Introduction to Algebraic Topology (Part-II)
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Module No # 03
Lecture No # 15
Cellular Maps – Continued

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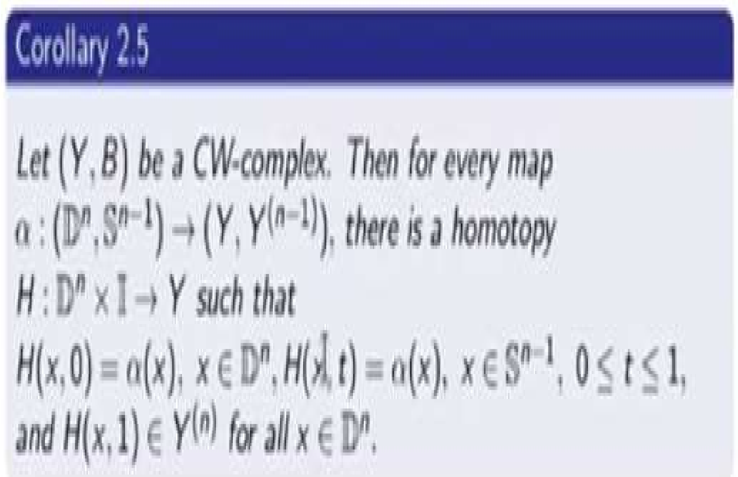

Carrying on with the proof of cellular approximation theorem, so far, we have proved that for the case when both domain and co domain have just one cell. So I will recall that lemma and then immediately go one step ahead.

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The basic idea is in the following lemma:

Lemma 2.14

Let $\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (Y, B)$ be a continuous map, where Y is got by attaching a single cell e^m to B. If $m > n$, then there is a homotopy $H : \mathbb{D}^n \times \mathbb{I} \to Y$ such that $H(x, 0) = \alpha(x), x \in \mathbb{D}^n, H(x, t) = \alpha(x), x \in \mathbb{S}^{n-1}$, $0 \leq t \leq 1$, and $H(x, 1) \in B$, for all $x \in \mathbb{D}^n$.

So let us see what was the lemma? The lemma was that given any map from $(\mathbb{D}^n, \mathbb{S}^{n-1})$ which is like a relative CW-complex with single cell of dimension n, right? To (Y, B) which is also a relative CW complex with a single cell of dimension $m > n$, can be homotoped to another map which takes values inside B; it can be pushed inside B. This is the lemma. Now we want to do the same thing taking (Y, B) to be any general CW complex, and keeping the domain the same. So this is the first corollary here.

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Corollary 2.5

Let (Y, B) be a CW-complex. Then for every map $\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (Y, Y^{(n-1)})$, there is a homotopy $H : \mathbb{D}^n \times \mathbb{I} \to Y$ such that $H(x, 0) = \alpha(x)$, $x \in \mathbb{D}^n, H(x, t) = \alpha(x)$, $x \in \mathbb{S}^{n-1}$, $0 \leq t \leq 1$, and $H(x, 1) \in Y^{(n)}$ for all $x \in \mathbb{D}^n$.

Let (Y, B) be a relative CW complex. Then for every map α from $(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $(Y, Y^{(n-1)})$, (I am replacing B with the $(n-1)$-skeleton of (Y, B)), there is homotopy H from $\mathbb{D}^n \times \mathbb{I}$ to Y such that $H(x,0) = \alpha(x)$, to begin with, $H(x,t)$ is also $\alpha(x)$ on the boundary $\mathbb{S}^{n-1}$ and $H(x,1)$ will be in the n-th skeleton $Y^{(n)}$ of (Y,B).

If (Y,B) is already of low dimension than n, there is nothing to prove here. The point is that I am implicitly assuming that there are cells of higher dimension in $Y \setminus B$. Moreover if α is already taking value inside $Y^{(n)}$, then also there is nothing to prove. So it is implicitly assumed that α maps some points into cells of dimension biggen than n. But this time there may be several such cells here as compared to the earlier lemma. How to handle this case?

