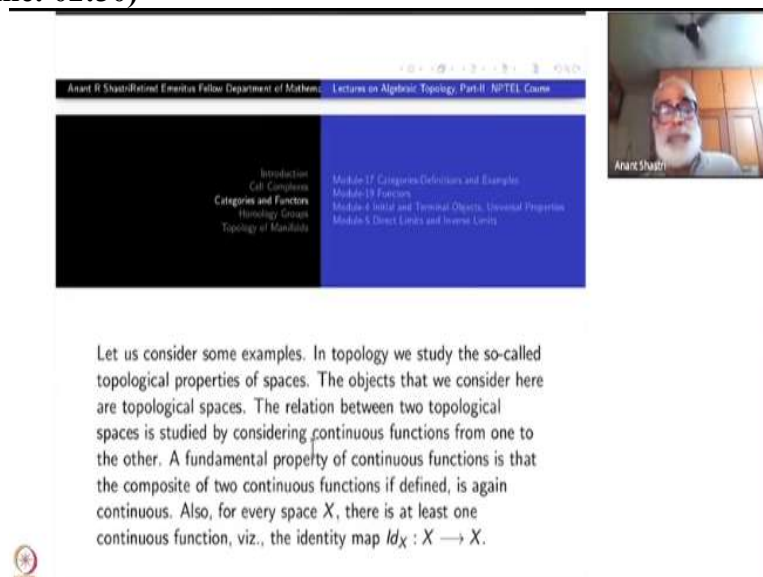
Anant R. Shastri, Retired Emeritus Fellow, Department of Mathematics, Lectures on Algebraic Topology, Part-II: NPTEL Course

Introduction	Module-17: Categories, Definitions and Examples
Cell Complexes	Module-18: Functors
Categories and Functors	Module-4: Initial and Terminal Objects, Universal Properties
Homology Groups	Module-5: Direct Limits and Inverse Limits
Topology of Manifolds	

All sciences are essentially the study of patterns. This is more so in the case of mathematics. The fundamental concepts such as number and space combine to produce enumerable patterns that we come across in day-to-day life. All mathematical concepts involve a certain family of objects which fit into a pattern and the study involves relations among these objects.

All sciences are essentially study of patterns. This is more so in mathematics. The fundamental concepts such as number and space combine to produce enumerable patterns that we come across in day-to-day life. All mathematics concepts involve a certain family of objects which fit into a pattern and then the study involves relations among those objects.

(Refer Slide Time: 02:30)



NPTEL

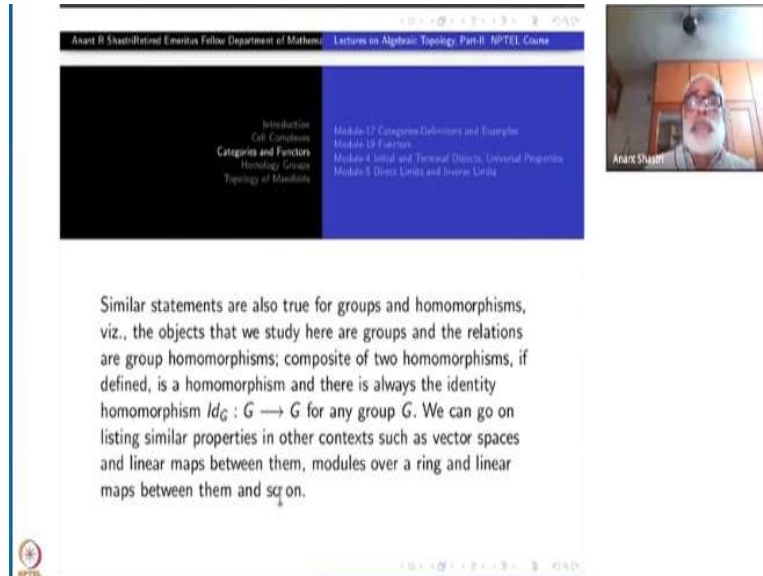
Anant R. Shastri, Retired Emeritus Fellow, Department of Mathematics, Lectures on Algebraic Topology, Part-II: NPTEL Course

Introduction	Module-17: Categories, Definitions and Examples
Cell Complexes	Module-18: Functors
Categories and Functors	Module-4: Initial and Terminal Objects, Universal Properties
Homology Groups	Module-5: Direct Limits and Inverse Limits
Topology of Manifolds	

Let us consider some examples. In topology we study the so-called topological properties of spaces. The objects that we consider here are topological spaces. The relation between two topological spaces is studied by considering continuous functions from one to the other. A fundamental property of continuous functions is that the composite of two continuous functions if defined, is again continuous. Also, for every space X , there is at least one continuous function, viz., the identity map $Id_X : X \rightarrow X$.

