

**Introduction to Algebraic Topology Part – II**  
**Prof. Anant R. Shastri**  
**Department of Mathematics**  
**Indian Institute of Technology – Bombay**

**Lecture – 16 B**  
**Homotopy Exact Sequence of a Fibration**

(Refer Slide Time: 00:11)

**Cell Complexes**  
Categories and Functors  
Homology Groups  
Other Homology groups  
Assorted Topics  
Topology of Manifolds

Module-2 Attaching cells  
Module-4B Lattice Structures  
Module-5 Topological Properties  
Module-8 Product of Cell Complexes  
Module-12 Homotopical Aspects  
Module-14 Cellular Maps

**Module 16 B Homotopy exact sequence of a fibration**

Being a fundamental result in homotopy theory, the homotopy exact sequence of a topological pair has several applications. Let us give just one such very important application.


NPTEL

Last time we started the study of relative homotopy groups, saw three different definitions of it, and then we established one of the fundamental results, namely, homotopy exact sequence of a topological pair. So, today we should give you one very important application of that, namely, homotopy exact sequence of a fibration, okay.

(Refer Slide Time: 01:00)

Other Homology groups  
 Assorted Topics  
 Topology of Manifolds

Module-8 Product of Cell Complexes  
 Module-12 Homotopical Aspects  
 Module-14 Cellular Maps



## Homotopy exact sequence of a fibration

Recall that by a fibration we mean a map  $p : E \rightarrow B$  which has the homotopy lifting property with respect to every space:

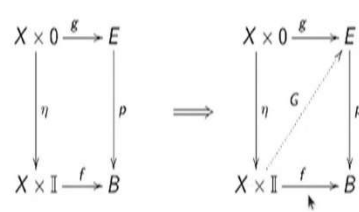


Figure 15: The homotopy lifting property

Anant R Shastri/Retired Emeritus Fellow Department of Mathematics

Lectures on Algebraic Topology, Part-II NPTEL Course


Cell Complexes

Module-2 Attaching cells

So, we recall that by a fibration we mean a certain type of a map of in topological spaces. Today it is customary to denote such a map by  $p$  from  $E$  to  $B$ ,  $E$  is called the total space,  $B$  is called the base space and the map  $p$  has homotopy lifting property with respect to every space. The homotopy lifting property, I just recall, means given a data here, namely, a homotopy  $f$  from say from  $X \times \mathbb{I}$  to  $B$ , into the base, and a lift  $g$  of  $f|_{X \times 0}$ ,  $g$  is a map from  $X \times 0$  into  $E$ ,  $f$  restricted to  $X \times 0$  is the initial state of the homotopy, there must be a lift of this entire homotopy  $f$ .



Lift of  $f$  means you have a map  $G$  such that  $p \circ G$  is  $f$  and  $g$  restricted to  $X \times 0$  is  $g$ . Here,  $\eta$  denotes the inclusion map of  $X \times 0$  into  $X \times \mathbb{I}$ , okay? Think of this as a copy of  $X$  here,  $p \circ g$  is the initial value of  $f$ . Then the conclusion is that there is a map  $G$  such that  $p \circ G$  is  $f$  and  $G$  restricted to  $X \times 0$  is little  $g$ . That is the homotopy lifting property. If this is true for every space  $X$  and every map  $f$  from  $X \times \mathbb{I}$  to  $B$ , then  $p$  is called a fibration.

**(Refer Slide Time: 02:52)**

<b>Cell Complexes</b> Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-2 Attaching cells Module-4B Lattice Structures Module-5 Topological Properties Module-8 Product of Cell Complexes Module-12 Homotopical Aspects Module-14 Cellular Maps	
--	---	---


  

**Proposition 1.5**  
 Any fibration  $p : (E, F, e_0) \rightarrow (B, \{b_0\}, b_0)$  induces isomorphisms on homotopy groups.  
 Here,  $F = p^{-1}(\{b_0\})$  is the fibre of  $p$  over  $b_0$ .



So, the proposition is that any fibration  $p$  from the triple  $(E, F, e_0)$  to  $(B, \{b_0\}, b_0)$  induces isomorphism of all homotopy groups. Here, it is customary to denote the inverse image under  $p$  of the single point  $b_0$  by  $F$ , where  $b_0$  is the base point in  $B$ . So, this is our basic result, very useful in the study of fibrations. Using this result we will get a very useful statement when we combine it with the homotopy exact sequence of the pair. Let us first prove this statement.

**(Refer Slide Time: 03:41)**

Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-5 Topological Properties Module-8 Product of Cell Complexes Module-12 Homotopical Aspects Module-14 Cellular Maps	
--	---	---

**Proof:** Let us first prove that  $p_{\#}$  is surjective. For a change, we shall use the definition (6) here. Consider an element  $[\alpha] \in \pi_n(B, b_0)$  represented by  $\alpha : (\mathbb{I}^n, \partial\mathbb{I}^n) \rightarrow (B, b_0)$ . Taking  $X = \mathbb{I}^{n-1}$ , think of  $\alpha$  as a homotopy,  $f = \alpha : X \times \mathbb{I} \rightarrow B$ . Take  $g : X \times 0 \rightarrow E$  to be the constant map at  $e_0$ . By HLP, we get a map  $G : X \times \mathbb{I} \rightarrow E$  fitting the above diagram. Clearly,  $G(\mathbb{I}^{n-1} \times 0) = g(\mathbb{I}^{n-1} \times 0) = \{e_0\}$  and  $p \circ G(\partial\mathbb{I}^n) = \alpha(\partial\mathbb{I}^n) = \{b_0\}$ . This means that  $G(\partial\mathbb{I}^n) \subset F$ . Therefore  $G$  represents an element of  $\pi_n(E, F, e_0)$  and  $p_{\#}([G]) = [\alpha]$ . This proves  $p_{\#}$  is surjective.