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Proof: Since $\alpha(\mathbb{D}^n)$ is a compact set, it meets the interior of only finitely many cells in Y. If all these cells are of dimension $\leq n$, there is nothing to prove. So, let $m > n$ be the maximum of the dimensions of these cells, and suppose e^m_i is one such cell.

The very first thing is to note that $\alpha(\mathbb{D}^n)$ is a compact set and so it will meet only finitely open cells of $Y \setminus B$, which we have seen. Among all these finitely many cells, pick up one cell of largest dimension. And if this largest dimension is itself less than equal to n then there is nothing to prove. So, let us say assume that this dimension is bigger than n. Fix up one such cell among all finitely many possibilities and denote it by e^m. We have assumed that alpha of $\mathbb{D}^n$ intersects the interior of e^m.

Now what do I do? It follows that α already takes values inside the m-th skeleton of (Y,B). So, I am concentrating my attention to just this one m-cells e^m.

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In the previous lemma, replace B by $Y^{(m-1)}$ and Y by $B \cup e^m$. Then the lemma gives a homotopy of α to a map α_1, relative to $\mathbb{S}^{n-1}$, so that $\alpha(\mathbb{D}^n) \subset Y^{(m-1)}$. In finitely many such steps, we will get a homotopy of α, relative to $\mathbb{S}^{n-1}$, with a map which sends e^n inside $Y^{(n)}$. ♠

In the lemma, I take B to be the union of $Y^{(m-1)}$ and the rest of the m-cells, and take Y to be the space obtained by attaching e^m to this B. (Here there is a small error in the slide). Then the lemma gives us a homotopy of α to a map α_1 relative to $\mathbb{S}^{n-1}$, so that $\alpha_1(\mathbb{D}^m)$ takes value inside B. Therefore the number of open m-cells intersecting the image of α is reduced at least by one. In finitely many steps you will actually get rid of all the m-dimensional cells and we may now assume that α takes values inside $Y^{(m-1)}$.

Repeating this process finitely many times, we would have homotoped α to a map which takes values inside $Y^{(n)}$.

So the cellular approximation theorem for the case, when the domain has only one cell is completed. Now we can generalize this to the case when (X, A) has finitely many cells, by going up with respect to the dimension of cells in $X \setminus B$. But the case of infinitely many cells in the domain is tricky. We got it very easily for the codomain. The presence of infinitely many cells in the codomain did not matter at all because, each time you have a compact thing which is contained in the union of finitely many cells. However, more care is needed when the domain has infinitely many cells.

But for this also, we have a readymade technique, when we proved local contractability of CW-complexes. We showed how to compose infinitely many homotopies. That is what we are going to use now. Added to that we need one more technical result also.

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Proof of Theorem 2.14

We shall construct inductively a family $H_n : X \times \mathbb{I} \to (Y, B)$ such that

(i) $H_0(x,0) = f(x),\ x \in X;\ H_n(x,t) = f(x),\ x \in X'$;

(ii) $H_n(x,0) = H_{n-1}(x,1),\ x \in X,\ n \geq 1$;

(iii) $H_n(x,t) = H_{n-1}(x,1)\ x \in X^{(n-1)}, 0 \leq t \leq 1$;

(iv) $H_n(-,1)$ is cellular on $X^{(n)}$.

Now inductively, we shall construct a family of homotopies, H_n from $(X, A) \times \mathbb{I}$ to (Y, B) such that:

(i) the starting point, $H_0(x,0) = f(x)$ and $H_n(x,t) = f(x)$, for $x \in X'$.

Remember X' was a subcomplex on which f is already cellular and so we do not want to disturb on X'. So, I may not mention this again and again, but this is to be assumed all the time that we were not going to disturb f on this part at all. We are going to change the map by a homotopy only on cells which are not in X'.

