Introduction to Algebraic Topology Part – II
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Lecture – 16 B
Homotopy Exact Sequence of a Fibration

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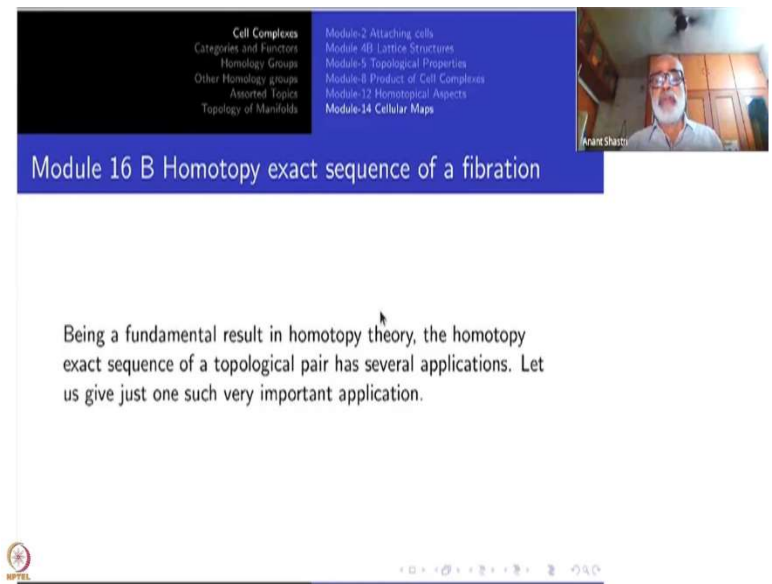

Last time we started the study of relative homotopy groups, saw three different definitions of it, and then we established one of the fundamental results, namely, homotopy exact sequence of a topological pair. So, today we should give you one very important application of that, namely, homotopy exact sequence of a fibration, okay.

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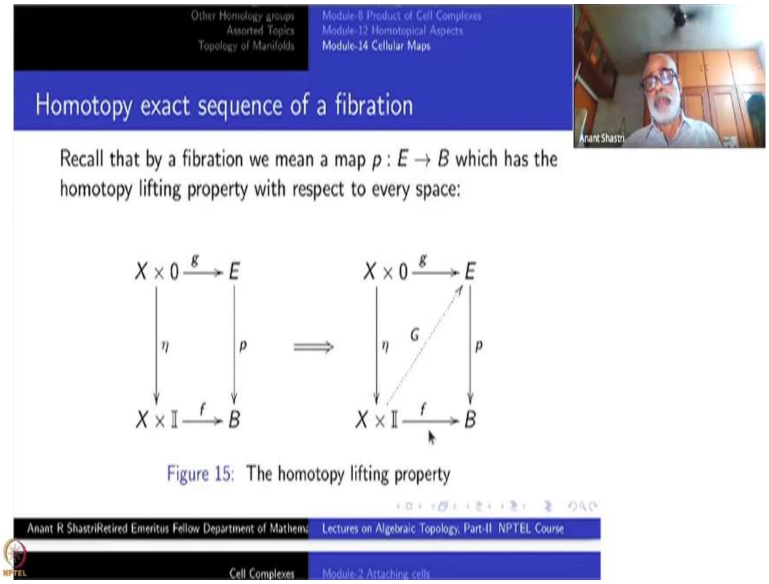

So, we recall that by a fibration we mean a certain type of a map of in topological spaces. Today it is customary to denote such a map by p from E to B, E is called the total space, B is called the base space and the map p has homotopy lifting property with respect to every space. The homotopy lifting property, I just recall, means given a data here, namely, a homotopy f from say from $X \times \mathbb{I}$ to B, into the base, and a lift g of $f_{X \times 0}$, g is a map from $X \times 0$ into E, f restricted to $X \times 0$ is the initial state of the homotopy, there must be a lift of this entire homotopy f.

Lift of f means you have a map G such that $p \circ G$ is f and g restricted to $X \times 0$ is g. Here, η denotes the inclusion map of $X \times 0$ into $X \times \mathbb{I}$, okay? Think of this as a copy of X here, $p \circ g$ is the initial value of f. Then the conclusion is that there is a map G such that $p \circ G$ is f and G restricted to $X \times 0$ is little g. That is the homotopy lifting property. If this is true for every space X and every map f from $X \times \mathbb{I}$ to B, then p is called a fibration.

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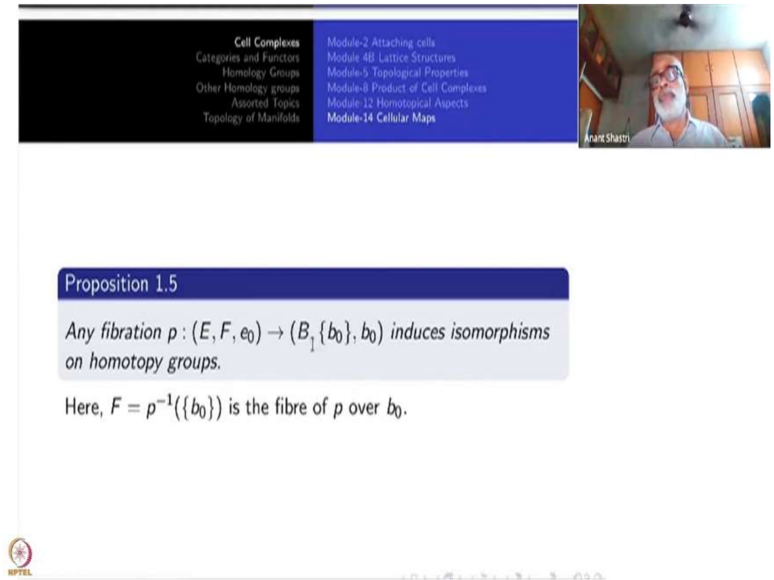

So, the proposition is that any fibration p from the triple (E, F, e_0) to $(B, \{b_0\}, b_0)$ induces isomorphism of all homotopy groups. Here, it is customary to denote the inverse image under p of the single point b_0 by F, where b_0 is the base point in B. So, this is our basic result, very useful in the in the study of fibrations. Using this result we will get a very useful statement when we combine it with the homotopy exact sequence of the pair. Let us first prove this statement.

