Introduction to Algebraic Topology (Part-II)
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Module No # 03
Lecture No # 14
Cellular Maps

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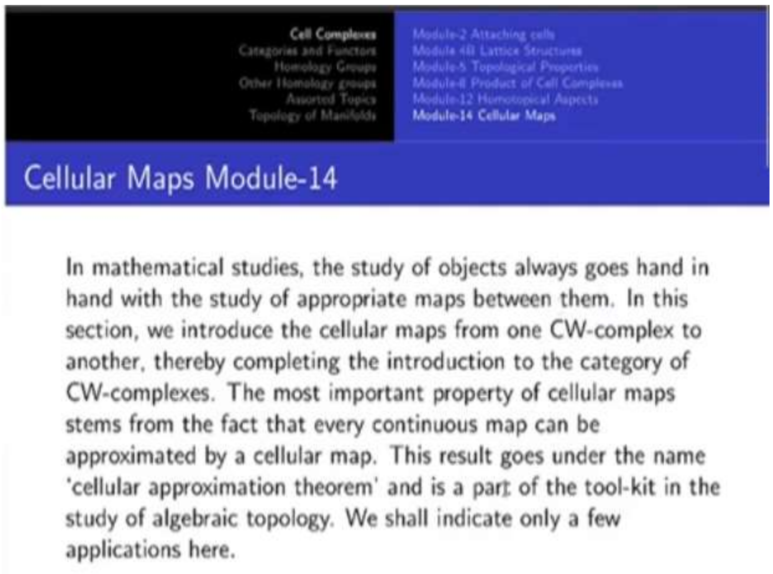

Welcome to module 14 today we can study what is called a cellular map. The subject of mathematics is such that object of study is not completed unless you study morphisms between them, the relations between them. For instance, if we are studying vector spaces, we have to study linear maps between them. If we are studying groups then we have to study homomorphisms between them. In the overall set-up of topological spaces we study continuous functions between them. But when topological spaces have extra structures like simplicial complexes you study simplicial maps between them. So here is the case wherein we have to study CW-complexes. So here we would like to introduce a notion of what is called `cellular map' between CW-complexes. These are the appropriate class of functions between CW-complexes. The most important property of cellular maps stems from the fact that every continuous map between two CW complexes can be approximated by a cellular map. So this result goes under the name cellular approximation theorem. And this theorem is part and parcel of the algebraic-topology-tool-kit. We shall indicate a couple of applications also in this course.

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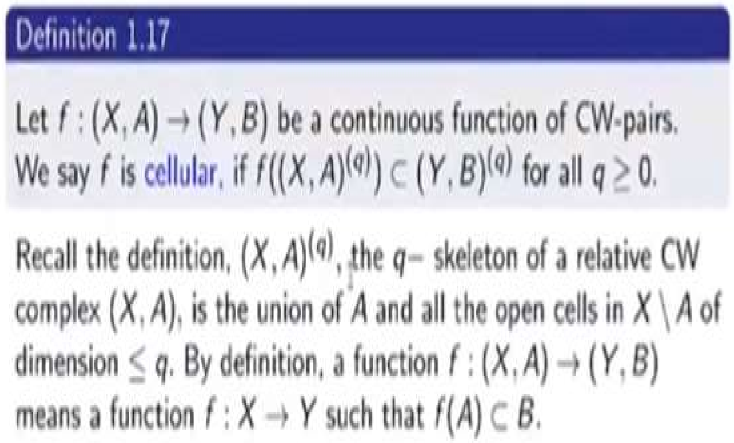

So here is the definition. Take two relative CW complexes (X, A) and (Y, B) and a function from (X, A) to (Y, B). We assume that the function is already continuous okay? And then we say f is cellular if the q-th skeleton of (X, A) goes inside the q-th skeleton of (Y, B) for every q, under the map f. Now you may recall what is the q-th skeleton of a relative CW complex (X, A) is.

It is the union of A along with all the open cell in $X \setminus A$, of dimensions less than or equal to q. Note that by definition, a function of the topological pairs (X, A) to (Y, B), is a function from X to Y such that A is taken inside B by the function.