Let us consider some examples. In topology, we study the so called topological properties of topological spaces. The objects that we consider here are topological spaces with the relations among them being what are called continuous functions. The fundamental property of continuous functions is that the composite of two continuous functions if defined, is a continuous function. And identity map from any space to itself is also continuous. Okay?

(Refer Slide Time: 03:14)

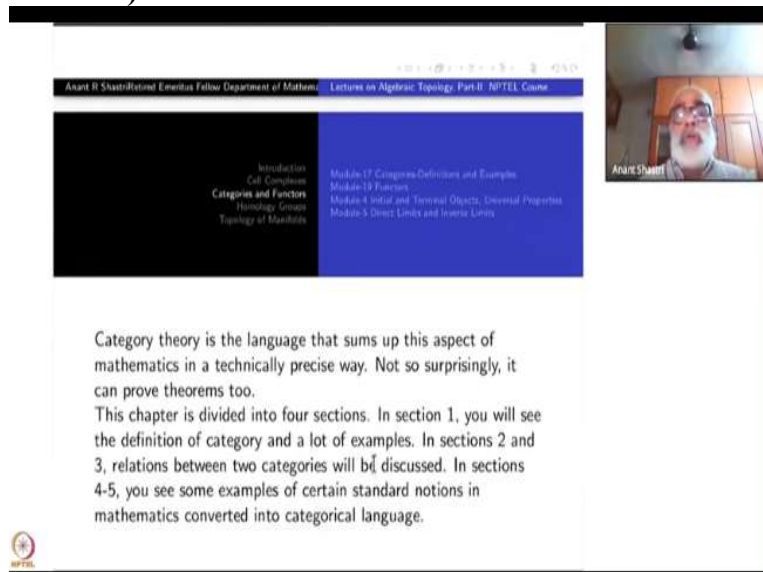


Similar statements are also true for groups and homomorphisms, viz., the objects that we study here are groups and the relations are group homomorphisms; composite of two homomorphisms, if defined, is a homomorphism and there is always the identity homomorphism $Id_G : G \rightarrow G$ for any group G . We can go on listing similar properties in other contexts such as vector spaces and linear maps between them, modules over a ring and linear maps between them and so on.

Similar statements can be done about groups. Suppose you are studying groups, then what are the functions from one group to another group, that we are going to study? Answer is homomorphisms. Once again, composite of two homomorphisms if defined, is again a homomorphism and identity map from any group to itself is also a homomorphism, Okay? We can go on listing such examples like vector spaces and linear map, modules over a ring and the module homomorphism which are also called linear maps, and so on. Okay?

So, what we shall do is give a strictly rigorous definition of categories and then re-examine some of these examples, in the light of this definition which extract certain basic properties that we are trying to explain.

(Refer Slide Time: 04:30)



Category theory is the language that sums up this aspect of mathematics in a technically precise way. Not so surprisingly, it can prove theorems too.

This chapter is divided into four sections. In section 1, you will see the definition of category and a lot of examples. In sections 2 and 3, relations between two categories will be discussed. In sections 4-5, you see some examples of certain standard notions in mathematics converted into categorical language.

Category theory is a language that sums up this aspect of mathematics in a technically precise way. Not so surprisingly, it can help you prove theorems also. This chapter is divided into

four sections. First section, you will see definition of a category and sufficiently many examples. In second and third, we will study relations between them under the name of functors. In 4th and 5th, we see more examples of certain standard notions which can be expressed, which have been expressed, which have been studied without the categorical language, but when you put them in the categorical language, how nice they become, okay? So, with this background, let us begin with the abstracts.