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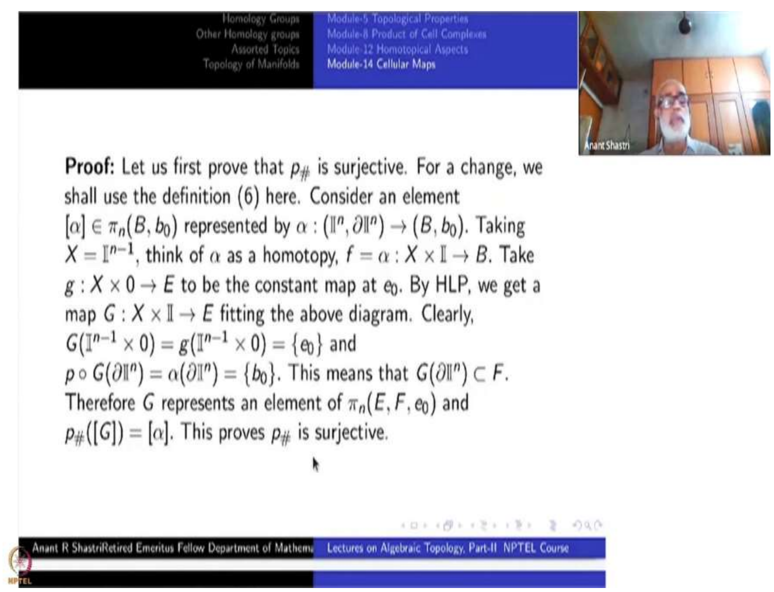

So, there are 2 parts here. One is that $p_\#$ is surjective and the other one is $p_\#$ is injective. To prove that $p_\#$ is surjective. Let us use the definition (6), namely, the simplest definition for the homotopy groups. So, start with an element $[\alpha]$ of $\pi_n(B, b_0)$. The element $[\alpha]$ is represented by a map α from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$) to (B, b_0).

Which means that α is a continuous function from $\mathbb{I}^n$ to B and takes the entire of the boundary to a single point. This is the simplest definition of the n-th homotopy group. Now you can think of this α as a homotopy on X equal to $\mathbb{I}^{n-1}$, okay? So take this as f in this picture here, okay? And then we try to apply this homotopy lifting property. We must have a map from $\mathbb{I}^{n-1} \times 0$ to E, but that is very easy here, because this particular f namely, the given α takes the entire of the boundary to a single point.

So, I can take g from $X \times 0 = \mathbb{I}^{n-1}$, which is a part of the boundary to be the constant function mapping to a single point here in E, viz, e_0. Then $p \circ g$ will be the constant b_0 here. So this diagram will be commutative. Therefore I can apply the conclusion to this situation and get a map G here that is the first conclusion that we get, okay. So, taking $X = \mathbb{I}^{n-1}$ and thinking of α as a homotopy and g to be the constant function to e_0, and applying the homotopy lifting property, we get a map G from $\mathbb{I}^n$ to E such that such that the diagram on the right is commutaive. That just means what G restricted to $\mathbb{I}^{n-1} \times 0$ is the constant function g, by our choice and $p \circ G$ is α. That the meaning that G is a lift of α. On the boundary, f is a constant map point that means $p \circ G$ of the boundary is the single point b_0. That means this entire thing is contained in the fiber F. G of boundary of $\mathbb{I}^n$, f is contained in F.

Therefore G represents an element of $\pi_n(E, F, e_0)$. That is the definition (6) that we are using. Elements of $\pi_n(E, F, e_0)$ are represented by maps from $\mathbb{I}^n$ to E such that the boundary of $\mathbb{I}^n$ goes into F and of course, the base point is going to e_0. Therefore, $p_\#[G] = [\alpha]$ okay? Because $p \circ G$ is α. This proves that $p_\#$ is the surjective.

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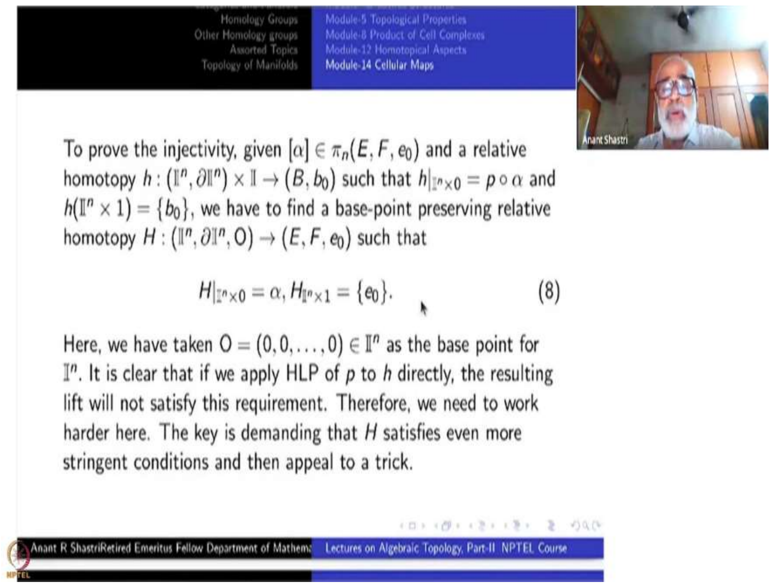