Anant R Shastri Retired Emeritus Fellow Department of Mathem
Lectures on Algebraic Topology, Part-II NPTEL Course

So, there are 2 parts here. One is that  $p_{\#}$  is surjective and the other one is  $p_{\#}$  is injective. To prove that  $p_{\#}$  is surjective. Let us use the definition (6), namely, the simplest definition for the


homotopy groups. So, start with an element  $[\alpha]$  of  $\pi_n(B, b_0)$ . The element  $[\alpha]$  is represented by a map  $\alpha$  from  $(\mathbb{I}^n, \text{boundary of } \mathbb{I}^n)$  to  $(B, b_0)$ .

Which means that  $\alpha$  is a continuous function from  $\mathbb{I}^n$  to  $B$  and takes the entire of the boundary to a single point. This is the simplest definition of the  $n$ -th homotopy group. Now you can think of this  $\alpha$  as a homotopy on  $X$  equal to  $\mathbb{I}^{n-1}$ , okay? So take this as  $f$  in this picture here, okay? And then we try to apply this homotopy lifting property. We must have a map from  $\mathbb{I}^{n-1} \times 0$  to  $E$ , but that is very easy here, because this particular  $f$  namely, the given  $\alpha$  takes the entire of the boundary to a single point.

So, I can take  $g$  from  $X \times 0 = \mathbb{I}^{n-1}$ , which is a part of the boundary to be the constant function mapping to a single point here in  $E$ , viz,  $e_0$ . Then  $p \circ g$  will be the constant  $b_0$  here. So this diagram will be commutative. Therefore I can apply the conclusion to this situation and get a map  $G$  here that is the first conclusion that we get, okay. So, taking  $X = \mathbb{I}^{n-1}$  and thinking of  $\alpha$  as a homotopy and  $g$  to be the constant function to  $e_0$ , and applying the homotopy lifting property, we get a map  $G$  from  $\mathbb{I}^n$  to  $E$  such that the diagram on the right is commutative. That just means what  $G$  restricted to  $\mathbb{I}^{n-1} \times 0$  is the constant function  $g$ , by our choice and  $p \circ G$  is  $\alpha$ . That the meaning that  $G$  is a lift of  $\alpha$ . On the boundary,  $f$  is a constant map point that means  $p \circ G$  of the boundary is the single point  $b_0$ . That means this entire thing is contained in the fiber  $F$ .  $G$  of boundary of  $\mathbb{I}^n$ ,  $f$  is contained in  $F$ .

Therefore  $G$  represents an element of  $\pi_n(E, F, e_0)$ . That is the definition (6) that we are using. Elements of  $\pi_n(E, F, e_0)$  are represented by maps from  $\mathbb{I}^n$  to  $E$  such that the boundary of  $\mathbb{I}^n$  goes into  $F$  and of course, the base point is going to  $e_0$ . Therefore,  $p_{\#}[G] = [\alpha]$  okay? Because  $p \circ G$  is  $\alpha$ . This proves that  $p_{\#}$  is the surjective.

**(Refer Slide Time: 07:31)**

Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module 5 Topological Properties Module 8 Product of Cell Complexes Module 12 Homotopical Aspects Module 14 Cellular Maps	
--	---	--

To prove the injectivity, given  $[\alpha] \in \pi_n(E, F, e_0)$  and a relative homotopy  $h : (\mathbb{I}^n, \partial\mathbb{I}^n) \times \mathbb{I} \rightarrow (B, b_0)$  such that  $h|_{\mathbb{I}^n \times 0} = p \circ \alpha$  and  $h(\mathbb{I}^n \times 1) = \{b_0\}$ , we have to find a base-point preserving relative homotopy  $H : (\mathbb{I}^n, \partial\mathbb{I}^n, O) \rightarrow (E, F, e_0)$  such that

$$H|_{\mathbb{I}^n \times 0} = \alpha, H|_{\mathbb{I}^n \times 1} = \{e_0\}. \quad (8)$$

Here, we have taken  $O = (0, 0, \dots, 0) \in \mathbb{I}^n$  as the base point for  $\mathbb{I}^n$ . It is clear that if we apply HLP of  $p$  to  $h$  directly, the resulting lift will not satisfy this requirement. Therefore, we need to work harder here. The key is demanding that  $H$  satisfies even more stringent conditions and then appeal to a trick.

Anant R Shashi/Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II NPTEL Course

Now we will prove injectivity. What is the meaning of injectivity? Start with the element  $[\alpha]$  in  $\pi_n(E, F, e_0)$  this time, okay, and a relative homotopy of  $p \circ \alpha$  to the constant function, okay, when you pass into the base space  $B$ . So, let  $h$  be a homotopy,  $h$  from  $(\mathbb{I}^n \times \mathbb{I}, \text{boundary of } \mathbb{I}^n \times \mathbb{I})$  to  $(B, b_0)$  such that this  $h$  restricted to  $\mathbb{I}^n \times 0$  is  $p(\alpha)$  and  $h(\mathbb{I}^n \times \mathbb{I}) = \{b\}$ ? It just means that  $p \circ \alpha$  is null homotopic in  $B$ , i.e.,  $p_{\#}[\alpha]$  is the trivial element. We want to show that  $[\alpha]$  itself is the trivial element in  $\pi_n(E, F, e_0)$ , right.