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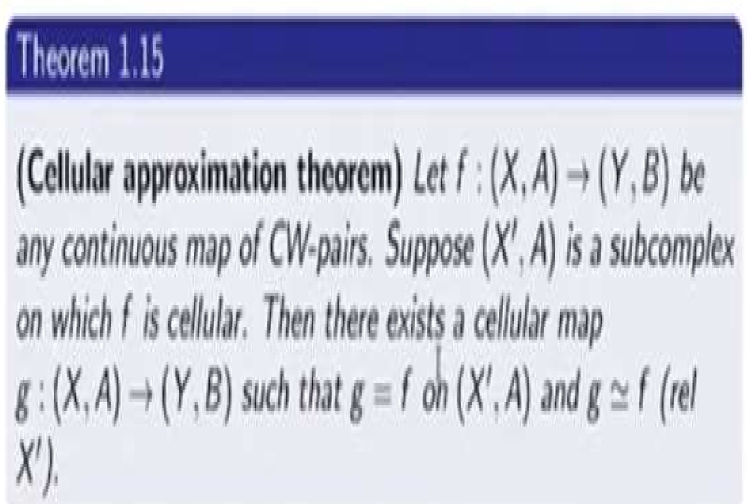

Now here is an example. Let K and L be simplicial complexes and phi from K to L is a simplicial map. Then if you look at mod K and mod L as CW complexes coming out of the simplicial complex structures, then the associated map mod phi from mod K to mod L will be a cellular map. k-skeleton of mod K will go inside the k-skeleton of mod L, for every k. That is very easy to see.

The cellular approximation theorem says the following. Take any continuous function from one relative CW complex to another, viz., $f : (X, A)$ to (Y, B). Suppose (X', A) is a sub-complex of (X, A), on which f is already cellular. Then there exists a cellular map $g : (X, A)$ to (Y, B) such that this g is identically f on (X', A) and g is homotopic to f relative to X'.

So starting with an arbitrary continuous function f from (X, A) to (Y, B) of relative CW complexes, you can replace it by a cellular map up to a relative homotopy. That is the statement okay? And moreover if this f is already cellular on a given subcomplex, there you do not have to change it. That is a strong statement. So wherever it is already a cellular map, the entire homotopy keeps that part fixed. So this is the statement of the theorem, which you may call a controlled homotopy theorem. Already on X' it is cellular, so there you do not want to change the function.

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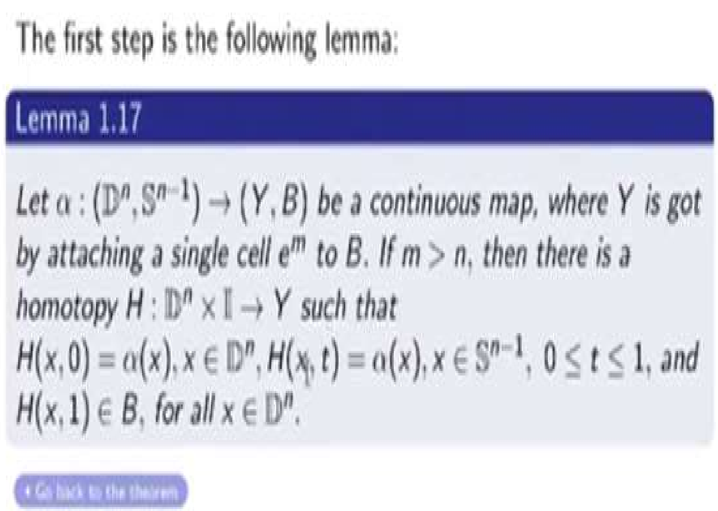

The first step is the following lemma. That itself takes some time to prove. After that the remaining proof of the theorem will be simpler.

So, let α from $(\mathbb{D}^n, \mathbb{S}^{n-1})$ to (Y, B) be a map, where Y itself is got by attaching a single cell e^m to B. In other words, $Y \setminus B$ is an open m-cell here. If further, $m > n$, then there exists a homotopy H on $\mathbb{D}^n \times \mathbb{I}$ to Y such that $H(x, 0) = \alpha(x)$ for all $x \in \mathbb{D}^n$, $H(x, t)$ is $\alpha(x)$ for all $x \in \mathbb{S}^{n-1}$ and for all t between 0 and 1, and finally $H(x, 1)$, the end map, takes values inside B for all $x \in \mathbb{D}^n$.

So, if the cell attached in the codomain is of higher dimension than the domain here, namely, $n < m$, then any map like this can be completely pushed inside B by a relative homotopy. That is the way to remember this lemma. So step by step we can do this in the general case also. But you will have to use this lemma which concentrates on one single cells on both sides, with $n < m$.

The given map may hit the interior of e^m and may not be just inside B. You do not want that for cellularity. So, we want to f to go inside B. So that is possible after a homotopy is the gist of this lemma.