(ii) Next thing is that $H_n(x,0) = H_{n-1}(x,1)$ for all $n \geq 1$. Now this is going to help us in taking the composition or better called concatenation of homotopies, just like composition of paths. So starting point of H_n is equal to in the end point of H_{n-1}.

(iii) Let us have this property, which is needed while passing from finite dimension to infinite dimension. Namely, $H_n(x,t) = H_{n-1}(x,t)$, for all $x \in X^{(n-1)}$ and for all t.

(iv) Finally the end point map $H_n(-,1)$ must be cellular on $X^{(n)}$. That means it takes the n-th skeleton of X into the n-th skeleton of Y. We do not know whether it is cellular on higher dimensional skeletons nor we need that.

So this is the inductive step. Having constructed H_0, you must be able to construct H_1 and then H_2 and so on. So you may treat this itself as a proposition.

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Then as in the proof of Lemma 2.12, we can piece them together to get a homotopy H as required, viz.,

$$H(x,t) = \begin{cases} H_{n-1}(x,(n+1)(nt-n+1)), & \frac{n-1}{n} \le t \le \frac{n}{n+1}; \\ H_n(x,1), & t = 1 \,\&\, x \in X^{(n)}. \end{cases}$$

Once we have done this, first let us see how we can complete the construction of H. That is an immediate consequence of the old technique, namely, concatenating various H_n's, in one single shot. (Remember what we did in the proof of lemma 1.15. It is similar here except that 0 and 1 are interchanged. That is the only difference. Deliberately I have done this so that you will now have better grip on this technique.)

So for each $t \in [0, 1)$, select the unique n such that $n/(n+1) \le t < (n+1)/(n+2)$. Then put H_n appropriately parameterized in this interval to get H. (So you are concatenating H_{n-1} with H_n for all n.) But so far H is defined only on $X \times [0, 1)$. Finally take $H(x, 1) = H_m(x, 1)$ where x belongs to $X^{(m)}$.

Because of (iii), it follows that $H(x, 1)$ is also well defined, and will not depend on the choice of m.

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If $x \in X^{(0)} \setminus X^{(-1)} := X'$, choose a path ω_x from $f(x)$ to some point in $Y^{(0)}$. Define $h_0 : X^{(0)} \times \mathbb{I} \to Y$ as follows:

$$h_0(x,t) = \begin{cases} f(x), & x \in X'; \\ \omega_x(t), & x \in X^{(0)}. \end{cases}$$

Then clearly, h_0 is continuous, and $h_0(-,1)$ is cellular. Since $X^{(0)} \hookrightarrow X$ is a cofibration, we can extend h_0 to $H_0 : X \times \mathbb{I} \to Y$ suhc that $H_0(x,0) = f(x)$, for all x.

First of all, continuity of H on $X \times [0,1)$ follows from (ii). To see the continuity of H on $X \times \mathbb{I}$, it is enough to check the continuity of H restricted $X^{(n)} \times \mathbb{I}$ for each n. But then this restriction is equal to H_n. Clearly, $H(x,0) = f(x)$ for $x \in X$, and $H(x,t) = f(x)$ for all $x \in X'$. Moreover $H(x,1)$ is clearly a cellular map. That will complete the proof of the theorem.

We have to do this job of defining H_n. So this we will do one by one. So first we have to define H_0. Since points of X' are not to be disturbed, we begin with any 0-cell which is not in $X_{-1} = X'$. (Essentially, we are working in the relative CW complex (X, X') and think of $X^{(0)}$ as obtained by attaching 0-cells to X_{-1}, and so on.) Choose a path ω_x from $f(x)$ to some point in $Y^{(0)}$. The entire CW-complex Y okay is built up from B by attaching cells inductively, Therefore every point in Y can be joined by a path to some point in $Y^{(0)}$. This is elementary result that for CW-complex, the number of path connected components is equal to the number of path connected components of $Y^{(1)}$. So you can join $f(x)$ first to a point in $Y^{(1)}$ and then it is easily seen that you can this point at a point of $Y^{(0)}$. A path can be treated as a homotopy of a point map. We put all these ω_x together and define $h_0 : X^{(0)} \times \mathbb{I}$ to Y by $h_0(x,t) = \omega_x(t)$ if $x \in X^{(0)} \setminus X'$ and $h_0(x,t) = f(x)$ if $x \in X'$. Since $X^{(0)} \setminus X'$ is discrete, it follows that h_0 is continuous. What is it for $t = 1$? This will be a point of $Y^{(0)}$. Therefore $h_0(X^{(0)} \times 1)$ subset of $Y^{(0)}$.

