# Introduction to Algebraic Topology Part – II

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## Lecture – 16 A
## Homotopy exact sequence of a pair

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Continuing with the topic introduced last time, viz., the study of higher homotopy groups let us now introduce the so called relative homotopy groups. This module 16 has been divided into 2 parts because it is a lengthy one. The idea is that I would like to take you a little deeper here to the extent that further study of homotopy groups is not possible within the scope of this course, because we need to use cohomology theory etc. So, at that level, I am going to leave you. Let us do a little more of higher homotopy groups here.

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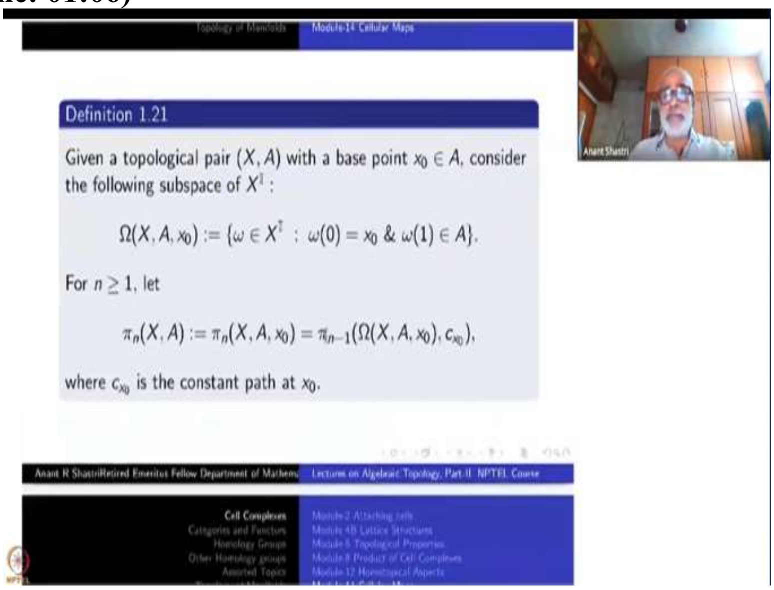

Begin with a topological pair $(X, A)$ with a base point $x_0$ belonging to $A$. Often I may not write down or mention the base point. But it is always there in the back of once mind, okay? Whenever you are discussing homotopy groups $\pi_1, \pi_2$ etc. Okay?

Let us consider the following subspace of all paths in $X$, namely continuous functions from the closed interval $\mathbb{I}$ to $X$, which is the function space. Remember that means that it is taken with the compact open topology. So now, I am writing a subspace which I am going to denote by $\Omega(X, A, x_0)$. (This capital $\Omega$ is usually used to denote the spae of all loops in a given space, in algebraic topology. Here we are using it in slightly general context.

So, $\Omega(X, A, x_0)$ is the set of all $\omega$ in $X^{\mathbb{I}}$, (that means $\omega$ is a map from $\mathbb{I}$ to $X$) with the starting point of $\omega$ being $x_0$ and the end point of $\omega$ belonging to $A$. $A$ is a subset of $X$, okay? So, that is $\Omega(X, A, x_0)$ with the subspace topology from $X^{\mathbb{I}}$.

For $n \geq 1$, we will have this notation $\pi_n(X, A)$, (which is a shorter notation for $\pi_n(X, A, x_0)$, okay, a lazy notation, the base point is always there but not mentioned), for the set $\pi_{n-1}(\Omega(X, A, x_0))$, with a specific base point being the constant function at $x_0$. For $n = 1$ this will be just a based set.

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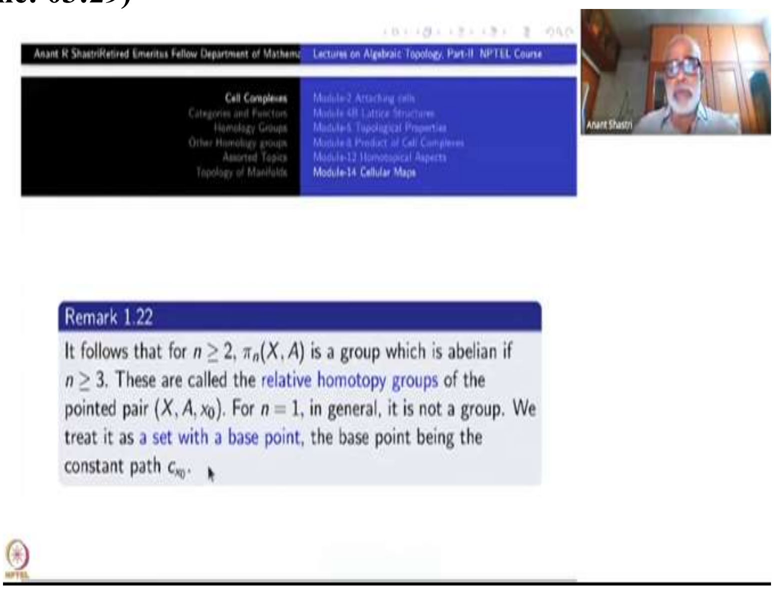


But remember that for $n$ greater than or equal to $\pi_1, \pi_2$, etc., have the standard group structure. So, that is just the definition of relative homotopy groups. So, for $n \geq 2, \pi_n(X, A)$ is a group which is abelian if $n \geq 3$. This is what we have seen last time. These are called relative homotopy groups of the pointed pair $(X, A, x_0)$.

Of course, if the base point $x_0$ is changed, then you know how to relate the different groups that arise. There may be some relation but the groups will be different. That is why the base point has to be mentioned.