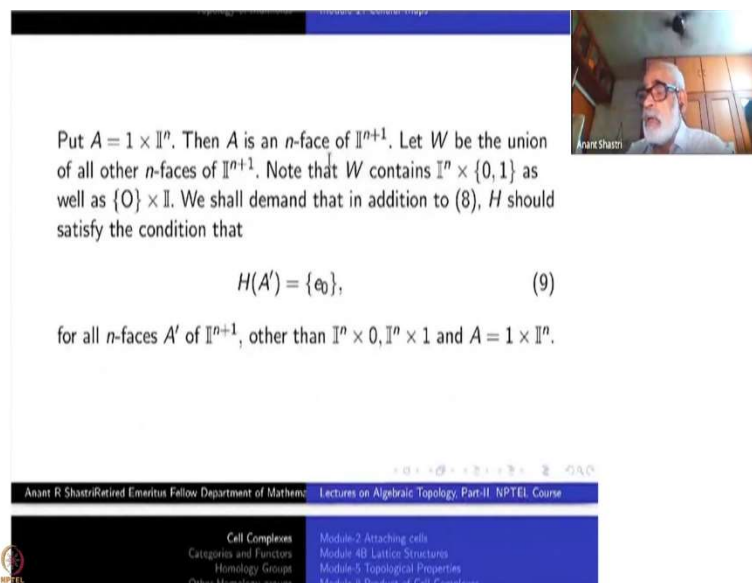
So that means we have to find a base point preserving, relative homotopy here, namely,  $H$  from  $(\mathbb{I}^n, \text{boundary of } \mathbb{I}^n, O)$  to  $(E, F, e_0)$  okay, such that  $H$  restricted to  $\mathbb{I}^n \times 0$  is  $\alpha$  and  $H$  should take  $\mathbb{I}^n \times 1$  to  $e_0$ , okay? This is what we have to find. Here we have taken  $O = (0, 0, \dots, 0)$  in  $\mathbb{I}^n$  okay as the base point for  $\mathbb{I}^n$ .

It is clear that if we apply homotopy lifting property of  $p$  to directly to  $h$ , resulting lift will not satisfy this requirement, namely, all of the set base point cross  $\mathbb{I}$  may not just go to the same base point  $e_0$  under the lifted function. That will not be guaranteed because all that we get is  $p \circ H$  is  $h$  and so it will say that this is contained in the set capital  $F$ . That is all, where as we want it to be actually just a single point  $e_0$ . That will not happen. That is an important point here.

So, in order to overcome that, we have to work a little harder. The key is in demanding that  $H$  satisfies even more stringent conditions and then appeal to a trick, okay? First, if you just try to

control this  $H$  only on  $O \times \mathbb{I}$ , that seems to be more difficult. So, demand that  $H$  satisfies a more stringent condition. So what is that? Let me tell you.

(Refer Slide Time: 10:40)



Put  $A = 1 \times \mathbb{I}^n$ . Then  $A$  is an  $n$ -face of  $\mathbb{I}^{n+1}$ . Let  $W$  be the union of all other  $n$ -faces of  $\mathbb{I}^{n+1}$ . Note that  $W$  contains  $\mathbb{I}^n \times \{0, 1\}$  as well as  $\{0\} \times \mathbb{I}$ . We shall demand that in addition to (8),  $H$  should satisfy the condition that

$$H(A') = \{e_0\}, \quad (9)$$

for all  $n$ -faces  $A'$  of  $\mathbb{I}^{n+1}$ , other than  $\mathbb{I}^n \times 0$ ,  $\mathbb{I}^n \times 1$  and  $A = 1 \times \mathbb{I}^n$ .

Anant R Shastri Retired Emeritus Fellow, Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course


Cell Complexes	Module-2 Attaching cells
Categories and Functors	Module-4B Lattice Structures
Homology Groups	Module-5 Topological Properties
Other Homology groups	Module-6 Product of Cell Complexes

So, first make a notation,  $A$  equal to one of the  $n$ -face of  $\mathbb{I}^{n+1}$ , to be specific, let  $A = \{1\} \times \mathbb{I}^n$ . Then  $A$  is an  $n$ -face of  $\mathbb{I}^{n+1}$ . Now let  $W$  be the union of all other  $n$ -faces of  $\mathbb{I}^{n+1}$ , okay? Note that this  $W$  contains  $\mathbb{I}^n \times \{0, 1\}$ , a pair of opposite  $n$ -faces. There are many other  $n$ -faces of  $\mathbb{I}^{n+1}$  in  $W$ , okay, but we pay special attention to these two. Also  $W$  contains the line segment  $\{0\} \times \mathbb{I}$ .

So, if I control the map  $H$  on  $W$ , automatically, it will be controlled on these subsets. In any case, I want the lift to be such that on  $\mathbb{I}^n \times 0$ , it is  $\alpha$ , on  $\mathbb{I}^n \times 1$ , it is the constant function  $e_0$ , right, and on this line segment also I want it to be a constant function  $e_0$ . So, I make this single demand that, in addition to the condition (8) viz., these two conditions,  $H(A')$  is the singleton  $e_0$ , where  $A'$  is any of the  $n$ -faces of  $\mathbb{I}^{n+1}$ , other than  $\mathbb{I}^n \times 0$ ,  $\mathbb{I}^n \times 1$  and  $1 \times \mathbb{I}^n$ . Of course the last one is not in  $W$ .  $W$  consists of all  $n$ -faces other than  $A = 1 \times \mathbb{I}^n$ .