So far h_0 is defined on $X^{(0)} \times \mathbb{I}$ only. But I want this to be defined on the whole of $X \times \mathbb{I}$. So how do I do that? Here I use the property that for every sub complex of a CW complex the inclusion map is a cofibration, i.e., it has the homotopy extension property. Just for your benefit I will recall this HEP from part 1 of the course.

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Let us just recall what a cofibration means:

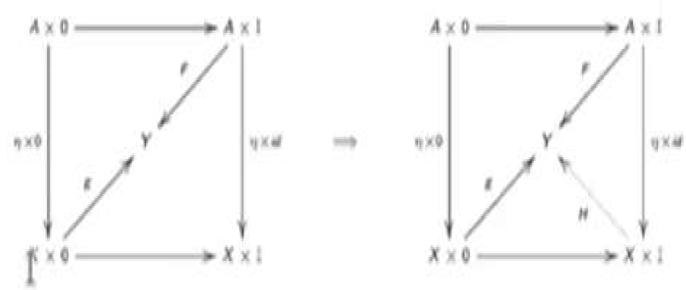

Figure 12: Homotopy Extension Property

The homotopy extension property involves this kind of commutative diagram of maps. Start with A contained inside X, there is this η which the inclusion map. It is called a cofibration, if whenever you have the diagram of maps on the left side here, it should imply the existence of the map H indicated by the dotted arrow on the right side so that the entire diagram is commutative. Here Y and g are totally arbitrary F should make the diagram commutative. This is much is the data. The rest of the diagram is automatically given. Suppose this much is given, then there is map H here which makes this entire diagram is commutative. That means restricted to $X \times 0, H$ is equal to g, i.e., H is a homotopy of g and restricted to $A \times \mathbb{I}$, it is equal to F. That means H extends the partial homotopy F of g. So that is why this is called homotopy extension property.

If A is a subcomplex of a CW complex X, this will be always true, which is we have proved in theorem 1.14. So I am going to use that property now.

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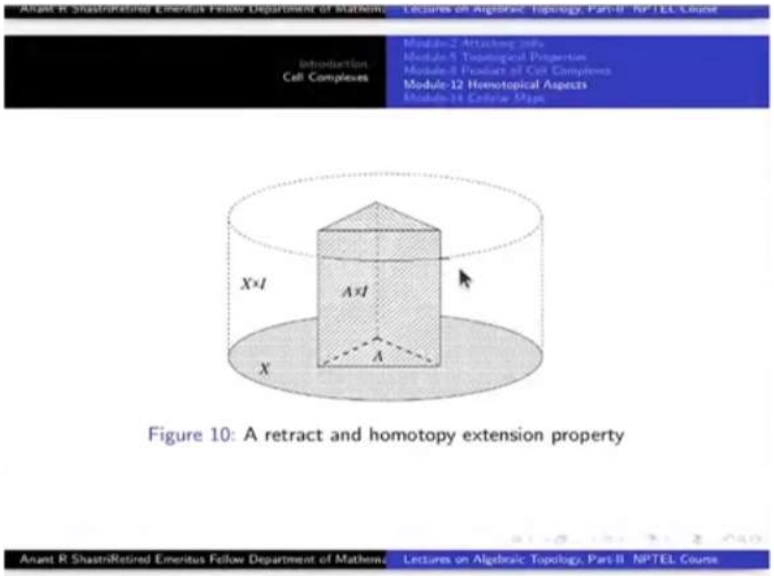

Figure 10: A retract and homotopy extension property

Start with h_0, which we have constructed on $X^{(0)} \times \mathbb{I}$. Now I extend it to whole of $X \times \mathbb{I}$, that means there is a map H_0 from $X \times \mathbb{I}$ to Y having the properties lists in (i). Automatically $H_0(-,1)$ satisfies (iv) also. That is all at the stage $n = 0$.