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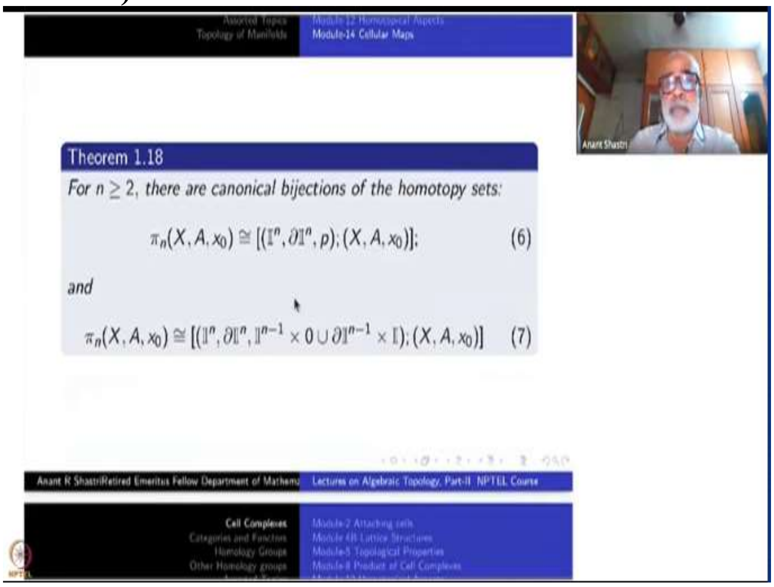


For $n = 1$ in general, this is not a group. You treat it as a set but a special kind of set, a set with a base point, the base point of the set being the constant path. okay? It is a set with a base point. The same constant loop at $x_0$ becomes the identity element of the group, when $n > 1$, okay? Now we have a theorem.

So, for $n \geq 2$, there are canonical bijections of the homotopy sets $\pi_n(X, A, x_0)$ with two other type of sets of homotopy classes of triples. The first one has the domain triple equal to ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, $p$), where $p = (0, 0, \ldots, 0)$ and the codomain is the triple $(X, A, x_0)$. Then the homotopies are defined on the triple ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$,$p$)$\times\mathbb{I}$, which is nothing but ( $\mathbb{I}^n \times \mathbb{I}$, boundary of $\mathbb{I}^n \times \mathbb{I}$, $\{p\} \times \mathbb{I}$). find cross $p$ for comma $p \times \mathbb{I}$. In the second type, we have taken only the domain triple slightly different. Okay?

So, according to your need, you may use any one of these three descriptions. In our definition, automatically, these sets get their group structure. In these two other descriptions, you not have any obvious way to define group structures but we take the group structure which makes the canonical bijections into an isomorphisms. The latter two descriptions have their own advantage.

I will now handle the second type. Here in the domain triple the third entry something different, the rest of the entries are as in the first type. Instead of just a single point $p = (0, 0, \ldots, 0)$, I am taking $\mathbb{I}^{n-1} \times 0$ union boundary of $\mathbb{I}^{n-1} \times \mathbb{I}$. All of it is a subset of the boundary of $\mathbb{I}^n$, and includes the point $p$. So, we take maps omega from $\mathbb{I}^n$ to $X$, which take this entire set into the single point $x_0$ and of course, take the boundary of $\mathbb{I}^n$ into $A$. And I will do of course, only sticks such maps and that homotopy case these 2 are equivalent and both of them are equivalent to $\pi_{n-1}(\Omega(X, A))$, that is the definition. Thus theorem 1.18 gives you 3 equivalent ways of looking at $\pi_n(X, A)$, okay?

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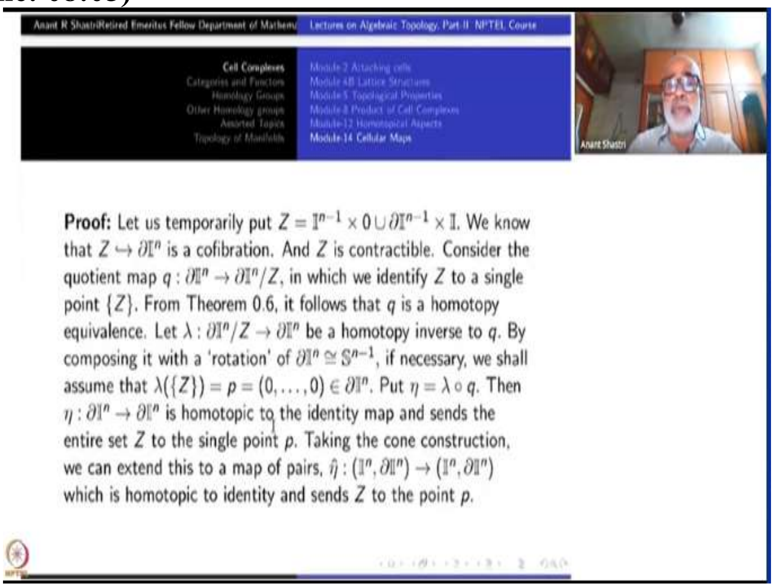


Let us prove this one. So let us have a temporary notation for this third set, $Z = \mathbb{I}^{n-1} \times 0$ union the boundary of $\mathbb{I}^{n-1} \times \mathbb{I}$. We know that the inclusion map from $Z$ to the boundary of $\mathbb{I}^n$ is a cofibration, because $Z$ is subcomplex. Remember that boundary of $\mathbb{I}^n$ is actually equal to $(Z \cup \mathbb{I}^{n-1}) \times 1$. That is the full boundary. What is missing in $Z$ is only the interior of the top face.

You take any $n$-cube, okay? Fix one of the faces and look at the union of all the other faces. That is what is happening here. And then the inclusion map into the entire boundary is a cofibration. We have proved this in part I actually. Not only that, this set is contractible also, which is easier to see. Under a homeomorphism to $(n-1)$-sphere, this set corresponds to half the sphere. That is one way to see that it is contractible. Consider the quotient map $q$ from boundary $\mathbb{I}^n$ to $\mathbb{I}^n/Z$.