Now we will prove injectivity. What is the meaning of injectivity? Start with the element $[\alpha]$ in $\pi_n(E, F, e_0)$ this time, okay, and a relative homotopy of $p \circ \alpha$ to the constant function, okay, when you pass into the base space B. So, let h be a homotopy, h from ($\mathbb{I}^n \times \mathbb{I}$, boundary of $\mathbb{I}^n \times \mathbb{I}$) to (B, b_0) such that this h restricted to $\mathbb{I}^n \times 0$ is $p(\alpha)$ and $h(\mathbb{I}^n \times \mathbb{I})$=$\{b\}$? It just means that $p \circ \alpha$ is null homotopic in B, i.e., $p_{\#}[\alpha]$ is the trivial element. We want to show that $[\alpha]$ itself is the trivial element in $\pi_n(E, F, e_0)$, right.

So that means we have to find a base point preserving, relative homotopy here, namely, H from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, O) to (E, F, e_0) okay, such that H restricted to $\mathbb{I}^n \times 0$ is α and H should take $\mathbb{I}^n \times 1$ to e_0, okay? This is what we have to find. Here we have taken $O = (0, 0, \dots, 0)$ in $\mathbb{I}^n$ okay as the base point for $\mathbb{I}^n$.

It is clear that if we apply homotopy lifting property of p to directly to h, resulting lift will not satisfy this requirement, namely, all of the set base point cross $\mathbb{I}$ may not just go to the same base point e_0 under the lifted function. That will not be guaranteed because all that we get is $p \circ H$ is h and so it will say that this is contained in the set capital F. That is all, where as we want it to be actually just a single point e_0. That will not happen. That is an important point here.

So, in order to overcome that, we have to work a little harder. The key is in demanding that H satisfies even more stringent conditions and then appeal to a trick, okay? First, if you just try to control this H only on $O \times \mathbb{I}$, that seems to be more difficult. So, demand that H satisfies a more stringent condition. So what is that? Let me tell you.

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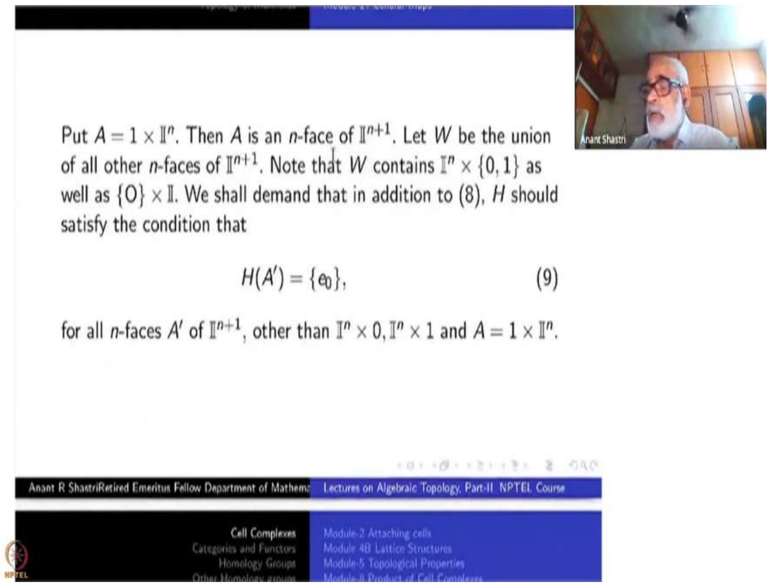

So, first make a notation, A equal to one of the n-face of $\mathbb{I}^{n+1}$, to be specific, let $A = \{1\} \times \mathbb{I}^n$. Then A is an n-face of $\mathbb{I}^{n+1}$. Now let W be the union of all other n-faces of $\mathbb{I}^{n+1}$, okay? Note that this W contains $\mathbb{I}^n \times \{0, 1\}$, a pair of opposite n-faces. There are many other n-faces of $\mathbb{I}^{n+1}$ in W, okay, but we pay special attention to these two. Also W contains the line segment $\{O\} \times \mathbb{I}$.

So, if I control the map H on W, automatically, it will be controlled on these subsets. In any case, I want the lift to be such that on $\mathbb{I}^n \times 0$, it is α, on $\mathbb{I}^n \times 1$, it is the constant function e_0, right, and on this line segment also I want it to be a constant function e_0. So, I make this single demand that, in addition to the condition (8) viz., these two conditions, $H(A')$ is the singleton e_0, where A' is any of the n-faces of $\mathbb{I}^{n+1}$, other than $\mathbb{I}^n \times 0$, $\mathbb{I}^n \times 1$ and $1 \times \mathbb{I}^n$. Of course the last one is not in W. W consists of all n-faces other than $A = 1 \times \mathbb{I}^n$.