(Refer Slide Time: 05:27)

Introduction
Cat. Constructions
Categories and Functors
Homology Groups
Topology of Manifolds

Module-17 Categories-Definitions and Examples
Module-18 Functors
Module-19 Natural and Universal Properties
Module-20 Direct Limits and Inverse Limits

Categories-Definitions and Examples Module-17

Definition 3.1

A category \mathcal{C} consists of a family $\text{Obj}(\mathcal{C})$ of objects with the following properties:

(i) To each ordered pair (A, B) of objects with $A, B \in \text{Obj}(\mathcal{C})$, there is a set denoted by $M(A, B)$; $M(A, B)$, $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$ in \mathcal{C} . The elements of $M(A, B)$ are called morphisms with domain A and codomain B .

Anant B Shastri-Rajendra Keshava Department of Mathematics, Lectures on Algebraic Topology, Part II, NPTEL Course

A category \mathcal{C} consists of, (I can say that it an ordered quadruple of certain things but, I do not want to use that kind of language), it consists of a family $\text{Obj}(\mathcal{C})$ (read it as objects \mathcal{C}), members of this family are called objects, okay, satisfying certain properties. That will come in the definition. The whole definition has to be understood together, before you understand individual stuffs here. So, what are the properties? One by one I am going to list them.

(i) For each ordered pair (A, B) of objects, i.e., A, B are members of this family or a class, (take note that we are careful not use the word 'set' this family, there is this slight difference here, $\text{Obj}(\mathcal{C})$ is a collection or you can say it a class but do not use the word set. However, what makes sense is the membership relation between objects), there is assigned a set (this time it is a set) denoted by $M(A, B)$ is a set with the property that the two sets $M(A, B)$, $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$. This must be true for all members A, A', B, B' of $\text{Obj}(\mathcal{C})$. These are all axioms, all are part of the definition, each word each comma here is a part of the definition that we are making, okay? The elements of $M(A, B)$ are called morphisms in \mathcal{C} , with domain A and codomain B . Okay?

(Refer Slide Time: 08:09)

there is a set denoted by $M(A, B)$; $M(A, B)$, $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$ in C . The elements of $M(A, B)$ are called **morphisms** with domain A and codomain B .

Anant R Shastri Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

Introduction Cell Complexes Categories and Functors Homology Groups Topology of Manifolds	Module 17: Categories Definitions and Examples Module 18: Functors Module 19: Initial and Terminal Objects, Universal Properties Module 20: Direct Limits and Inverse Limits
--	--

We follow the practice of writing $f : A \rightarrow B$ whenever,
 $f \in M(A, B)$.

NPTEL

(Refer Slide Time: 09:05)

Definition continued

(ii) For each triple (A, B, C) of objects, there is a binary operation

$$\circ : M(A, B) \times M(B, C) \rightarrow M(A, C) \quad (f, g) \mapsto g \circ f;$$

these are, collectively associative in the following sense: for $f \in M(A, B)$, $g \in M(B, C)$, $h \in M(C, D)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Anant R Shastri Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

(ii) If you take a triple of objects, A, B, C , then you have three sets $M(A, B)$, $M(B, C)$, and $M(A, C)$. There is a binary operation from $M(A, B) \times M(B, C)$ to $M(A, C)$. Now these three are all sets and so binary operation makes sense. So I write it as (f, g) goes to $g \circ f$. Okay? Similar to the standard practice of writing compositions of functions. These binary operations are for each triple there is one okay? So, if you change those triples the operations also change but we are using the same notation \circ . They are collectively associative. What is the meaning of 'collectively associative'? In the following sense namely, given f from A to B , g from B to C , h from C to D , then we have $g \circ f$, right, we also have $h \circ g$, these two things make sense.

What we want is $g \circ f$ is followed by h to get an element of $M(A, D)$ that element should be the same f followed by $h \circ g$. That is the brackets can be interchanged like this, just like associativity of compositions of functions. Collectively associative means whenever the compositions are defined then we have the associative law. The compositions may not be defined unless the arrows match, just like the case of sets and functions. Okay?

(Refer Slide Time: 11:09)

Definition continued

(iii) For each object A in \mathcal{C} , there exists a (unique) $Id_A \in M(A, A)$ such that for any $f \in M(A, B)$ and $g \in M(C, A)$ we have,

$$f \circ Id_A = f; \quad Id_A \circ g = g.$$

A morphism $f : A \longrightarrow B$ in \mathcal{C} is called an **equivalence** or **invertible** if there exists a morphism $g : B \longrightarrow A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Then g is said to be an inverse to f . If there exists an equivalence $f : A \longrightarrow B$ then we say A and B are **equivalent objects** in \mathcal{C} .