Here you are `collapsing' $Z$ to a point. All the points of $Z$ are identified with each other to give a single class. Rest of the point of boundary of $\mathbb{I}^n$ are undisturbed. So, that is what we mean `collapsing $Z$' to a single point. From theorem 0.6, which you can have from the notes of part I course. Anyway, I will just you this reference here, show you this statement.

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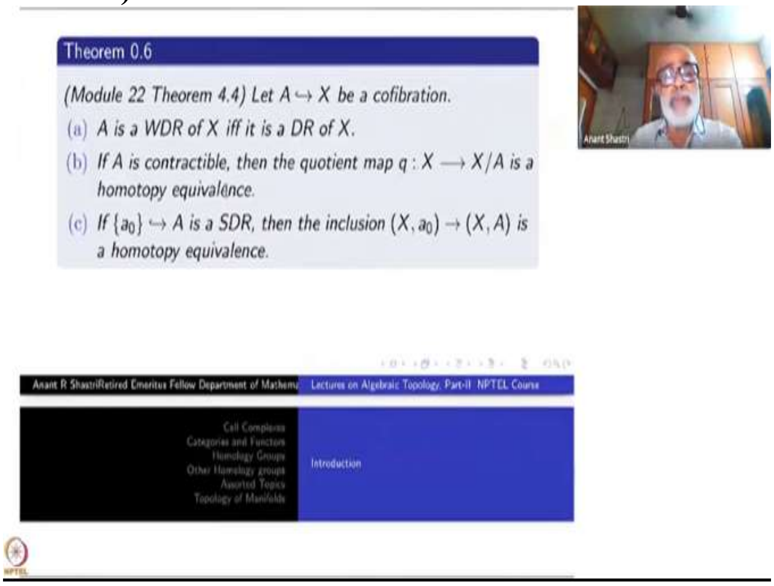


So, let $A$ to $X$ be a cofibration, $A$ is contractible then the quotient map $X$ to $X/A$ is a homotopy equivalence, okay? So, this was a theorem there. Alright? Let us come back to the present. Now this $q$ is a homotopy equivalence from that theorem, so, we let $\lambda$ from boundary $\mathbb{I}^n/Z$ to the boundary of $\mathbb{I}^n$ be a homotopy inverse of $q$, okay? So, that means that $\lambda \circ q$ and $q \circ \lambda$ are homotopy equivalent to the identity maps of the respective spaces.

By composing with a `rotation' of boundary of $\mathbb{I}^n$, (for this, you think of boundary of $\mathbb{I}^n$ as the sphere $\mathbb{S}^{n-1}$ and then you can have a rotation which takes any point to any other point and use the fact that rotations of $\mathbb{S}^{n-1}$ are homotopic to the identity map), you can assume that $\lambda(Z)$ is the point $p$. $\lambda$ is any homotopy inverse of $q$, you do not know where it maps the point $\{Z\}$. If it is equal to $p$, then fine.

Otherwise you just compose with an appropriate rotation and assume that $\lambda$ itself maps $\{Z\}$ to $p$. Now take put $\eta = \lambda \circ q$, okay? Then $\eta$ is a map from boundary of $\mathbb{I}^n$ to itself, is homotopic identity map and maps the entire of $Z$ to the single point $p$. Understand?

Now, whenever you have a map from $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$, you can extend it, by the cone construction, to a map $\mathbb{D}^n$ to $\mathbb{D}^n$. That is what we do now. Let $\hat{\eta}$ from $\mathbb{I}^n$ to $\mathbb{I}^n$ be the extension of $\eta$ obtained by taking the cone construction.

Automatically, $\hat{\eta}$ will be also a homotopy equivalence, Indeed it is homotopic to the identity map of $\mathbb{I}^n$. Okay? This is a bit stronger than saying that it is a homotopy equivalence. Of course, $\hat{\eta}$ also sends the entire of $Z$ to a single point.

Now, you might have appreciated why I am doing all this. Upto homotopy, the entire $Z$ can be treated as single point. Of course every conclusion that follows is true up to homotopy.

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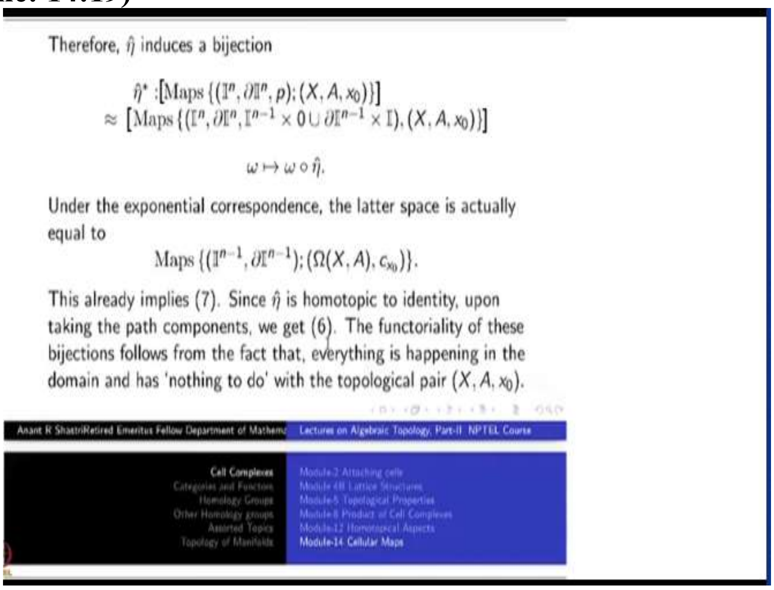


Look at the map induced by $\hat{\eta}$ from the space of all maps from the triple ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, $p$) to $(X, A, x_0)$. Suppose you have a map here into $(X, A, x_0)$, okay? Then by composing with $\hat{\eta}$, you will get a map from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, $Z$), because this whole $Z$ goes into a single point under $\hat{\eta}$. Okay? So $\omega$ going to $\omega \circ \hat{\eta}$ is the function $\hat{\eta}^\star$, from the first set to second set. Both are function spaces and $\hat{\eta}^\star$ is continuous.