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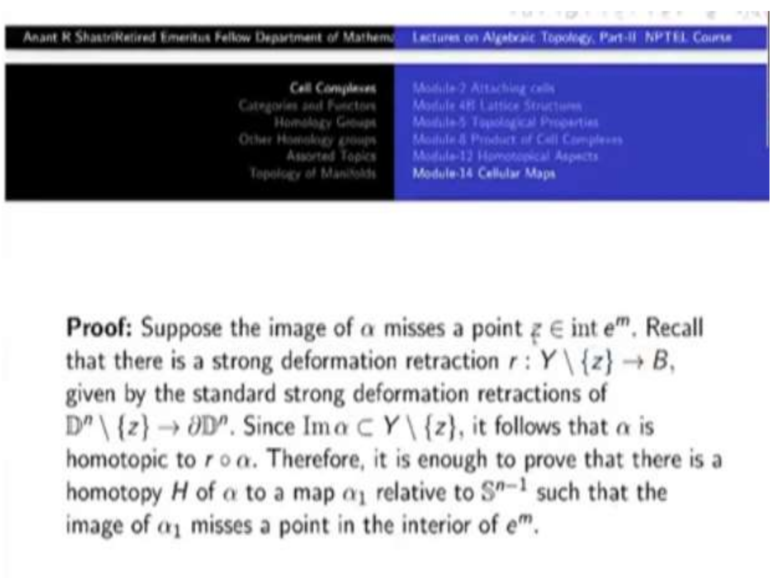

So let us start doing this one. Suppose first, that the image of α misses a point z in the interior of e^m, i.e., the image of α is not covering the whole of the interior of e^m. (In general, this may not be true, we are only assuming it, temporarily). Recall that there is a strong deformation retraction r: from $Y \setminus \{z\}$ to B, i.e., the entire thing outside B which is a part of interior of e^m, can be pushed back into B by a strong deformation retraction. This retraction is given by the standard retraction of $\mathbb{D}^m \setminus \{z\}$ to the boundary of $\mathbb{D}^m$, that is $\mathbb{S}^{m-1}$. Since the boundary of $\mathbb{D}^m$ is mapped into B by the attaching map, so from this standard deformation retraction you get a strong deformation retraction of $Y \setminus \{z\}$ to B.

Now if α misses the point z, namely, the image of α is contained in $Y \setminus \{z\}$, it follows that α is homotopic to $r \circ \alpha$. You see, image of α is contained here and therefore, $r \circ \alpha$ make sense, otherwise I would do not be able to take this composite. And r being homotopic to identity being a strong deformation retraction, $r \circ \alpha$ is homotopic to α relative to $\mathbb{S}^{n-1}$.

Therefore it is enough to prove that there is a homotopy H of α to a map α_1 relative to $\mathbb{S}^{n-1}$ such that the image of α_1 misses a point in the interior of e^m. Once the image misses a point, then I can take this homotopy. So, first we observe that if image α miss a point then the lemma is over.

So now what we want to do is to establish that there is such a possibility that we can homotope the given map into another map α_1 which has this property, namely, atleast one point in the interior of e^m is not in the image of α_1. So that is the strategy alright?

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Put $\mathbb{J} = [-1, 1]$. First of all, by fixing a homeomorphism $\mathbb{J}^k \to \mathbb{D}^k$, we shall assume that the characteristic map of the m-cell e^m in Y is $\phi : \mathbb{J}^m \to Y$ and $\alpha : (\mathbb{J}^n, \partial(\mathbb{J}^n)) \to (Y, B)$.
Recall the Lattice simplicial structure from example 1.4.

Go to Lattice structure

$$L_k^N := \{x \in \mathbb{R}^N \ : \ x_i = \frac{r_i}{2^k}, \ r_i \in \mathbb{Z}, \ \ i = 1, 2, \ldots, M\}$$

Now consider L_2^m. This will cut $\mathbb{J}^m$ into 8^m little cubes of side length 1/4 each. By starring each face of these little cubes and the cubes themselves, we get a simplicial structure on $\mathbb{J}^m$ so that each little cube is a subcomplex. See example 1.5.)

So now instead of using the round disk $\mathbb{D}^k$'s, we would like to use the square or cubic models, namely, $\mathbb{J}^k$ equal to $[-1, 1]^k$, for all k. So put $\mathbb{J}$ the closed interval $[-1, +1]$. First of all we fix a homeomorphism from $\mathbb{J}^k$ to $\mathbb{D}^k$ for each k. We can and shall assume that the characteristic map of the m-cell e^m in Y is from $\mathbb{J}^m$, instead of $\mathbb{D}^m$, to Y. And similarly, we assume that α is a map from ($\mathbb{J}^n$, boundary of $\mathbb{J}^n$) to (Y, B), instead of from $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

Recall the lattice structure that we have introduced in example 1.4, I am just recalling it here, so we do not need to go back to the look slides. This is inside $\mathbb{R}^N$, L_k^N denotes all those points x of $\mathbb{R}^N$ with the i-th coordinate x_i equal to some integer r_i divided by 2^k, for all i from 1 to n.