(iii) The third condition is that for each object $A \in \mathcal{C}$, there exists a unique Id_A (note this notation) inside $M(A, A)$, the set of morphisms from A to A , there is a unique element which has the property that it is a 2-sided identity; for all f from A to B and for all g from C to A , you have $f \circ Id_A$ (this makes sense) is equal to f and $Id_A \circ g$ is equal to g . Thus Id_A is a 2-sided identity for the binary operations okay? This completes the definition of a category.

A morphism f from A to B is called an equivalence in the category \mathcal{C} , if there exists a morphism g from B to A such that $g \circ f$ is equal to Id_A and $f \circ g$ is equal to Id_B . Then g is said to be the inverse of f and f is said to be the inverse of g . In fact, it is an elementary algebra to verify that the identity elements in $M(A, A)$ as well as inverse of morphism it exist are unique. There may not be any inverse of a given morphism in general. Morphisms with inverse are also called invertible morphisms.

So, if there is an equivalence f from A to B , then we say A and B are equivalent objects in $\text{Obj}(\mathcal{C})$. Okay? That is the end of some definitions. So, we have defined a category? Objects in it, morphisms, equivalences and equivalent objects. Okay?

(Refer Slide Time: 13:44)

Asmit B Shastri-Rajendran Emeritus Fellow Department of Mathem... Lectures in Algebraic Topology, Part-II, NPTEL Course

Introduction	Module 17: Category Definitions and Examples
Cell Complexes	Module 18: Functors
Categories and Functors	Module 19: Initial and Terminal Objects, Universal Properties
Homology, Spectra	Module 20: Direct Limits and Inverse Limits
Topology of Manifolds	

Remark 3.1

(i) Often when more than one category is involved in the discussion, we write $M_{\mathcal{C}}(X, Y)$, $\text{Mor}(X, Y)$ or $\text{Hom}_{\mathcal{C}}(X, Y)$ to denote the set of all morphisms from X to Y , where X and Y are considered as objects in a particular category \mathcal{C} .

(ii) It follows easily that the identity morphism Id_A is unique in $M(A, A)$ for all A . Note that $M(A, B)$ is a set, may or may not be empty, whereas objects of \mathcal{C} need not be sets. Neither is the collection of objects a set, in general. You need not bother about these purely 'logical' problems at this stage.

Asmit B Shastri-Rajendran Emeritus Fellow Department of Mathem... Lectures in Algebraic Topology, Part-II, NPTEL Course

Now, I want to make a few elementary remarks. Often when more than one category is involved at a time in the discussion, for object $X, Y \in \mathcal{C}$, instead of just the simple notation $M(X, Y)$, we write $M_{\mathcal{C}}(X, Y)$ to indicate which category we are working in. Also the notation $\text{Mor}(X, Y)$ (or $\text{Mor}_{\mathcal{C}}(X, Y)$) is used in place of $M(X, Y)$ (respectively, $M_{\mathcal{C}}(X, Y)$).

Suppose, I am having two different categories \mathcal{C} and \mathcal{D} at hand. It may happen that objects under discussion are objects in both of them. Therefore, if I just use $M(X, Y)$, you do not know whether I am talking about morphism is it is you category \mathcal{C} or category \mathcal{D} , right? So, that is why in that case, we use the elaborate notation. When you know that the discussion is going on in single particular category, then we use the simpler notation as usual is a practice. But this kind of thing should not be done when you are dealing with a computer for instance. Computers would not understand unless you are strictly following the notation that you have fed into it. By the way, computer scientists use the category theory very much okay?

So, it follows easily that identity morphism is unique in $M(A, A)$ for all A . Try to write down a proof yourself. Note that $M(A, B)$ is a set but it may be empty also. Clearly, $M(A, A)$ is non empty, because Id_A is there.

The collection of all objects in \mathcal{C} need not be sets, Okay? We will see some examples later on. We have already remarked that the family $\text{Obj}(\mathcal{C})$ need not be a set. You need not bother about these purely logical problems at this stage. That is the key word here. Because if you get into this logical problem at this stage, you will never learn the category theory. You may bother about it only when you are talking about foundational mathematics, like logic, machine learning etc., The logicians are more interested in this aspect of category okay? Unless, we have some problems we should not bother about that aspect. Right now you learn the language.