Upon passing to homotopy classes, the corresponding function induced by $\hat{\eta}^\star$ is a bijection.

But now under the exponential correspondence the latter space is actually equal to the this one, viz, the space of all maps from the pair ($\mathbb{I}^{n-1}$, boundary of $\mathbb{I}^n$, $p$) to $\Omega(X, A, x_0)$. Remember that $\Omega(X, A, x_0)$ consists of all paths in $X$ starting at $x_0$ and with endpoint in $A$, right? Look at a function $f$ from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, $Z$) into $(X, Ax_0)$ fix a point

$x \in \mathbb{I}^{n-1} \times 0$ and take the restriction of $f$ to $x \times I$. That will give you an element of $\Omega(X, A, x_0)$, under the exponential correspondence. Other set theoretic conditions are easy to check. That is not a problem. Boundary of $\mathbb{I}^n$ goes inside $A$ okay? More important is that the entire is that the entire of $Z$ goes to the constant map at $x_0$, okay?

So, this implies assertion (7) Okay? Now, (6) follows from the previous observation that eta hat star induces bijection of homotopy classes. Finally, the functoriality of these bijections follows from the fact that everything is happening in the domain and has nothing to do with the internal structure of the topological triple $(X, A, x_0)$. The actual meaning of this is as follows.

Suppose you have another triple $(Y, B, y_0)$, okay? And a map $\alpha$ from here to here. Composing with that map alpha from $(XA, x_0)$, to $(Y, B, y_0)$ will give you a function $\alpha' : \pi_n(X, A) \to \pi_n(Y, B)$ on the one hand and another function $\alpha'' : [(\mathbb{I}^n,$ boundary $\mathbb{I}^n, p); (X, Ax_0)]$ to $[\mathbb{I}^n,$ boundary of $\mathbb{I}^n, p); (Y, By_0)]$, The assertion is that the entire diagram commutes: $\hat{\eta}^\star \circ \alpha'$ is equal to $\alpha'' \circ \hat{\eta}^\star$. The is a simple consequence of associativity of compositions of maps.

Anyway, functoriality will be explained completely when you study categories and functors in the next chapter. Okay? After that you may come back here and see whether it makes better sense for you. Okay?

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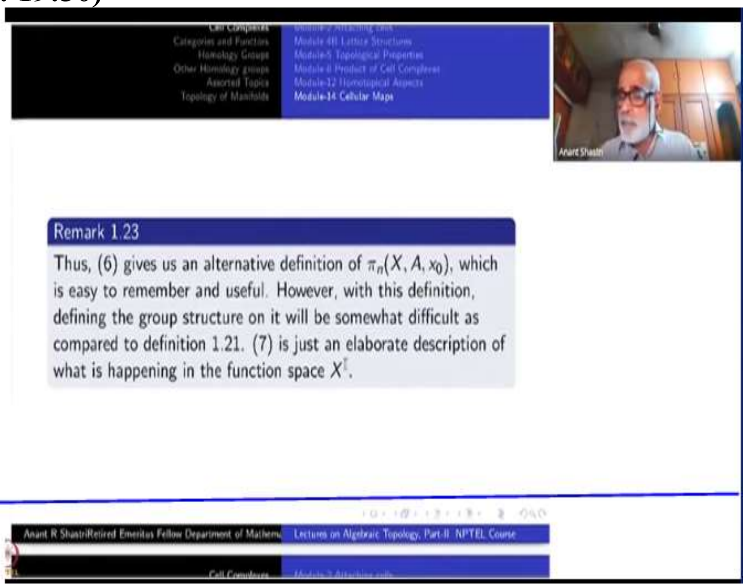

So, (6) gives an alternative definition of $\pi_n(X, A, x_0)$ which is easy to remember and useful. However, with this definition, defining the group structure will be somewhat difficult as compared to the first definition 1.21 wherein the group structures on $\pi_n$ are automatically defined right?

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Now, take a special case when $A$ is the singleton $\{x_0\}$. It follows that from this definition (6) now mentioning single point $\{p\}$ is redundant okay? Since the entire boundary is going to to the single point $x_0$. Therefore, it is same thing as $\pi_n(X, x_0, x_0)$ which is just $\pi_n(X, x_0)$, okay? So, that is the So, $\pi_n(X, A, x_0) = \pi_n(X, x_0)$.

We can now state and prove one of the most fundamental results in the homotopy theory.

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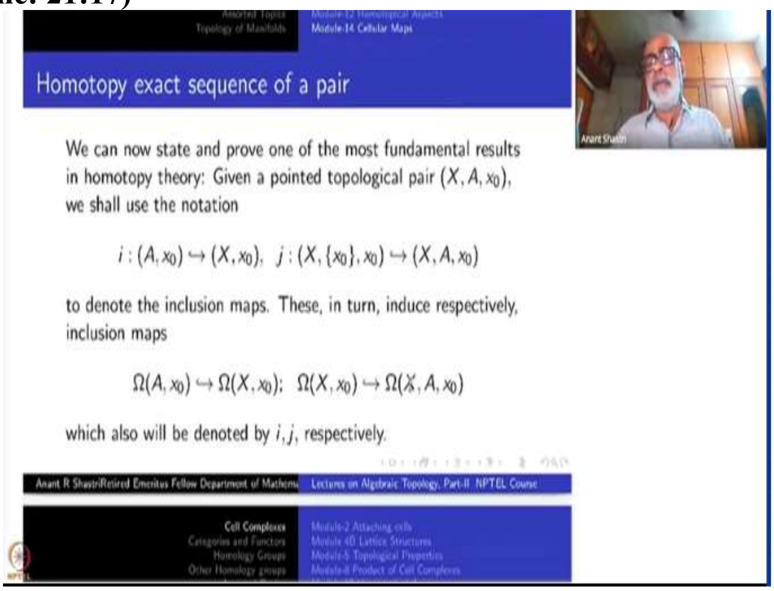

Given a pointed topological pair $(X, A)$ with a base point, we shall use the notation $i$ to denote the inclusion map $(A, x_0)$ into $(X, x_0)$ of pointed topological spaces and $j$ for the inclusion map from $(X, x_0, x_0)$ into $(X, A, x_0)$. These map, in turn, induces respectively inclusion maps when you pass on to the path spaces, $\Omega(A, x_0)$ into $\Omega(X, x_0)$ and $\Omega(X, x_0)$ into $\Omega(X, A, x_0)$. We shall denote them also by $i$ and $j$ respectively and hope there will not be any confusion.