So, can we get such an  $H$  is a question, by using homotopy lifting property. Directly, it does not give you that. So we have to appeal to a trick here, okay? So what is that trick? I will tell you okay.

(Refer Slide Time: 13:13)

Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module 4B: Lattice Structures Module 5: Topological Properties Module 8: Product of Cell Complexes Module 12: Homotopical Aspects Module 14: Cellular Maps	
---	--	---

Define  $\lambda : W \rightarrow E$  by taking

$$\lambda|_{\mathbb{I}^n \times 0} = \alpha; \lambda(\mathbb{I}^n \times 1) = \{e_0\}; \text{ \& } \lambda(A') = \{e_0\}$$

for all other  $n$ -faces  $A' \subset W$ . We have to find  $H : \mathbb{I}^{n+1} \rightarrow E$  such that  $H|_W = \lambda$  and  $p \circ H = h$ . Since  $h(\partial \mathbb{I}^n \times \mathbb{I}) = \{b_0\}$ , it will follow that  $H(\partial \mathbb{I}^n \times \mathbb{I}) \subset F$ . That will complete the proof. Now we are going to reveal the trick.

Anant R Shastri Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

Define  $\lambda$  from  $W$  to  $E$  by taking  $\lambda$  equal to  $\alpha$  on  $\mathbb{I}^n \times 0$  and the constant function  $e_0$  on  $\mathbb{I}^n \times 1$ , and also on all other  $n$ -faces  $A'$ , just like what we want  $H$  to be. I want to find  $H$  from  $\mathbb{I}^{n+1}$  to  $E$  to be  $\lambda$  on  $W$ , and  $p \circ H$  is  $h$ . Since  $h$  of boundary of  $\mathbb{I}^n \times \mathbb{I}$ , by the very choice of this homotopy, is singleton  $b_0$ , this demand is consistent.

It will then follow that  $H(\text{boundary of } \mathbb{I}^n \times \mathbb{I})$  is inside  $F$ . That is also a requirement. However, that comes freely for us. That will complete the proof okay? So, how to apply homotopy lifting property for this  $W$  instead of just  $\mathbb{I}^n \times 0$ ? That is the trick.

**(Refer Slide Time: 14:18)**

Topology of Manifolds	Module 14: Cellular Maps
-----------------------	--------------------------

We appeal to the fact that there is a homeomorphism  $\tau : A \rightarrow W$  given by the radial projection from the point  $(2, 1/2, \dots, 1/2) \in \mathbb{I}^{n+1}$  which is the identity map on the common boundary. (See the picture). Take  $\phi : \partial \mathbb{I}^{n+1} \rightarrow \partial \mathbb{I}^{n+1}$  such that

$$\phi|_W = \tau^{-1}; \phi|_A = \tau.$$

Then  $\phi$  is a homeomorphism. Let  $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  be the map

$$T(t_1, t_2, \dots, t_n, t_{n+1}) = (1 - t_{n+1}, t_2, \dots, t_n, t_1).$$

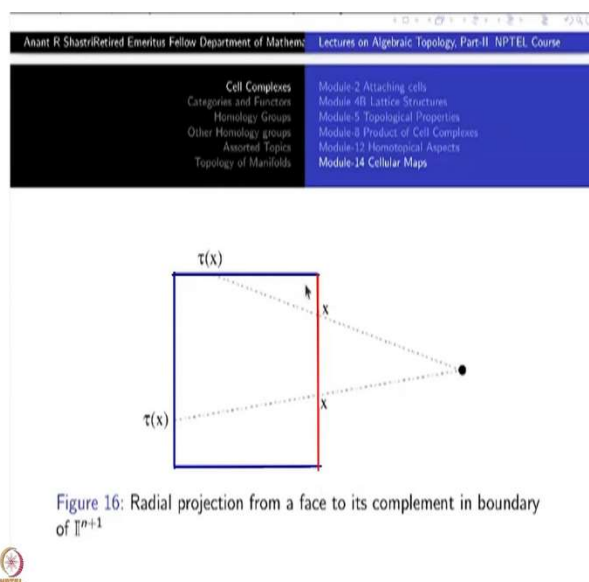
Then  $T$  is an affine linear isomorphism taking  $\partial \mathbb{I}^{n+1}$  onto itself. Put  $\psi = \phi \circ T : \partial \mathbb{I}^{n+1} \rightarrow \partial \mathbb{I}^{n+1}$  and extend this to a homeomorphism  $\hat{\psi} : \mathbb{I}^{n+1} \rightarrow \mathbb{I}^{n+1}$  by cone-construction. Note that  $\psi$  and  $\hat{\psi}$  map  $\mathbb{I}^n \times 0$  homeomorphically onto  $W$ .