So, can we get such an H is a question, by using homotopy lifting property. Directly, it does not give you that. So we have to appeal to a trick here, okay? So what is that trick? I will tell you okay.

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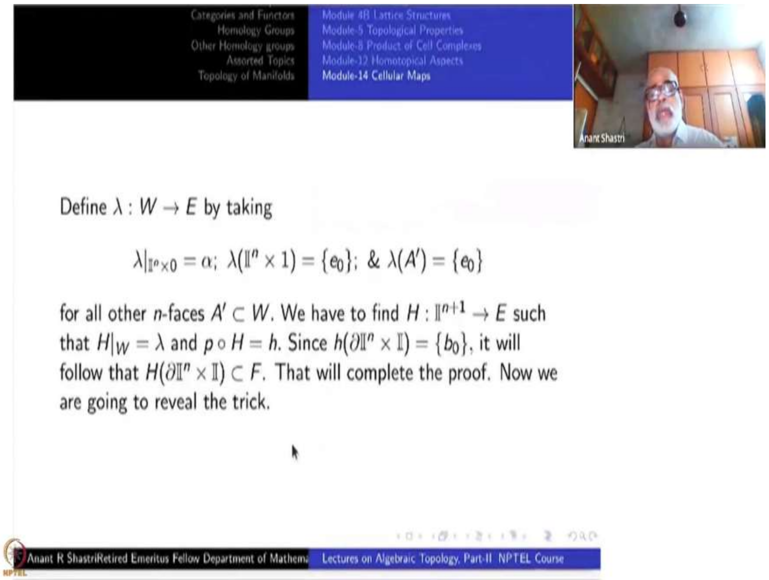

Define λ from W to E by taking λ equal to α on $\mathbb{I}^n \times 0$ and the constant function e_0 on $\mathbb{I}^n \times 0$, and also on all other n-faces A', just like what we want H to be. I want to find H from $\mathbb{I}^{n+1}$ to E to be λ on W, and $p \circ H$ is h. Since h of boundary of $\mathbb{I}^n \times \mathbb{I}$, by the very choice of this homotopy, is singleton b_0, this demand is consistent.

It will then follow that H(boundary of $\mathbb{I}^n \times \mathbb{I}$) is inside F. That is also a requirement. However, that comes freely for us. That will complete the proof okay? So, how to apply homotopy lifting property for this W instead of just $\mathbb{I}^n \times 0$? That is the trick.

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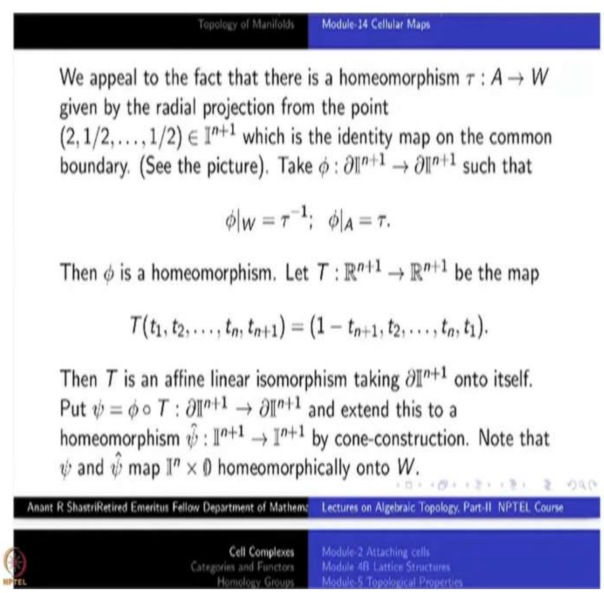

We appeal to the fact that there is a homeomorphism tau from A to W. Remember A is equal to $1 \times \mathbb{I}^n$, one of the faces of $\mathbb{I}^{n+1}$. This homeomorphism is given by the radial projection from the point $(2, 1/2, 1/2, \ldots, 1/2)$ belonging to $2 \times \mathbb{I}^n$, okay? This radial projection a retraction of $\mathbb{I}^{n+1}$ onto W. It is the identity map on the common boundary, boundary of A and boundary of W, okay. So, let us look at how this is got in the case of when $n = 1$. Look at the picture.

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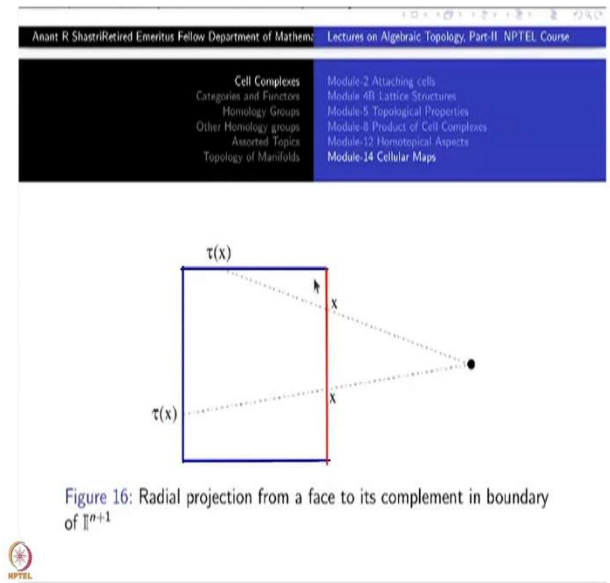

So, this is my picture of $\mathbb{I} \times \mathbb{I}$. Here, $\mathbb{I}^n \times \mathbb{I}$ is drawn for $n = 2$. This red thing is $\mathbb{I}^n \times 1$ and this is point is $p = (2, 1/2, \ldots, 1/2)$. You project points of $\mathbb{I}^{n+1}$ radially means each point $x \in 1 \times \mathbb{I}^n$ goes to the unique point $\tau(x) \in W$, shown by this blue color, which lies on the extended line through p and x. Why is τ a homeomorphism? Why this is a bijection? All this is very clear. First of all, τ is a projection so there is no problem with continuity.