Now you assume that you have done this for $i \leq n-1$, viz., we have constructed H_i for $i \leq n-1$ so as to satisfy the properties (i) to (iv). First you get h_n, using our corollary 1.5, cell by cell, you get the extension on the whole of the n-th skeleton, by patching up h_n's defined on each n-cell. So I will repeat this part.

Inductively suppose I have defined H_{n-1} with the property as specified in (i) to (iv). For each n-cell e_j^n, okay, in $X^{(n)}$, (remember we do not touch any n-cells in X' at all because X' is $X^{(-1)}$ and each $X^{(n-1)}$ contains X'), with characteristic map say ϕ_i from $(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $(X^{(n)}, X^{(n-1)})$, put α_i equal to $H_{n-1} \circ (\phi_i \times Id)$.$H_{n-1}$ itself is defined on whole of X. Remember that. So this makes sense and now use that corollary to get a homotopy $H_{n,i}$ on $e_i^n \times \mathbb{I}$ to Y such that the final map takes value in $Y^{(n)}$ and it is identity on $X^{(n-1)} \times \mathbb{I}$.

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Inductively, suppose we have defined H_{n-1} with properties as specified. For each n-cell e_i^n in $X^{(n)} \setminus X_1^{\prime}$, with the characteristic map $\phi_i : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X^{(n)}, X^{(n-1)})$, put $\alpha = H_{n-1}(-, 1) \circ \phi$ in the corollary above, to get a homotopy $h_{n,i} : e^n \times \mathbb{I} \to Y$ such that $h_{n,i}|_{\partial e_i^n \times 1} = H_{n-1}|_{\partial e_i^n \times 1}$, $h_n(x, 0) = H_{n-1}(x, 0)$ and $h_n(e_{n,i} \times 1) \subset Y^{(n)}$. It follows that all these $h_{n,i}$ will pacth-up to define a homotopy $h_n : X^{(n)} \times \mathbb{I} \to Y$ with the required properties (i)-(iv)(above in place of H_n).

You can now put all of them together to cover whole of $X^{(n)}$. What is definition? Take any point (x, t) here. If x is not in $X^{(n-1)}$, then it will in the interior of some e_j^n for a unique j. Then put $h_n(x, t) = h_{n,i}(x, t)$. So they will all patch up because they all agree with H_{n-1}. Therefore, we have h_n. Now as before, you extend h_n, using the cofibration property, to a homotopy H_n on the whole of X.

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Following the important Lemma 2.14 we now make a definition.

Definition 2.12

Let $n \geq 0$ be an integer. A topological pair (X, A) is said to be n-connected, if for each $0 \leq k \leq n$, every map $f : (\mathbb{D}^k, \mathbb{S}^{k-1}) \to (X, A)$ is homotopic relative to $\mathbb{S}^{k-1}$ to a map $g : \mathbb{D}^k \to A$. (Here by convention, for $n = 0$, $\mathbb{D}^0$ is a singleton space and $\mathbb{S}^{-1} = \emptyset$.) In particular, if $A = \{x_0\}$, where x_0 is the base point then we say X is n-connected.

So inductive step is over. Therefore the proof of the theorem is over.

Following this fundamental lemma 1.17, we will make a couple of definition and then mention some very interesting results, which can be deduced by applying your mind and the results proved so far. They will be your exercises. I will give them as assignments or exercises to you. Think about them. I hope there won't be any difficulty, now that I have prepared you well.

So let me just introduce these definitions.