Anant R Shastri Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

Cell Complexes Categories and Functors Homology Groups	Module 2: Attaching cells Module 4B: Lattice Structures Module 5: Topological Properties
--	--

We appeal to the fact that there is a homeomorphism  $\tau$  from  $A$  to  $W$ . Remember  $A$  is equal to  $1 \times \mathbb{I}^n$ , one of the faces of  $\mathbb{I}^{n+1}$ . This homeomorphism is given by the radial projection from the point  $(2, 1/2, 1/2, \dots, 1/2)$  belonging to  $2 \times \mathbb{I}^n$ , okay? This radial projection is a retraction of  $\mathbb{I}^{n+1}$  onto  $W$ . It is the identity map on the common boundary, boundary of  $A$  and boundary of  $W$ , okay. So, let us look at how this is got in the case of when  $n = 1$ . Look at the picture.

(Refer Slide Time: 15:13)



So, this is my picture of  $\mathbb{I} \times \mathbb{I}$ . Here,  $\mathbb{I}^n \times \mathbb{I}$  is drawn for  $n = 2$ . This red thing is  $\mathbb{I}^n \times 1$  and this point is  $p = (2, 1/2, \dots, 1/2)$ . You project points of  $\mathbb{I}^{n+1}$  radially means each point  $x \in 1 \times \mathbb{I}^n$  goes to the unique point  $\tau(x) \in W$ , shown by this blue color, which lies on the extended line through  $p$  and  $x$ . Why is  $\tau$  a homeomorphism? Why this is a bijection? All this is very clear. First of all,  $\tau$  is a projection so there is no problem with continuity.

So, take a point in  $W$  here okay? I am going to define  $\tau^{-1}$  now. Take a point  $y$  here. How to determine the point  $x \in A$ ? Write down the line segment  $[p, y]$  in the parametric form viz.,  $tp + (1 - t)y, t \in [0, 1]$  and put the first coordinate equal to 1 to determine the value of  $t$ . We want a point on  $1 \times \mathbb{I}^n$  so put that condition. Immediately it gives you a unique solution. That is your  $x$  so that  $\tau(x) = y$ . This is just an elementary linear algebra problem. Okay?

Since you can write down the formula for tau inverse, that completes the claim that  $\tau$  is a homeomorphism. Obviously, when you take the boundary point here or here, okay, namely 1



cross boundary of  $\mathbb{I}^n$  is the boundary of this one okay, there your original point and its projection coincide. So,  $\tau$  is identity on the boundary of  $A$ , okay? Now take  $\phi$  from boundary of  $\mathbb{I}^{n+1}$  to itself, to be the map such that  $\phi$  is  $\tau^{-1}$  on  $W$  and  $\tau$  on  $A$ . On the intersection both are identity maps and so, they patch up to define a homeomorphism  $\phi$  from boundary of  $\mathbb{I}^{n+1}$  to itself.

Finally, take the affine linear transformation  $T$  from  $\mathbb{R}^{n+1}$  to itself which merely interchanges the first coordinate and  $(n+1)$ -th coordinate and followed by the reflection in  $1/2 \times \mathbb{I}^n$ . The formula is:  $(t_1, t_2, \dots, t_n, t_{n+1}) \mapsto (1 - t_{n+1}, t_2, \dots, t_{n-1}, t_1)$ . The first coordinate has come to last coordinate that is just a permutation, and then the reflection which is an affine linear isomorphism. Obviously,  $\mathbb{I}^{n+1}$  goes to itself under  $T$  and its boundary goes to the boundary.

Therefore I can take  $T$  restricted to boundary of  $\mathbb{I}^{n+1}$  and compose it with  $\phi$  and call it  $\psi$ , so that I get a map from boundary of  $\mathbb{I}^{n+1}$  which is a homeomorphism. After that you can extend it to a homeomorphism  $\hat{\psi}$  of the entire  $\mathbb{I}^{n+1}$  to itself, by taking the cone construction.


So, this homeomorphism is  $\hat{\psi}$ , an extension of  $\psi$ , okay? This homeomorphism has the following properties, which justifies why we have done all this:

On  $\mathbb{I}^n \times 0$ , both  $\psi$  and  $\hat{\psi}$  are homeomorphisms onto  $W$ . Now trick is revealed. You wanted to control the map on  $W$ , but now this can be done by controlling it on a single face  $\mathbb{I}^n \times 0$ , which is done, by the very definition of homotopy lifting property. So now, go back to our homotopy lifting diagram.

**(Refer Slide Time: 20:18)**

**Cell Complexes**  
Categories and Functors  
Homology Groups  
Other Homology groups  
Assorted Topics  
Topology of Manifolds


Module 2 Attaching cells  
Module 4B Lattice Structures  
Module 5 Topological Properties  
Module 8 Product of Cell Complexes  
Module 12 Homotopical Aspects  
Module 14 Cellular Maps



Consider the following diagram:

$$\begin{array}{ccccc}
 \mathbb{I}^n \times 0 & \xrightarrow[\cong]{\psi} & W & \xrightarrow{\lambda} & E \\
 \downarrow & & \downarrow G & \nearrow H & \downarrow p \\
 \mathbb{I}^n \times \mathbb{I} & \xrightarrow[\cong]{\hat{\psi}} & \mathbb{I}^n \times \mathbb{I} & \xrightarrow{h} & B
 \end{array}$$

We want to get  $H$  to fit the rectangle on the right. But we cannot apply HLP here directly. So, we take the larger rectangle and apply HLP to the map  $f = h \circ \hat{\psi}$  with  $g = \lambda \circ \psi$ , to get  $G$  as shown. Now we take  $H = G \circ \hat{\psi}^{-1}$ .