For instance, $\Omega(A, x_0)$ consists of paths in $A$ starting at $x_0$, right?

So, you compose with $i$, to get a path in $X$, again starting at $x_0$. In other words, a path in $A$ can be thought of as a path in $X$. That is also an inclusion map of $\Omega(A, x_0)$ into $\Omega(X, x_0)$. Similarly each element of $\Omega(X, x_0)$ can be thought of a member of $\Omega(X, A, x_0)$, okay? Since $x_0$ is a point inside $A$ okay? So, these are the inclusion maps which we shall denote again by $i$ and $j$. Right?

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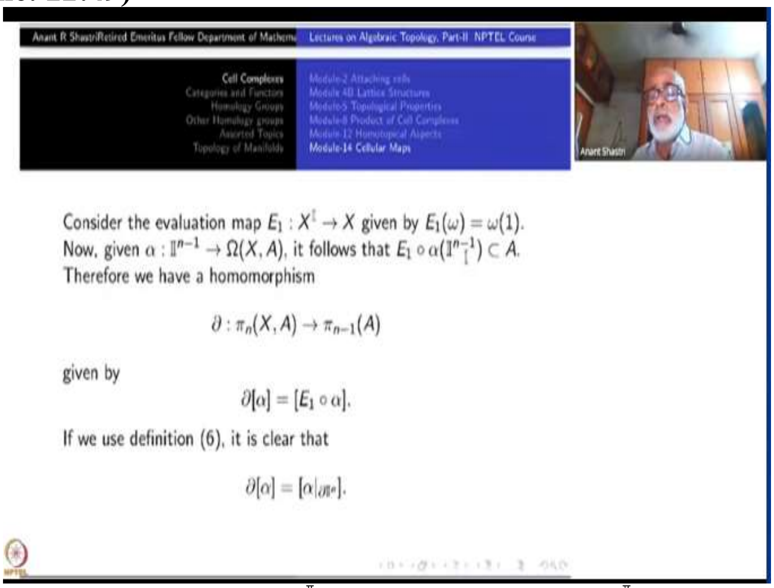


Consider the evaluation map $E_1$ from $X^{\mathbb{I}}$ to $X$. An element of $X^{\mathbb{I}}$ is a map $\omega : \mathbb{I} \to X$. If it is in $\Omega(X, A)$, then the starting point is $x_0$, right? But you can take its value at 1, the end point, that becomes important for us. So, look at the end-point map $E_1$ from $\Omega(X, A)$ to $A$ given by $E_1(\omega) = \omega(1)$.

Now, given $\alpha$ from $\mathbb{I}^{n-1}$ to $\Omega(X, A)$, we have $E_1 \circ \alpha$ of $\mathbb{I}^{n-1}$ is a subset of $A$, by the definition of $\Omega(X, A)$. Therefore, we have a homomorphism $\partial$ from $\pi_n(X, A)$ to $\pi_{n-1}$, defined by $\partial$ of the class of $\alpha$ equal to the class of $E_1 \circ \alpha$. Okay?

We are writing simplified notation here. This $\partial$ is classical notation. So, the definition of boundary of $\alpha$ is nothing but the class of $E_1 \circ \alpha$, okay? Going through the definition (6) carefully, if you view $\alpha$ as a map from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$, $p$) to $(X, A, x_0)$, it is clear that $\partial[\alpha]$ is nothing but just the class of $\alpha$ restricted to boundary of $\mathbb{I}^n$.

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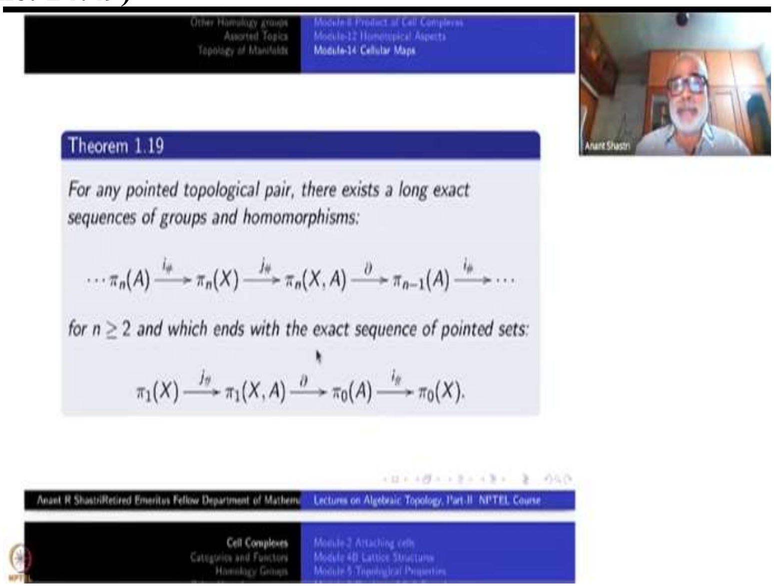


Now the statement. For any pointed topological pair $(X, A, x_0)$, there exists a long exact sequence of groups and homomorphism induced by this topological pair, using the maps $i$ and $j$ and $\partial$. All these three maps induce homomoephism at each stage $n$, and they will form a chain (possibly an infinite one) of homotopy groups and homomorphisms, $i_\#, j_\#$ and $\partial$, in that order: $i_\#$ from $\pi_n(A)$ to $\pi_n(X)$, then $j_\#$ to $\pi_n(X, A)$ and then $\partial$ to $\pi_{n-1}(A)$ and so on. I have not mentioned the base point, which is always $x_0$.