So, take a point in W here okay? I am going to define τ^{-1} now. Take a point y here. How to determine the point $x \in A$? Write down the line segment $[p, y]$ in the parametric form viz., $tp + (1-t)y, t \in [0, 1]$ and put the first coordinate equal to 1 to determine the value of t. We want a point on $1 \times \mathbb{I}^n$ so put that condition. Immediately it gives you a unique solution. That is your x so that $\tau(x) = y$. This is just an elementary linear algebra problem. Okay?

Since you can write down the formula for tau inverse, that completes the claim that τ is a homeomorphism. Obviously, when you take the boundary point here or here, okay, namely 1 cross boundary of $\mathbb{I}^n$ is the boundary of this one okay, there your original point and its projection coincide. So, τ is identity on the boundary of A, okay? Now take ϕ from boundary of $\mathbb{I}^{n+1}$ to itself, to be the map such that ϕ is τ^{-1} on W and τ on A. On the intersection both are identity maps and so, they patch up to define a homeomorphism ϕ from boundary of $\mathbb{I}^{n+1}$ to itself.

Finally, take the affine linear transformation T from $\mathbb{R}^{n+1}$ to itself which merely interchanges the first coordinate and $(n+1)$-th coordinate and followed by the reflection in $1/2 \times \mathbb{I}^n$. The formula is: $(t_1, t_2, \ldots, t_n, t_{n+1}) \mapsto (1 - t_{n+1}, t_2, \ldots, t_{n-1}, t_1)$. The first coordinate has come to last coordinate that is just a permutation, and then the reflection which is an affine linear isomorphism. Obviously, $\mathbb{I}^{n+1}$ goes to itself under T and its boundary goes to the boundary.

Therefore I can take T restricted to boundary of $\mathbb{I}^{n+1}$ and compose it with ϕ and call it ψ, so that I get a map from boundary of $\mathbb{I}^{n+1}$ which is a homeomorphism. After that you can extend it to a homeomorphism $\hat{\psi}$ of the entire $\mathbb{I}^{n+1}$ to itself, by taking the cone construction.

So, this homeomprhism is $\hat{\psi}$, an extension of ψ, okay? This homeomorphism has the following properties, which justifies why we have done all this:

On $\mathbb{I}^n \times 0$, both ψ and $\hat{\psi}$ are homeomorphisms onto W. Now trick is revealed. You wanted to control the map on W, but now this can be done by controlling it on a single face $\mathbb{I}^n \times 0$, which is done, by the very definition of homotopy lifting property. So now, go back to our homotopy lifting diagram.

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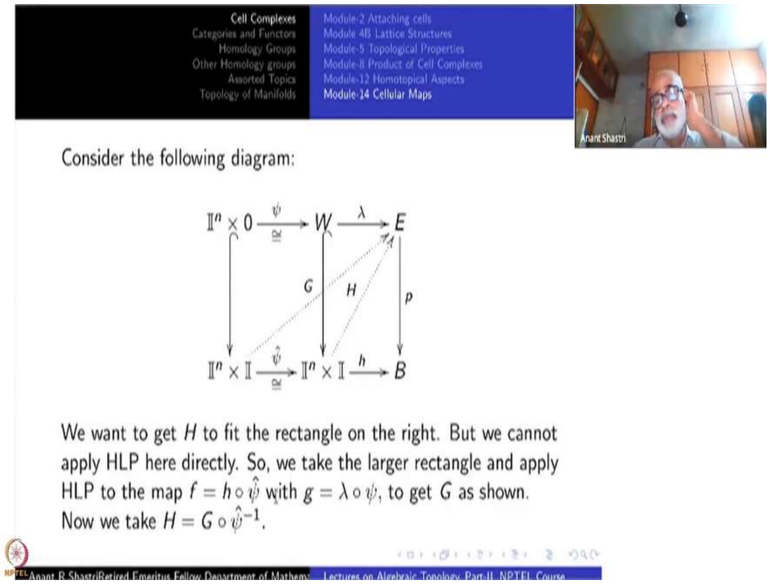

I have this map p which is a fibration here, I have an h here, I want capital H here such that restricted to W it is λ and $p \circ H$ is h. I cannot apply the homotopy lifting property directly here, because this W is not just $\mathbb{I}^n \times 0$. So, what I do? I take $\hat{\psi} \circ h$ and come to the larger rectangle here. At the bottom, on $\mathbb{I}^{n+1}$, I take $\hat{\psi} \circ h$ and at the top I take $\lambda \circ \psi$ restricted to $\mathbb{I}^n \times 0$, okay. You have to check that $p \circ \lambda \circ \psi$ is equal to $h \circ \hat{\psi}$ restricted to $\mathbb{I}^n \times \mathbb{I}$, which is obvious.

Therefore the HLP gives you this map G. Now all that I do is start here, come to here via $\hat{\psi}^{-1}$ and follow it by G, i.e., take capital H equal to $G \circ \hat{\psi}^{-1}$, okay? So that will give you whatever H we wanted. Okay?