Anant R. Shastri/Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

I have this map  $p$  which is a fibration here, I have an  $h$  here, I want capital  $H$  here such that restricted to  $W$  it is  $\lambda$  and  $p \circ H$  is  $h$ . I cannot apply the homotopy lifting property directly here, because this  $W$  is not just  $\mathbb{I}^n \times 0$ . So, what I do? I take  $\hat{\psi} \circ h$  and come to the larger rectangle here. At the bottom, on  $\mathbb{I}^n \times \mathbb{I}$ , I take  $\hat{\psi} \circ h$  and at the top I take  $\lambda \circ \psi$  restricted to  $\mathbb{I}^n \times 0$ , okay. You have to check that  $p \circ \lambda \circ \psi$  is equal to  $h \circ \hat{\psi}$  restricted to  $\mathbb{I}^n \times \mathbb{I}$ , which is obvious.

Therefore the HLP gives you this map  $G$ . Now all that I do is start here, come to here via  $\hat{\psi}^{-1}$  and follow it by  $G$ , i.e., take capital  $H$  equal to  $G \circ \hat{\psi}^{-1}$ , okay? So that will give you whatever  $H$  we wanted. Okay?

**(Refer Slide Time: 22:06)**


**Theorem 1.20**


Let  $p : E \rightarrow B$  be a fibration,  $e_0 \in E, p(e_0) = b_0$  and  $F = p^{-1}(b_0)$ . Then we have a long exact sequence of homotopy groups and homomorphisms

$$\cdots \pi_n(F, e_0) \xrightarrow{i_{\#}} \pi_n(E, e_0) \xrightarrow{p_{\#}} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \cdots$$

which ends with the exact sequence of pointed sets:

$$\pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_{\#}} \pi_0(E, e_0).$$





Anant R. Shastri/Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course

**Cell Complexes**  
Categories and Functors  
Homology Groups  
Other Homology groups  
Assorted Topics  
Topology of Manifolds

Module 2 Attaching cells  
Module 4B Lattice Structures  
Module 5 Topological Properties  
Module 8 Product of Cell Complexes  
Module 12 Homotopical Aspects  
Module 14 Cellular Maps

Now, let us come to the final statement of the theorem 1.20. The proof is very easy. Take any fibration, take a base point  $e_0$  in the top space. And then the base point in  $B$  must be taken  $b_0 = p(e_0)$ . And let  $F = p^{-1}(b_0)$ . This is the convention. Then we have a long exact sequence of homotopy groups and homomorphisms,  $\pi_n(F, e_0)$  to  $\pi_n(E, e_0)$  as in the previous theorem but then suddenly, instead of a relative homotopy here, we have  $\pi_n(B, b_0)$  and so on....

The homomorphisms are  $i_\#$  and  $p_\#$  instead of  $j_\#$  and then instead of the old boundary operator  $\partial$ , a new one  $\delta$  to  $\pi_{n-1}(F)$  so on. Finally, it will end up with  $\pi_1(B, b_0)$  to  $\pi_0(F)$  to  $\pi_0(E)$ . Remember the last entries are not groups but just pointed sets, the set of path components of  $F$  and  $E$  respectively and the last function is the inclusion induced map, okay?

The proof is now very clear. In the long exact sequence of relative homotopy groups of the pair  $(E, F, e_0)$ , I replace all  $\pi_n(E, F, e_0)$  by  $\pi_n(B, b_0)$ . I need to replace the two adjacent homomorphisms also properly. The proposition says that the relative homotopy group is isomorphic to  $\pi_n(B, b_0)$  under  $p_\#$ . So, I am going to take  $p_\# \circ j_\#$ , which I denote again by  $p_\#$ , because  $p \circ j = p$ .

**(Refer Slide Time: 24:09)**

**Cell Complexes**

Categories and Functors

Homology Groups

Other Homology groups

Assorted Topics

Topology of Manifolds

Module-2 Attaching cells


Module-4B Lattice Structures

Module-5 Topological Properties

Module-8 Product of Cell Complexes

Module-12 Homotopical Aspects

Module-14 Cellular Maps




Anurag Shastri

Starting with the homotopy exact sequence of the pair  $(E, F, e_0)$ , we replace  $\pi_n(E, F, e_0)$  by  $\pi_n(B, b_0)$  through the isomorphism  $p_\#$  and define  $\delta = \partial \circ p_\#^{-1}$  to get the result.

$$\begin{array}{ccccccc}
 \cdots & \pi_n(F, e_0) & \xrightarrow{i_\#} & \pi_n(E, e_0) & \xrightarrow{j_\#} & \pi_n(E, F, e_0) & \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{i_\#} \cdots \\
 & & & \searrow p_\# & & \downarrow p_\# & \nearrow \delta \\
 & & & & & \pi_n(B, b_0) & 
 \end{array}$$

This completes the proof. ♣



⏮ ⏪ ⏩ ⏭ ⏴ ⏵ ⏶ ⏷ ⏸ ⏹ ⏺ ⏻ ⏼ ⏽ ⏾ ⏿

Similarly, I define the homomorphism  $\delta$  to be  $p_\#^{-1}$  followed by  $\partial$ . That is all. That is why we use a different notation here. Automatically, this sequence will now be an exact sequence. Coming

from here to here and come down here go that way and that is the precise statement of this okay. So, that completes the proof of the big theorem okay homotopy exact sequence of a fibration.