(Refer Slide Time: 16:56)

empty, whereas objects of \mathcal{C} need not be sets. Neither is the collection of objects a set, in general. You need not bother about these purely 'logical' problems at this stage.

ANANT SHARMA
Assistant Professor, Department of Mathematics, IIT Bombay

Introduction
Cat. Concepts
Categories and Functors
Homology Groups
Topology of Manifolds

Module 17 Categories Definitions and Examples
Module 19 Functors
Module 4 Initial and Terminal Objects, Universal Properties
Module 5 Direct Limits and Inverse Limits

(iii) It also follows that if f is invertible, then its inverse is unique and hence we can write it as f^{-1} .

(iv) The equivalence of any two objects defines an equivalence relation on the family of all objects of a category. The central theme in any categorical study is to determine this set.

We have also remarked that if a morphism f is invertible then its inverse is unique. therefore, I can use the notation f^{-1} for the inverse of f , Okay? The equivalence of any two objects in $\text{Obj}(\mathcal{C})$ defines an equivalence relation, namely, it is reflexive, transitive and symmetric. The central problems in mathematics is to determine the equivalence classes in any category that you have chosen.

(Refer Slide Time: 18:09)

Introduction
Cat. Conclusions
Categories and Functors
Homotopy Groups
Topology of Manifolds

Module 17: Categories, Definitions and Examples
Module 18: Functors
Module 19: Initial and Terminal Objects, Universal Properties
Module 20: Direct Limits and Inverse Limits

(v) The condition of disjointness of $M(A, B)$ and $M(A', B')$ unless $A = A'$, $B = B'$ is important especially in the categorical language. We are aware of the common practice of denoting a restriction of a function $f : X \rightarrow Y$ to a subset $A \subset X$ by the same notation, and often we too will follow this 'convenient' practice. For example the functions $\sin : \mathbb{R} \rightarrow [-1, 1]$, $\sin : \mathbb{R} \rightarrow \mathbb{R}$, $\sin : \mathbb{C} \rightarrow \mathbb{C}$ are all denoted by \sin , (which is a sin according to categorical language). The justification is that we are cutting down on clumsy notations which may obscure the mathematical ideas that we want to present. On the other hand, one of the basic objects of the language of category theory is to provide us rigor without being too verbose.

Anant R Shrivastava, Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

One more remark. The condition of disjointness. Remember that I told you $M(A, B)$ and $M(A', B')$ are disjoint unless $A = A'$ and $B = B'$. There is no overlap among them. That is very important here right now, you may not understand the importance of this right now, because in standard mathematics such as in calculus and so on, right from the ancient days of Euler, Gauss and so on, we are not doing this, and we have to keep that tradition and cannot give it up, okay? For example, I will give an example. You are writing 'sin' for the sine function on an open interval. The interval could be the whole of \mathbb{R} or just $[-1, 1]$, okay? You can treat the same function as from \mathbb{R} to \mathbb{R} also, even though its image will be only in $[-1, 1]$. Moreover, there is also a complex valued sin function defined all over the complex numbers and we use the same notation 'sin' for it okay? In all these examples of the function denoted by 'sin', you take a function, you restrict it to a subset of the domain and still use the same notation. This practice occurs all the time in mathematics, right? According to categorical language this is not allowed. The moment domain or codomain are different, morphisms have to be treated as different. They are different elements.

That is what we have chosen in the definition of a category. However, I told you that cutting down clumsy notations is a must in all mathematics that we do, okay? But, for example, this is not at all done in computer science, by computers, okay? On the other hand, one of the basic objects of language of category theory is to provide us 'rigor without being too verbose'. However, to achieve this one, right in the beginning, we need to be very very verbose. Right? The reward will come later.

Insisting on having $M(A, B)$ and $M(A', B')$ disjoint all the time and so on, in the beginning seems to you to be very, very verbose. You will see the reward coming in a little later. Just tell one single statement in the categorical language, it will mean thousands of theorems. That is the kind of achievement which category theory has done. All right.

(Refer Slide Time: 21:50)

Introduction
Set-Combinatorics
Categories and Functors
Homology Groups
Topology of Manifolds

Module 17: Categories-Definitions and Examples
Module 18: Functors
Module 19: Initial and Terminal Objects, Universal Properties
Module 20: Direct Limits and Inverse Limits

Definition 3.2

Let \mathcal{C} be a category. By a subcategory \mathcal{D} of \mathcal{C} , we mean a category \mathcal{D} : satisfying the following conditions:

(i) Each object in \mathcal{D} is an object in \mathcal{C} .