Let n be a non negative integer. A topological pair (X, A) (with A non empty) is said to be n-connected if for each $0 \leq k \leq n$, every map from $(\mathbb{D}^k, \mathbb{S}^{k-1})$ to (X, A) is homotopic relative to $\mathbb{S}^{k-1}$ to a map from $\mathbb{D}^k$ to A. Note how the conclusion of the lemma has been converted into a definition. With additional hypothesis as in the lemma such as $m > n$ and so on, we have an assertion there.

If that assertion happens every time, then you call this pair $(X, A), n$-connected. Remember that for every $k \leq n$ this should happen. You do not know what (X, A) is. We have proved this assertion for the case where X is got by attaching m-cells for $m > n - 1$. So, I am making that conclusion as a hypothesis in the definition of n-connectedness.

You have to have some convention for the case $n = 0$, viz., $\mathbb{D}^0$ is a singleton space and $\mathbb{S}^{-1}$ is empty. In particular, if A consists of single point x_0, that point is the base point and base point and then instead of saying (X, A) is n-connected, we say X is n-connected.

Note that when we take a topological pair (X, A), the subset A is never empty. When A is a single point, then it is common practice not to mention that point specifically and just say X is 0-connected. Otherwise we mention the pair (X, A) is n-connected. The condition is same of course.

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Next I want to introduce the concept of higher homotopy groups, somewhat similar to the concept of fundamental group. Some miracles happen here. The definitions are more or less similar, but miracles happen here from dimension 1 to dimension $2, 3$ and so on $2, 3$ etc., I mean the case $n = 1$ is quite different from the case $n > 1$.

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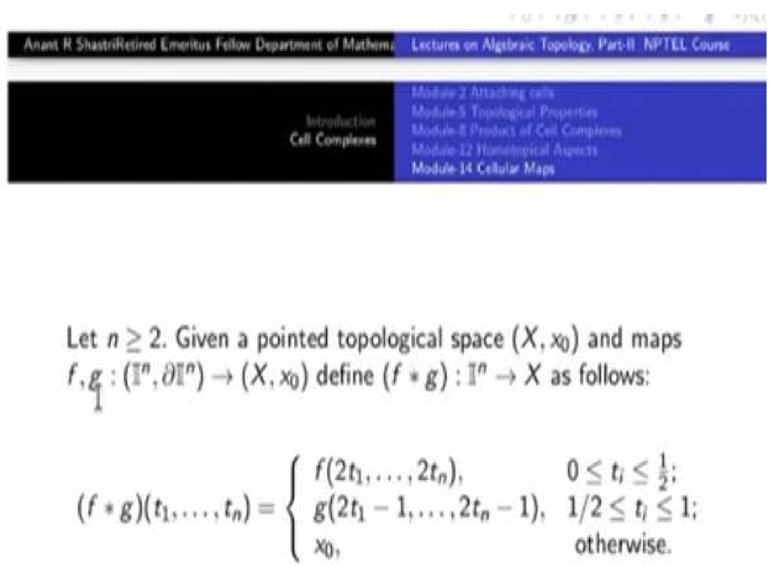

So let n be an integer greater than or equal to 2. Instead of the round model for the discs, let us work with the square models and consider two maps f, g from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$) to a pointed space (X, x_0). If $n = 1$, these maps represent some loops in X based at x_0, right? Remember that a path with both end points equal is a loop. So how did we compose loops? Exactly same way I want to do in the general case also.

Define $f \star g$ from $\mathbb{I}^n$ to X as follows: For $n = 2$, if both t_1 and t_2 are less than equal to $1/2$, take $(f \star g)(t_1, t_2) = f(2t_1, 2t_2)$; if both t_i are bigger that or equal to $1/2$, take $(f \star g)(t_1, t_2)$ equal to $g(2t_1 - 1, 2t_2 - 1)$; otherwise put $(f \star g)(t_1, t_2) = x_0$.