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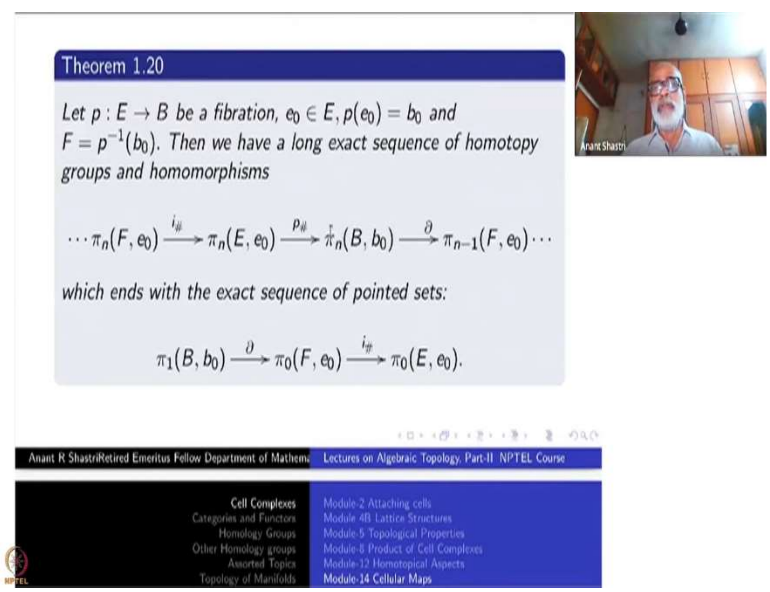

Now, let us come to the final statement of the theorem 1.20. The proof is very easy. Take any fibration, take a base point e_0 in the top space. And then the base point in B must be taken $b_0 = p(e_0)$. And let $F = p^{-1}(b_0)$. This is the convention. Then we have a long exact sequence of homotopy groups and homomorphisms, $\pi_n(F, e_0)$ to $\pi_n(E, e_0)$ as in the previuos theorem but then suddenly, instead of a relative homotopy here, we have $\pi_n(B, b_0)$ and so on....

The homomorphisms are $i_\#$ and $p_\#$ instead of $j_\#$ and then instead of the old boundary operator ∂ , a new one δ to $\pi_{n-1}(F)$ so on. Finally, it will end up with $\pi_1(B, b_0)$ to $\pi_0(F)$ to $\pi_0(E)$. Remember the last entries are not groups but just pointed sets, the set of path components of F and E respectively and the last function is the inclusion induced map, okay?

The proof is now very clear. In the long exact sequence of relative homotopy groups of the pair (E, F, e_0), I replace all $\pi_n(E, F, e_0)$ by $\pi_n(B, b_0)$. I need to replace the two adjacent homomorphisms also properly. The proposition says that the relative homotopy group is isomorphic to $\pi_n(B, b_0)$. under $p_\#$. So, I am going to take $p_\# \circ j_\#$, which I denote again by $p_\#$, because $p \circ j = p$.

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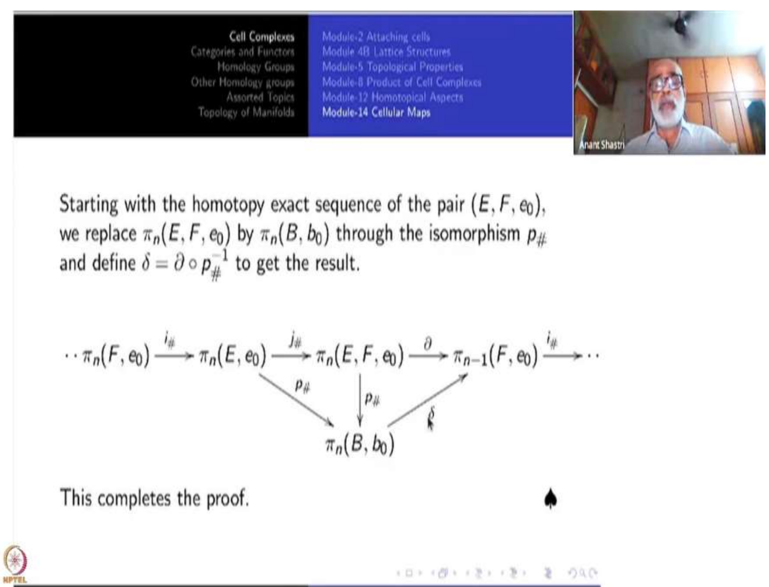

Similarly, I define the homomorphism δ to be $p_\#^{-1}$ followed by ∂. That is all. That is why we use a different notation here. Automatically, this sequence will now be an exact sequence. Coming from here to here and come down here go that way and that is the precise statement of this okay. So, that completes the proof of the big theorem okay homotopy exact sequence of a fibration.