(ii) For each pair of objects $A, B \in \mathcal{D}$, the set of all morphisms from A to B is a subset of the set of all \mathcal{C} -morphisms from A to B , i.e.,

$$M_{\mathcal{D}}(A, B) \subset M_{\mathcal{C}}(A, B) \quad (5)$$

(iii) The binary operations in \mathcal{D} are just the restrictions of the same in \mathcal{C} .

Asant R Shastri Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

Let \mathcal{C} be a category. Let us see what is the meaning of a subcategory that is what I am introducing right now. Okay? By a subcategory of \mathcal{C} , we mean a category \mathcal{D} which satisfies the following conditions.

- (i) So, \mathcal{D} is going to be a category by itself. But what is this relation between $\text{Obj}(\mathcal{D})$ and $\text{Obj}(\mathcal{C})$? Each object of \mathcal{D} is an object in \mathcal{C} also.
- (ii) For each pair of objects A, B inside \mathcal{D} , the set of morphisms in \mathcal{D} from A to B is a subset of the set of all morphisms in \mathcal{C} from A to B , i.e., $M_{\mathcal{D}}(A, B)$ is a subset of $M_{\mathcal{C}}(A, B)$.

The third condition is: the binary operation inside \mathcal{D} are the restriction of the corresponding binary operations inside \mathcal{C} . This condition is similar to the definition of a subgroup, a subring, vector subspace etc. Okay?

(Refer Slide Time: 23:09)

Further if equality holds in this one, namely, for A and B in $\text{Obj}(\mathcal{D})$, $M_{\mathcal{D}}(A, B) = M_{\mathcal{C}}(A, B)$, if this happens for all pairs of A and B , whenever A and B are inside \mathcal{D} , then we call \mathcal{D} a full subcategory of \mathcal{C} .

(Refer Slide Time: 23:38)

So, we will soon have examples of this no problem. Right now we will give examples of categories. As you might have guessed, the most important and easiest category, which you may call the mother of many, many categories, mother category, (or daddy category) which is denoted by **Ens**, (**Ens** is a short for 'ensemble' a French word, which means 'sets').

Take the collection of all sets, that is the $\text{Obj}(\mathbf{Ens})$. for this category. All objects are sets. Okay? What are the morphisms? Morphisms are the usual set functions. The binary operation is the usual composition of functions.

So, verification of the axioms is totally easy because the axioms have been modelled on what is happening in the sets okay? So, you see that many other examples that we are going to discuss, they are all in some sense, some subcategories of this category **Ens** or some slight modifications. That is why I call **Ens** the mother category.

(Refer Slide Time: 24:56)

Introduction Cat. Conventions Categories and Functors Homology Groups Topology of Manifolds	Module 17: Categories Definitions and Examples Module 18: Functors Modules 19: Initial and Terminal Objects, Universal Properties Module 20: Direct Limits and Inverse Limits
--	---

2 The singleton category \mathbf{P} Consider a category with a single object denoted by $*$ and with the morphism set $M(*, *)$ consisting of the single element, viz., the identity morphism. In some sense this category is the smallest of all categories which are 'non empty'.

3 The opposite category Given any category \mathcal{C} , there is another category called the *opposite* of \mathcal{C} and denoted by \mathcal{C}^{op} . Its objects are the same as that of \mathcal{C} . The morphisms are defined by $M_{\mathcal{C}^{op}}(A, B) := M_{\mathcal{C}}(B, A)$ and the composition law is also the same. Note that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Asmita B. Shastri/Faculty Emeritus Fellow Department of Math., Lectures on Algebraic Topology, Part II, NPTEL Course

So, the next example I will give you a very simple one. Consider the singleton category **P**, a category with a single object. How to denote this single object? Put just a $*$. There is only this object. It has no structure, nothing. but it is the only object in **P**, okay? Then I have to define morphisms $M_P(*, *)$. So, take that also equal to the singleton set. You have to take at least one element there because the axiom says that $M(A, A)$ has to be nonempty. And that element is also determined viz., the identity morphism.