For $n \geq 2$, these are all groups and homomorphisms. What is the meaning of exact sequence? Any two consecutive composition is the trivial homomorphism. That is a part of the meaning, but it is stronger than that, viz, kernel of $j_\#$ is equal to image of $i_\#$, kernel of $\partial$ is equal to image of $j_\#$ and kernel of $i_\#$ is equal to image of $i_\#$. This should happen at every stage. It goes on an as the number $n$ keeps coming down, down $n-1, n-2$ and so on.

More precisely, the image of $j_\#$ is equal to the set of all classes which are mapped to the constant function by $\partial$ and similarly the image of $\partial$ is precisely equal to the set of all classes which are mapped to to the constant function by $i_\#$.

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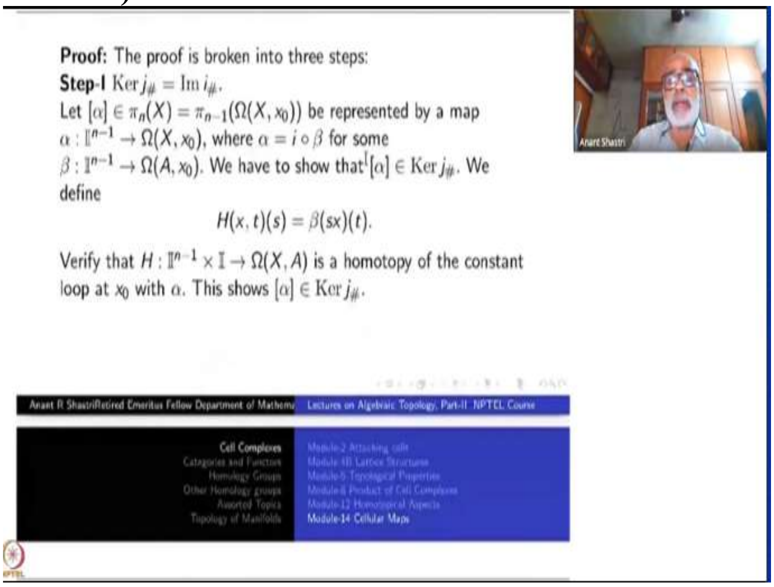


So, let us prove this theorem. Obviously, the proof will be divided into 3 parts, namely, exactness here, exactness here and exactness here, Okay? So, one by one. The proof is broken into 3 steps.

The first step is to show that kernel of $j_\#$ is equal to image of $i_\#$. Kernel of $j_\#$ here, the $i_\#$ will come from here this one $j_\#$. Okay? So, start with an element $[\alpha]\pi_n(X)$ which is by definition, equal to $\pi_{n-1}(\Omega(X, x_0))$, represented by a map $\alpha$ from $\mathbb{I}^{n-1}$ to $\Omega(X, x_0)$. Suppose $\alpha = i \circ \beta$ for some $\beta$ from $\mathbb{I}^{n-1}$ to $\Omega(A, x_0)$, you are taking an element here in the image of $i_\#$. I have to show that $j \circ \alpha$ is null homotopic. That will prove that the image of $i_\#$ is contained in the kernel of $j_\#$. Okay?

So, I have picked up this $\alpha$ such that $\alpha$ is $i \circ \beta$. Just define a homotopy $H$ from $\mathbb{I}^{n-1} \times \mathbb{I}$ to $\Omega(X, A)$, by the formula, $H(x, t)(s) = \beta(tx)(s)$.
(There is an error here in the slide).

Here $x$ is in $\mathbb{I}^{n-1}$ and $t$ and $s$ are in $\mathbb{I}$. Therefore, $tx$ is in $\mathbb{I}^{n-1}$ and hence $s$ mapsto $H(x, t)(s) = \beta(tx)(s)$ is a path in $A$ starting at $x_0$. Therefore, $H(x, t)$ is an element of $\Omega(X, A, x_0)$. Also, for $t = 0, \beta(tx)(s) = \beta(p) - c_{x_0}$ the constant path at $x_0$. For $t = 1$, $H(x, 1) = \beta(x)$, which can be viewed as the $j \circ i \circ \beta(x)$ and hence van also be viewed as $j \circ \alpha(x)$. That proves that $j \circ \alpha$ is null homotopic.

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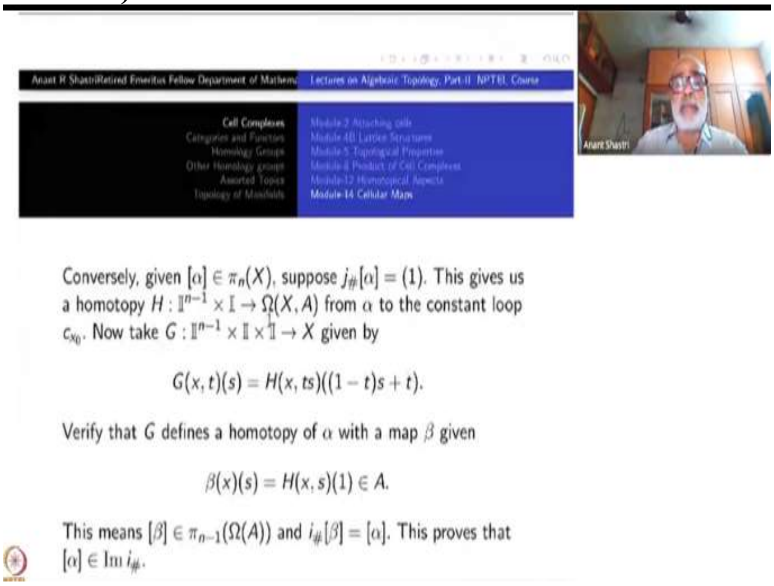


So, one part of Step I is over. Let us go ahead to the second part, the converse part. Given $[\alpha] \in \pi_n(X)$, suppose $j_\#(\alpha)$ is the trivial element in $\pi_n(X, A, x_0)$. This gives us a homotopy $H$ from $\mathbb{I}^{n-1\times\mathbb{I}}$ to $\Omega(X, A)$ of $j \circ \alpha$ with the constant loop at $x_0$. Now, I take $G$ from $\mathbb{I}^{n-1} \times \mathbb{I} \times \mathbb{I}$ to $X$ given by (we are reversing the roles of what we did the previous step but a little carefully) equal to $H(x, ts)((1-t)s + t)$, okay?