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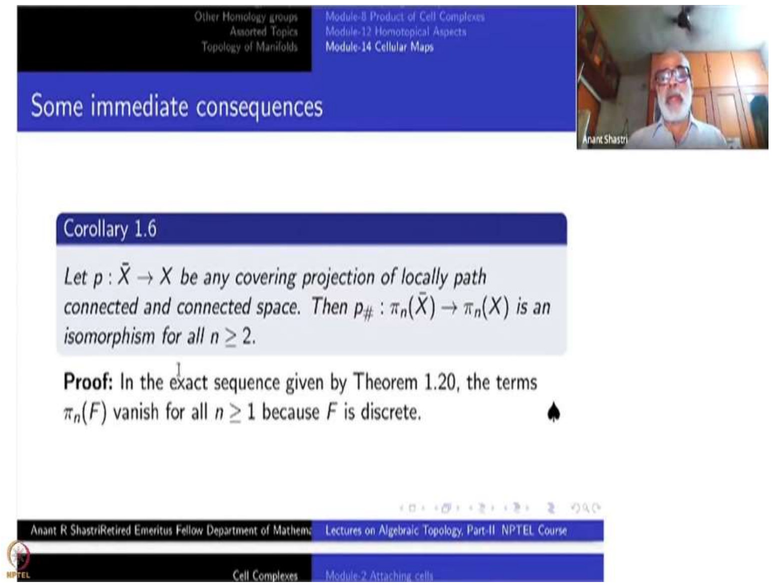

Of course it has several applications. We will only mention a few of them which are immediate consequences obtained without much trouble. You should be able to get them easily.

One of them is that a covering projection is always a fibration. It is a very peculiar fibration, a very, very important fibration. It has more properties than an ordinary fibration. In particular, the fiber of a covering projection is a discrete space. When F is a discrete space, what happens to its homotopy groups? π_0 will correspond to the set F itself, and all other higher homotopy groups π_1, π_2, etc., all of them are all 0. So, if you use that hypothesis here in the long exact sequence, this $\pi_n(F)$ will keep appearing at every third term and hence this $p_\#$ will be an isomorphism for all $n \geq 2$. When $n = 1$, you have a problem because $\pi_0(F)$ is not a singleton in general.

So, when $n = 1$, it will not work. But for $n = 1$, we know exactly what happens. This we have studied under covering projection theory.

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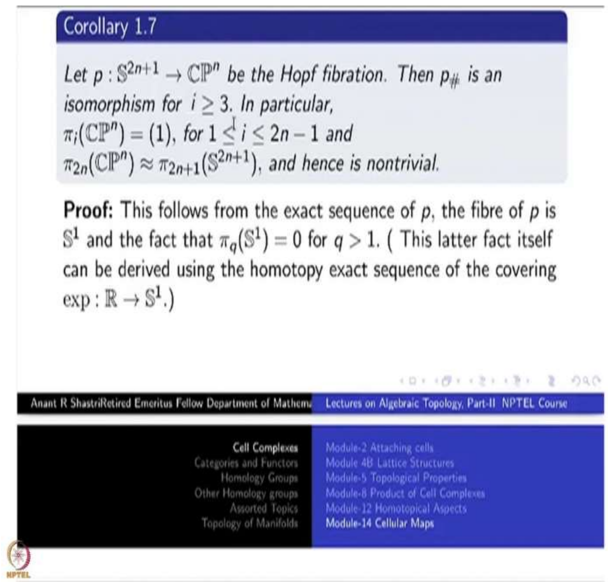

So, here is a case wherein we have even much better looking result. Consider the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$, by thinking of $\mathbb{S}^{2n+1}$ as the space of a units in $\mathbb{C}^n$, product of copies of the complex numbers taken n times. So, $\mathbb{S}^1$ is the space of unit complex numbers. The action is just by coordinatewise multiplication. The quotient space $\mathbb{C}P^n$, as we know, is the complex projective space of dimension n. Then the quotient map p from $\mathbb{S}^{2n+1}$ to $\mathbb{C}P^n$ is a fibration. That is not very easy to see though. We cannot prove that one here. It is a standard result in differential and algebraic topology.

A more general standard result is the following: If we have a compact Lie group acting on a smooth compact manifold then the quotient is a fibration okay? This result is the richest source of fibrations, by the way, but in this course, we cannot cover that one, okay? So, this p from $\mathbb{S}^{2n+1}$ to $\mathbb{C}P^n$ is a fibration and it has a name, Hopf fibration, because for the case $n = 1$, viz., p from $\mathbb{S}^3$ to $\mathbb{C}P^1 = \mathbb{S}^2$, it was an important contribution by H. Hopf.

Here, $p_{\#}$ is an isomorphism for $i \geq 3$. In the general case, it is so for $i \geq 2n + 1$. That is $\pi_i(\mathbb{C}P^n)$ is 1 for $1 \leq i \leq 2n - 1$. And $\pi_{2n}(\mathbb{C}P^n)$ is isomorphic to $\pi_{2n+1}(\mathbb{S}^{2n+1})$, okay? And hence not trivial. Non triviality of this one can be seen in many ways. The way that we have seen it is, namely, the identity map of a sphere cannot be null homotopic, okay? This fact was proved while proving Brouwer's fixed point theorem, okay?

Apply this here with $E = \mathbb{S}^{2n+1}$ and $B = \mathbb{C}P^n$, the fiber $F(p) = \mathbb{S}^1$. And we know that the homotopy groups $\pi_1(\mathbb{S}^1)$ are trivial for $i \geq 2$. Okay? So, we have isomorphisms for $i \geq 3$. For 2, there will be a problem because $\pi_1(\mathbb{S}^1)$ is nontrivial. It is infinite cyclic.