So, the composition is also well defined by the 2-sided identity axiom. There is no other choice okay. So, this is, in some sense the smallest category. Maybe one can make a category where the family of objects is empty. I am not going to discuss it further, but that is also allowed okay? Empty objects. Other than that, this is **P** is a nice category. Sometimes category theorist write this as **1** and the empty category as **0** okay? I am not going to do all that, but I am aware of such things okay?

The opposite category of a given category. Opposite category means that there is already a category \mathcal{C} , I am going to construct a category, call \mathcal{C}^{op} , the opposite of \mathcal{C} . Objects of opposite category are the same as the objects of the original category \mathcal{C} , but whenever A and B are

members of this category \mathcal{C} , $M_{\mathcal{C}^{op}}(A, B)$ is taken to be $M_{\mathcal{C}}(B, A)$, okay? And the binary operation law is also the same except that you write it in the reverse order:

So, you can easily verify that left identity in \mathcal{C} becomes right identity in \mathcal{C}^{op} and vice versa. So, the 2-sided identities are there, no problem, okay? Associativity is also no problem. So, these are all easy to verify. Not only that, if you take $(\mathcal{C}^{op})^{op}$, i.e., the opposite of the opposite category of \mathcal{C} , then you get the category \mathcal{C} back. This idea just looks like a totally useless thing. But proof of some of very deep results in category theory use this idea, which goes under the name 'duality principles'. We would not have time for discussing that one. But, because of the importance of this idea in general category, I have just introduced it here, a very simple idea of constructing in the opposite category. Similarly, the singleton category is also very important in some sense, though we may not have much use of them.

(Refer Slide Time: 28:43)

Anant B. Shastri, Emeritus Fellow, Department of Mathematics, IIT Madras. Lectures on Algebraic Topology, Part-II, NPTEL Course.

Introduction
Cell Complexes
Categories and Functors
Homology Groups
Topology of Manifolds

Module 17: Categories, Definitions and Examples
Module 18: Functors
Module 4: Initial and Terminal Objects, Universal Properties
Module 5: Direct Limits and Inverse Limits

4 The category of topological spaces, **Top** As discussed in a Remark in PArt-I, and in the beginning of this section, there is a category whose objects are topological spaces, for each pair $\{X, Y\}$ of spaces, $M(X, Y)$ is the set of all continuous functions from X to Y and the composition is the usual composition of functions as the binary operations. This is called the category of topological spaces and is usually denoted by **Top**. An equivalence in this category is called a **homeomorphism**. The equivalence class of objects in this category are called **homeomorphism types**.

The category of topological spaces is one that is what we are interested in. What are the objects? All topological spaces. That collection by the way, is not a set okay? The collection of all topological spaces is not a set but it is a class. What are morphisms? Continuous functions. Okay? Binary operations are compositions of functions. What are the equivalences? Homeomorphisms. What are equivalence classes of objects? They are homeomorphism types, okay? So, this is the first example we started with, even before giving the definition. So, now we can see that it is a good example.

(Refer Slide Time: 29:42)

Introduction Set-Combinatorics Categories and Functors Homotopy Groups Topology of Manifolds	Module 17: Categories-Definitions and Examples Module 18: Functors Module 4: Initial and Terminal Objects, Universal Properties Module 5: Direct Limits and Inverse Limits
---	---

5 The homotopy category of topological spaces \mathcal{T} :
 Consider the category in which objects are all topological spaces and $M(X, Y)$ is the set of all homotopy class of maps from X to Y . This category is called the **homotopy category** which we shall denote by \mathcal{T} . (Notice that the objects in this category are the same as objects in **Top**. In some literature, this category is denoted by **Top** instead of the one we have introduced in Example 4 above.)

Anant R Shastri Emeritus Fellow Department of Mathematics, Lecture on Algebraic Topology, Part II, NPTEL Course