$t$ and $s$ are general elements of $\mathbb{I}$ and hence $(1-t)s + t$ is a convex combination of $s$ and 1. And hence is an element of $\mathbb{I}$. So the RHS makes sense. We have to see that $H$ is a homotopy of $\alpha$ with a map $\beta$ taking values inside $A$, viz., $\beta(s) = G(x, 1)(s) = H(x, s)(1)$ which is clearly in $A$. When $t = 0$, we get $G(x, 0)(s) = H(x, 0)(s) = \alpha(x)$ by the choice of $H$.

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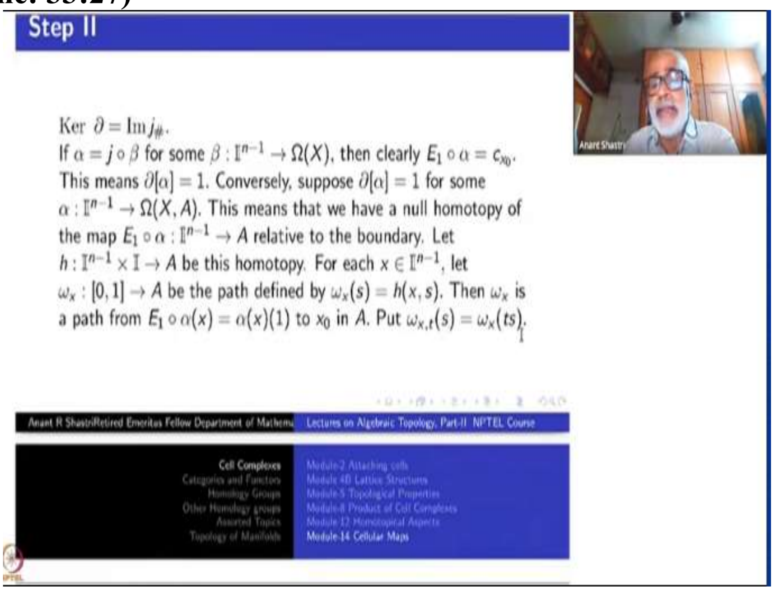

So, coming to Step II, we have to show that kernel of $\partial$ is equal to image of $j_{\#}$, okay? So now, we are coming here this part kernel here is equal to image of $j_{\#}$. Elements are inside of $\pi_n(X, A)$, okay? If $\alpha$ is $j \circ \beta$ for some $\beta$ from $\mathbb{I}^{n-1}$ to $\Omega(X, x_0)$, okay? Then $E_1 \circ \alpha = \alpha(1) = x_0$. But by definition of $\partial$, $[E_1 \circ \alpha]$ is equal to $\partial[\alpha]$. So, one way was very easy.

Now, conversely suppose $\partial[\alpha]$ is the trivial element, for some $\alpha$ from $\mathbb{I}^{n-1}$ to $\Omega(X, A, x_0)$. This means that we have a null homotopy of $E_1 \circ \alpha$, okay? You see, $E_1 \circ \alpha$ may not be the constant function but it is homotopic to the constant function at $x_0$. That is what we get, okay? A relative homotopy. That is the meaning of that something is a trivial element in this group okay? So, what we have a null homotopy of $E_1 \circ \alpha$, okay? Let $H$ be such a homotopy. For each fixed $x \in X$, let us define a path $\omega_x$ from $[0, 1]$ to $A$, okay? By the formula, $\omega_x(s) = h(x, s)$.

Then $\omega_x$ is a path in $A$ starting from $h(x, 0) = E_1 \circ \alpha(x) = \alpha(1)$ to $h(x, 1) = x_0$. So, you know $\alpha(x)(1)$ may not be exactly $x_0$ all the time, right? This is $\alpha(x)$ is homotopic to constant function at $x_0$. So, $\alpha(x)(1)$ is some point in $A$, and from there it is joined to the point $x_0$. That is what the path $\omega_x$ is doing.

Then omega x is a path from starting from E 1 of alpha x to alpha x of 1 which x naught inside A so, you know alpha of x is not exactly all the time x naught. This is alpha x of 1 by definition and it is homotopic to constant function to x naught you get each of them will be an endpoint to be just x naught. So, alpha x 1 is some point and from there and joining it to x naught that is the meaning of this omega x what is that is what this one. Now put omega x t of s equal to this omega x ts so, I am defining another path here temper path.

Now for each $t \in I$, I can define another path $\omega_{x,t}(s) = \omega_{tx}(s) = h(tx, s)$.