**(Refer Slide Time: 25:29)**

Other Homology groups  
Assorted Topics  
Topology of Manifolds

Module-8 Product of Cell Complexes  
Module-12 Homotopical Aspects  
Module-14 Cellular Maps

Some immediate consequences

**Corollary 1.6**

Let  $p : \tilde{X} \rightarrow X$  be any covering projection of locally path connected and connected space. Then  $p_{\#} : \pi_n(\tilde{X}) \rightarrow \pi_n(X)$  is an isomorphism for all  $n \geq 2$ .

**Proof:** In the exact sequence given by Theorem 1.20, the terms  $\pi_n(F)$  vanish for all  $n \geq 1$  because  $F$  is discrete.

Anant R. Shastri Retired Emeritus Fellow Department of Mathematics  
Lectures on Algebraic Topology, Part-II, NPTEL Course

Cell Complexes  
Module-2 Attaching cells

Of course it has several applications. We will only mention a few of them which are immediate consequences obtained without much trouble. You should be able to get them easily.

One of them is that a covering projection is always a fibration. It is a very peculiar fibration, a very, very important fibration. It has more properties than an ordinary fibration. In particular, the fiber of a covering projection is a discrete space. When  $F$  is a discrete space, what happens to its homotopy groups?  $\pi_0$  will correspond to the set  $F$  itself, and all other higher homotopy groups  $\pi_1, \pi_2$ , etc., all of them are all 0. So, if you use that hypothesis here in the long exact sequence, this  $\pi_n(F)$  will keep appearing at every third term and hence this  $p_{\#}$  will be an isomorphism for all  $n \geq 2$ . When  $n = 1$ , you have a problem because  $\pi_0(F)$  is not a singleton in general.

So, when  $n = 1$ , it will not work. But for  $n = 1$ , we know exactly what happens. This we have studied under covering projection theory.

**(Refer Slide Time: 27:12)**

### Corollary 1.7

Let  $p : \mathbb{S}^{2n+1} \rightarrow \mathbb{CP}^n$  be the Hopf fibration. Then  $p_{\#}$  is an isomorphism for  $i \geq 3$ . In particular,  $\pi_i(\mathbb{CP}^n) = (1)$ , for  $1 \leq i \leq 2n-1$  and  $\pi_{2n}(\mathbb{CP}^n) \approx \pi_{2n+1}(\mathbb{S}^{2n+1})$ , and hence is nontrivial.

**Proof:** This follows from the exact sequence of  $p$ , the fibre of  $p$  is  $\mathbb{S}^1$  and the fact that  $\pi_q(\mathbb{S}^1) = 0$  for  $q > 1$ . (This latter fact itself can be derived using the homotopy exact sequence of the covering  $\exp : \mathbb{R} \rightarrow \mathbb{S}^1$ .)

Anant R. Shastri/Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II: NPTEL Course

<b>Cell Complexes</b> Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-2 Attaching cells Module-4B Lattice Structures Module-5 Topological Properties Module-6 Product of Cell Complexes Module-12 Homotopical Aspects Module-14 Cellular Maps
--	---

NPTEL

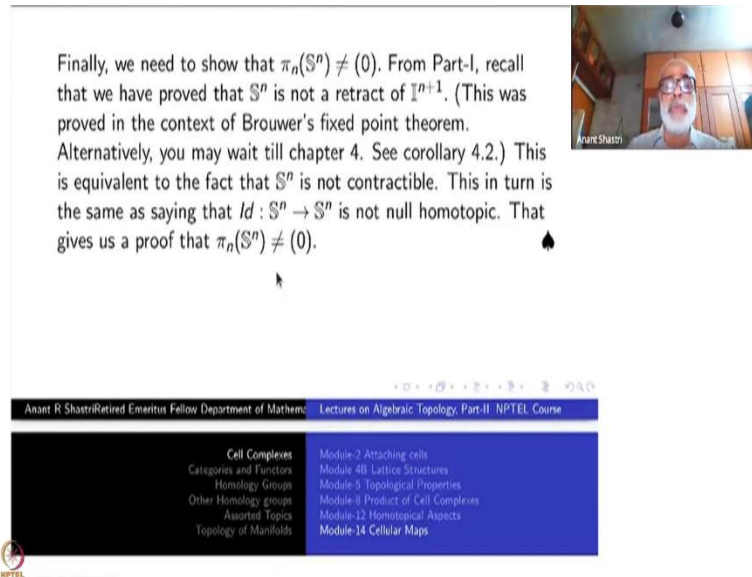
So, here is a case wherein we have even much better looking result. Consider the action of  $\mathbb{S}^1$  on  $\mathbb{S}^{2n+1}$ , by thinking of  $\mathbb{S}^{2n+1}$  as the space of a units in  $\mathbb{C}^n$ , product of copies of the complex numbers taken  $n$  times. So,  $\mathbb{S}^1$  is the space of unit complex numbers. The action is just by coordinatewise multiplication. The quotient space  $\mathbb{CP}^n$ , as we know, is the complex projective space of dimension  $n$ . Then the quotient map  $p$  from  $\mathbb{S}^{2n+1}$  to  $\mathbb{CP}^n$  is a fibration. That is not very easy to see though. We cannot prove that one here. It is a standard result in differential and algebraic topology.

A more general standard result is the following: If we have a compact Lie group acting on a smooth compact manifold then the quotient is a fibration okay? This result is the richest source of fibrations, by the way, but in this course, we cannot cover that one, okay? So, this  $p$  from  $\mathbb{S}^{2n+1}$  to  $\mathbb{CP}^n$  is a fibration and it has a name, Hopf fibration, because for the case  $n = 1$ , viz.,  $p$  from  $\mathbb{S}^3$  to  $\mathbb{CP}^1 = \mathbb{S}^2$ , it was an important contribution by H. Hopf.