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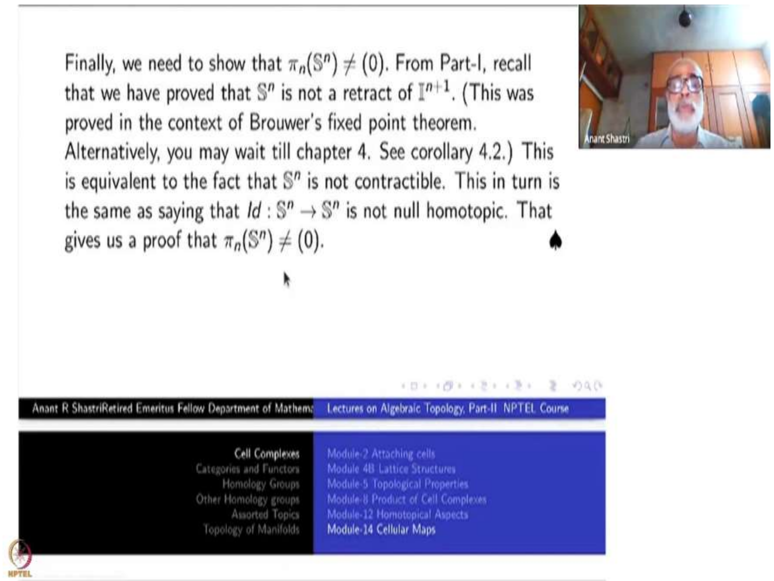

So that is what I am telling you. I repeated. $\pi_n(\mathbb{S}^n)$ is not trivial is a fact that we have proved while proving Brouwer's fixed point theorem, namely, by proving that the identity map itself is not null homotopic, okay? Indeed what happens is just like $\pi_1(\mathbb{S}^1)$ is infinite cyclic, one can prove that $\pi_n(\mathbb{S}^n)$ is also infinite cyclic. Actually that result goes under the name Hopf's theorem, okay? Hopf's a degree theorem. But this is not a part of this course okay? Usually this result is proved in differential topology, okay?

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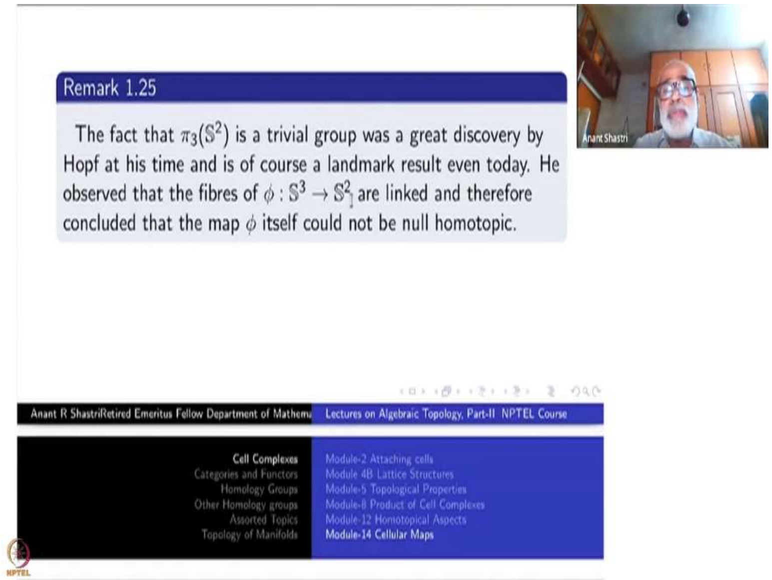

So, one more remark. The proof that $\pi_3(\mathbb{S}^2)$ is not trivial was a great discovery by Hopf when he did it, in his time, okay? It is a landmark result even today. So, he observed that the fibers of p from $\mathbb{S}^3$ to $\mathbb{S}^1$, they are all copies of $\mathbb{S}^1$ and they are inter-linked, in a very nice way, namely in a simple way like this and that is precisely what we call nowadays the Hopf link. Okay? So, this is a non trivial link that is easy to see. But fom this Hopf concludes that the map p itself is not null homotopic. That part, I cannot explain here. Okay?

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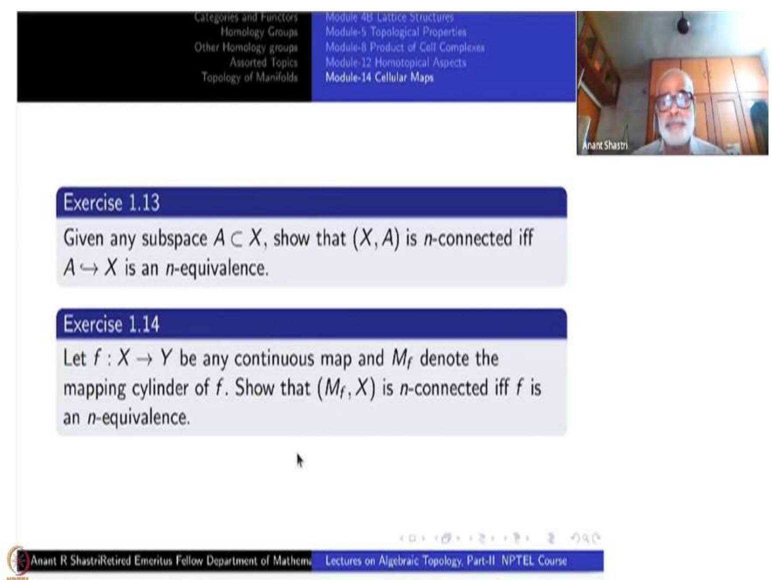

So, here are a few exercises which you can try on your own. Trying to solve exercises is a part of the learning process, okay? Trying itself is more important than just knowing the solutions okay? So keep doing exercises, in the hope that you learn more. Thank you.