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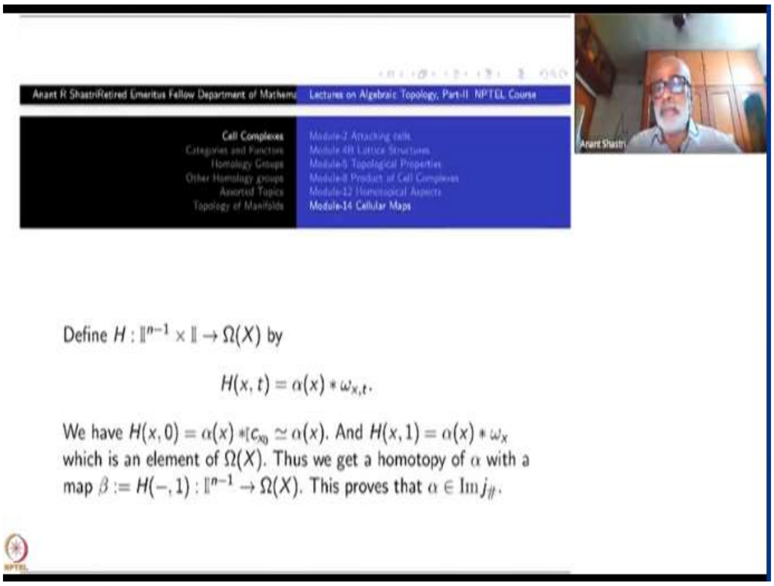


Next, define $H$ from $\mathbb{I}^{n-1} \times \mathbb{I}$ to $\Omega(X, A)$ by as follows: $H(x,t)$ equal to the composite of two paths; first trace the path $\alpha(x)$, remember this is a path in $X$ from $x_0$ to its end point $\alpha(x)(1)$, from that point now trace the path $\omega$ only upto $\omega(t)$. When $t = 1$, clearly this path will be a loop and hence is an element of $\Omega(X)$. So, take this path okay? I want to say that they are all loop. Also, when $t = 0$, we get $\alpha(x)$ and for $t = 1$, we get an element of $\Omega(X)$. So, $H$ will be the required homotopy.

(Editors note) Here is the precise formula for $H$, which is given wrongly in the slides.

For each $(x,t) \in \mathbb{I}^{n-1} \times \mathbb{I}$, define $H(x,t)(s)$ equal to $\alpha(x)(2s/(2-t))$ in the interval $0 \le s \le (2-t)/2$ and equal to $h(x, 2st + t - 2)$, in the interval $(2-t)/2 \le s \le 1$. We leave checking the details that this $H$ is the correct homotopy to you as an exercise.

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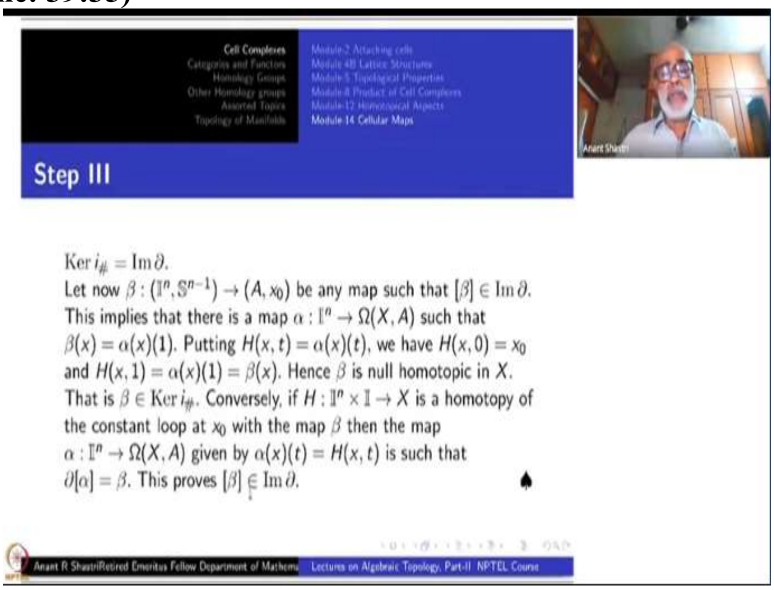

Now the third step is to prove that kernel of $i_\#$ is equal to image of $\partial$. This last thing here. Now the elements are inside $\pi_{n-1}(A)$. Okay? Assume that $[\beta] \in \pi_{n-1}(A)$ is in the image of $\partial$. (Then I have to show that $i_\#[\beta]$ is the trivial element, i.e., homotopic to the constant function, right?) This implies that there is a map $\alpha$ from $\mathbb{I}^n$ to $\Omega(X, A)$ such that $\beta(x)$ is $\alpha(x)(1)$, by the definition of $\partial$, okay?

Putting $H(x,t) = \alpha(x)(t)$, (remember $\alpha(x)$ is a path in $X$, so, $H(x,t)$ is an element of $X$. We have $H(x_0)$ is equal to $\alpha(x)(0) = x_0$ and $H(x,1)$ is $\alpha(x)(1) = \beta(x)$. Therefore $\beta$ is null homotopic in $X$, okay? So, that means $[\beta]$ is in the kernel of $i_\#$, where $i$ is the inclusion of $A$ to $i(X)$, okay? So, one way we have proved.

Conversely, suppose you have a homotopy $H$ from $\mathbb{I}^n \times \mathbb{I}$ to $X$ of the constant loop at $x_0$ with the map $\beta$, i.e., $i \circ \beta$ is null homotopic in $X$. Then look at $\alpha$ from $\mathbb{I}^n$ to $\Omega(X, A)$ given by $\alpha(x)(t) = H(x,t)$, the same homotopy. Then, $\partial([\alpha])$ is equal to $\alpha(x)(1) = H(x,t) = \beta(x)$ because by definition, $\partial[\alpha] = [E_1 \circ \alpha]$, where $E_1$ is evaluation at the end point. So, $[\alpha]$ is an element in $\pi_n(\Omega(X, A))$ such that $\partial[\alpha] = [\beta]$. That proves $[\beta]$ is in the image of $\partial$.

So, to sum up, steps I and III were easier, step II was somewhat difficult. Though, in nature, all the three steps similar, the proof are slightly different. That is why I taken pain to write down all these proofs, okay? So, go through them carefully. It is not that if you prove one of them then you can leave others because they are similar. That is not the case here okay. So, go through them carefully. That is the only way to understand the various interrelations here and the definitions and so on. Okay? So we will take a rest here and start the second part of the model 16 A little later. Thank you.