I told the definition for the case $n = 2$. You do the same thing for all n as well. Note that if $n = 1$ this definition coincides with the loop composition of f with g. Take this definition and just like in the case of fundamental group verify that this operation is homotopy associative, it has a 2-sided homotopy identity, namely, the constant map, etc. And one more thing, namely what will be the homotopy inverse of any map f? You just try $g = f(1 - t_1, t_2, ..., t_n)$, viz., just reverse one of the coordinates. That will be the homotopy inverse. So the set of homotopy classes of maps from ($\mathbb{I}^n$, boundary of $\mathbb{I}^n$) to (X, x_0) become a group. That group is going to be called the n-homotopy group of X with base point x_0, and denoted by $\pi_n(X, x_0)$. So that is the definition of higher homotopy groups.

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As usual, one shows that this operation factors down to define an associative binary operation on the set $[(\mathbb{I}^n, \partial(\mathbb{I}^n)); (X, x_0)]$ which has the class of the constant map as the two sided identity and for each $[f]$ the inverse is given by 'inverting' in one of the coordinates:

$$(-f)(t_1, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n).$$

We may denote the homotopy inverse of f by $(-f)$ which is given by $(-f)(t_1, t_2, ..., t_n) = f(1 - t_1, t2, ..., t_n)$, just one of the coordinates say, t_1 is replaced by $1 - t_1$. Verify that $(f \star -f)$ is homotopic to the constant map relative to the boundary of $\mathbb{I}^n$. So the group obtained this way, is denoted by $\pi_n(X, x_0)$ and is called the n-th homotopy group of X.

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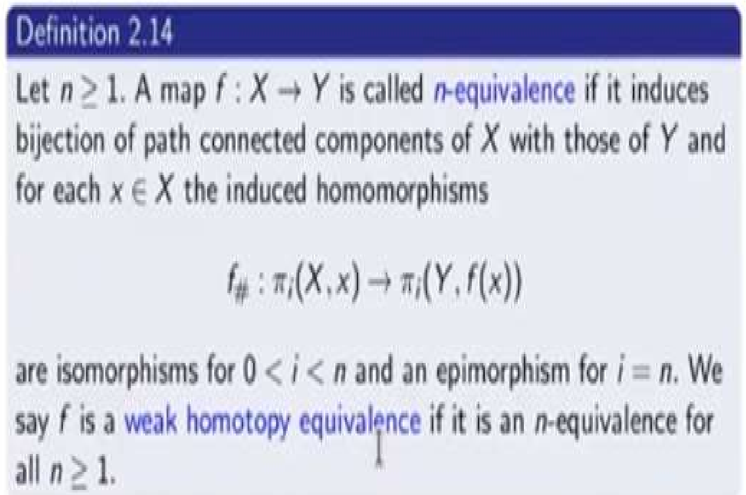

So one more definition: a map f from X to Y is called an n-equivalence (just like the term homotopy equivalence but this one depends upon n) if it induces a bijection of path components of X and Y and on each point x in X, the induced homomorphism from i^{th} homotopy group $\pi_i(X, x)$ which you have defined just now, to $\pi_i(Y, f(x))$, must be an isomorphism for all $i = 1, 2, ..., n - 1$ and for $i = n$ it must be surjective, an epimorphism. Such a map is called an n-equivalence. If f is an n-equivalence for every n, then we call it a weak homotopy equivalence.

The idea is this. It is very easy to see that if f is a homotopy equivalence then it is a weak homotopy equivalence. So, a weak homotopy equivalence is very close to being a homotopy equivalence. There is a big theorem known as Whitehead's theorem which says that for CW complexes X and Y, a weak homotopy equivalence is actually a homotopy equivalence.

So whatever I have stated so far canbe given as an exercise to you. It is not just one step exercise, but a number of short exercises, all of them you can solve one by one. So that is what I feel that if you have understood various points of whatever has been taught to you so far, you will be able to do these exercises. Such preparation I have done for you. Thank you.