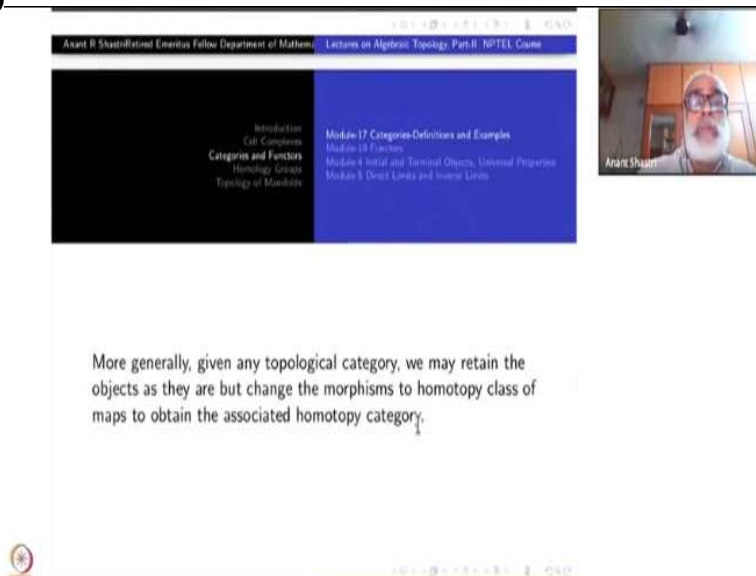
In the next example, you will start seeing the power of category theory --- how one can just change a little bit and express a lot of ideas. Consider the category in which objects are all topological spaces but morphisms from X to Y , where X and Y are topological spaces, are not continuous maps, but the homotopy classes of continuous functions. A homotopy class of a continuous function from X to Y is treated as a morphism from X to Y , okay? This category is called the homotopy category.

Recall an elementary property of homotopy that if f is homotopic to g and $g \circ h$ and $g \circ h'$ are defined, then $f \circ h$ is homotopic to $f \circ h'$. We have verified that in part I. Such properties are necessary to verify that this homotopy category makes sense, viz., we can define the binary operations on homotopy classes to be the homotopy classes of composition of representative functions.

What is an equivalence here? A homotopy equivalence. What are the equivalence classes of objects? Homotopy equivalent spaces?

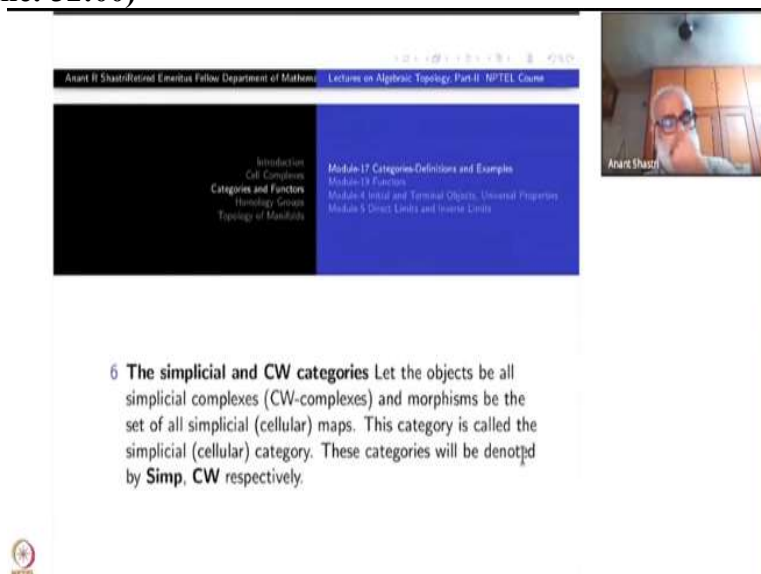
Suddenly, it has a totally different meaning okay? And this is the category in which algebraic topology is all the time interested in, okay?

(Refer Slide Time: 31:23)



So, such a change can happen in different ways. For example, suppose you start with the category of topological spaces, **Top**. Change the family of objects slightly. There are various ways of doing it, like, you can take objects to be simplicial complexes, or CW complexes, (I am going to discuss them below in detail,) okay? Or just take all Hausdorff spaces as objects. But remain all continuous functions as morphisms. Then you may change them to homotopy classes, okay and so on. So, there is a lot of scope in this language. Okay?

(Refer Slide Time: 32:00)



So, now these two categories are of importance. The simplicial category was essentially introduced in part one, though we did not have this terminology of category then. Now we will just verify all those definitions terminology. Similarly, the CW-category was introduced just a couple of days back. Let go through them again.

What are the objects in CW? Topological spaces which have an extra structure namely a CW-complex structure. Remember that topological space itself is a set with an extra structure. Now we have one more extra structure, namely the CW-structure. What are the morphisms? cellular maps between CW complexes. Composite of cellular maps is cellular, and identity map is always cellular. This is what we need to verify but that is already done. Similarly the simplicial category in which morphisms are simplicial maps. Okay So we will discuss more examples next time. Thank you.