Here,  $p_{\#}$  is an isomorphism for  $i \geq 3$ . In the general case, it is so for  $i \geq 2n+1$ . That is  $\pi_i(\mathbb{CP}^n)$  is 1 for  $1 \leq i \leq 2n-1$ . And  $\pi_{2n}(\mathbb{CP}^n)$  is isomorphic to  $\pi_{2n+1}(\mathbb{S}^{2n+1})$ , okay? And hence not trivial. Non triviality of this one can be seen in many ways. The way that we have seen it is, namely, the identity map of a sphere cannot be null homotopic, okay? This fact was proved while proving Brouwer's fixed point theorem, okay?

Apply this here with  $E = \mathbb{S}^{2n+1}$  and  $B = \mathbb{C}P^n$ , the fiber  $F(p) = \mathbb{S}^1$ . And we know that the homotopy groups  $\pi_1(\mathbb{S}^1)$  are trivial for  $i \geq 2$ . Okay? So, we have isomorphisms for  $i \geq 3$ . For 2, there will be a problem because  $\pi_1(\mathbb{S}^1)$  is nontrivial. It is infinite cyclic.

**(Refer Slide Time: 30:33)**



Finally, we need to show that  $\pi_n(\mathbb{S}^n) \neq (0)$ . From Part-I, recall that we have proved that  $\mathbb{S}^n$  is not a retract of  $\mathbb{I}^{n+1}$ . (This was proved in the context of Brouwer's fixed point theorem. Alternatively, you may wait till chapter 4. See corollary 4.2.) This is equivalent to the fact that  $\mathbb{S}^n$  is not contractible. This in turn is the same as saying that  $Id : \mathbb{S}^n \rightarrow \mathbb{S}^n$  is not null homotopic. That gives us a proof that  $\pi_n(\mathbb{S}^n) \neq (0)$ .

Anant R Shastri Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II NPTEL Course


Cell Complexes	Module 2 Attaching cells
Categories and Functors	Module 4b Lattice Structures
Homology Groups	Module 5 Topological Properties
Other Homology groups	Module 6 Product of Cell Complexes
Assorted Topics	Module 12 Homotopical Aspects
Topology of Manifolds	Module 14 Cellular Maps

So that is what I am telling you. I repeated.  $\pi_n(\mathbb{S}^n)$  is not trivial is a fact that we have proved while proving Brouwer's fixed point theorem, namely, by proving that the identity map itself is not null homotopic, okay? Indeed what happens is just like  $\pi_1(\mathbb{S}^1)$  is infinite cyclic, one can prove that  $\pi_n(\mathbb{S}^n)$  is also infinite cyclic. Actually that result goes under the name Hopf's theorem, okay? Hopf's a degree theorem. But this is not a part of this course okay? Usually this result is proved in differential topology, okay?

**(Refer Slide Time: 31:24)**

**Remark 1.25**

The fact that  $\pi_3(\mathbb{S}^2)$  is a trivial group was a great discovery by Hopf at his time and is of course a landmark result even today. He observed that the fibres of  $\phi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  are linked and therefore concluded that the map  $\phi$  itself could not be null homotopic.



Anant R Shastri Retired Emeritus Fellow Department of Mathem

<b>Cell Complexes</b> Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-2 Attaching cells Module-4B Lattice Structures Module-5 Topological Properties Module-8 Product of Cell Complexes Module-12 Homotopical Aspects Module-14 Cellular Maps
--	---

Lectures on Algebraic Topology, Part-II NPTEL Course

So, one more remark. The proof that  $\pi_3(\mathbb{S}^2)$  is not trivial was a great discovery by Hopf when he did it, in his time, okay? It is a landmark result even today. So, he observed that the fibers of  $p$  from  $\mathbb{S}^3$  to  $\mathbb{S}^1$ , they are all copies of  $\mathbb{S}^1$  and they are inter-linked, in a very nice way, namely in a simple way like this and that is precisely what we call nowadays the Hopf link. Okay? So, this is a non trivial link that is easy to see. But from this Hopf concludes that the map  $p$  itself is not null homotopic. That part, I cannot explain here. Okay?


**(Refer Slide Time: 32:18)**

**Exercise 1.13**

Given any subspace  $A \subset X$ , show that  $(X, A)$  is  $n$ -connected iff  $A \hookrightarrow X$  is an  $n$ -equivalence.

**Exercise 1.14**

Let  $f : X \rightarrow Y$  be any continuous map and  $M_f$  denote the mapping cylinder of  $f$ . Show that  $(M_f, X)$  is  $n$ -connected iff  $f$  is an  $n$ -equivalence.



Anant R Shastri Retired Emeritus Fellow Department of Mathem

<b>Cell Complexes</b> Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-2 Attaching cells Module-4B Lattice Structures Module-5 Topological Properties Module-8 Product of Cell Complexes Module-12 Homotopical Aspects Module-14 Cellular Maps
--	---

Lectures on Algebraic Topology, Part-II NPTEL Course

So, here are a few exercises which you can try on your own. Trying to solve exercises is a part of the learning process, okay? Trying itself is more important than just knowing the solutions okay? So keep doing exercises, in the hope that you learn more. Thank you.