Now consider L_2^m, i.e. put $k = 2$, and $m = N$. So the denominators here are either 2 or 4. Therefore, all these points will cut $\mathbb{J}^m$ into 8^m little m-cubes of side length $1/4$. By starring each little edge then square and so on, all the upto the little m-dimensional cubes, you will get a simplicial structure on $\mathbb{J}^m$. This structure, we studied it before.

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Put $B_i = \frac{i}{4}J^m, i = 1, 2, 3, 4$, and $K_i = \alpha^{-1}\phi(B_i)$. Then each K_i is a compact subset of $\mathbb{J}^n$ and we have

$$K_1 \subset K_2 \subset K_3 \subset \text{int } K_4 \subset \alpha^{-1}\phi(\text{int } B_4) \subset \mathbb{J}^n.$$

Put $\beta = \alpha \circ \phi^{-1} : \text{int } K_4 \to \text{int } \mathbb{J}^m$.

Put B_i equal to $i/4$ times J^m, where i ranges from 1 to 4. $1/4, 2/4, 3/4, 4/4$ times J^m. Let K_i equal to $\alpha^{-1}\phi(B_i)$. Remember that ϕ is the characteristic map J^m to Y, so, $\phi(B_i)$ are subspaces of Y and α is a map from $\mathbb{J}^n$ to Y, so I can talk about alpha inverse of these subspaces. K_i are closed subspaces of $\mathbb{J}^n$, being inverse images of closed subspaces. So, each K_i is a compact subspace of $\mathbb{J}^n$. Because B_1 is contained in B_2 contained in B_3 etc. actually B_i is contained in the interior of B_{i+1}, the next one, it follows that K_1 will be contained in interior K_2 that is contained in K_3 contained in interior of K_4. Each K_i is contained the interior of K_{i+1}. And interior of K_4 is contained inside $\alpha^{-1}\phi(\text{int } B_4)$ which is contained inside $\mathbb{J}^n$. Because α is a map from $\mathbb{J}^n$ to Y and B_4 is equal to J^m, it follows that interior of K_4 is contained in $\alpha^{-1}\phi(\text{int } B_4)$.

Introduce an auxiliary notation now. Put β equal to $\phi^{-1} \circ \alpha$ from interior of K_4 to interior of B_4, Note that ϕ^{-1} makes sense of $\phi(\text{int } J^m)$. The whole idea is now I am looking at a map β from J^n to J^m so that (Y, B) have gone away. We have come to subspaces of Euclidean spaces both in the domain and codomain. This β is map from here to here so, we will do some analysis here then go back to our space (Y, B) via the map ϕ.

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We shall construct a homotopy $h : \text{int } K_4 \times \mathbb{I} \to \text{int } \mathbb{J}^m$ such that $h(x, t) = \beta(x)$, if $t = 0$, or if $x \in \text{int } K_4 \setminus \text{int } K_2$ and such that $\beta_1 = h(x, 1)$ has the property that

$$\text{int } \mathbb{J}^m \setminus \beta_1(\text{int } K_4) \neq \emptyset.$$

So we shall construct a homotopy h from interior of $K_4 \times \mathbb{I}$ to interior of J^m such that $h(x, t) = \beta(x)$, if $t = 0$, the starting point this homotopy, or if if x is in the interior of K_4 minus interior of K_2. So it will not be changing outside interior K_2. Only inside interior of K_2, we are going to modify β, which we define by $\beta_1(x) = h(x, 1)$ has the property that interior of $J^m \setminus \beta_1$ of interior of K_4 is non-empty. That means β_1 misses at least one point in the interior of J^m. Once you have such a β_1, you compose it with ϕ to get α_1, whatever you get you will have property that you want. So that will complete the proof of the lemma.

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We can then extend $\phi \circ h$ to a homotopy H by putting $H(x,t) \equiv \alpha(x)$ for all x outside int K_3. This H will be as required.

In fact, we can then extend $\phi \circ h$ to a homotopy H defined on the entire $J^n \times I$ by putting $H(x,t) = \alpha(x)$, for all x outside K_3. Because outside $K_2, h(x,t)$ is identically $\beta(x)$ and so composite with ϕ, it is α. That will complete the proof alright.

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Choose k large enough so that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} d(K_2, J^n \setminus K_3)$. Note that the diameter of any little cube in L_k^n is equal to $\frac{\sqrt{n}}{2^k}$. Therefore, if L is the union of all littles cubes in L_k^n, which intersect K_2, then

$$K_2 \subset L \subset \text{int } K_3.$$

Now choose k large enough so that $1/2^k$ is less than $1/\sqrt{n}$ times the distance between K_2 and $\mathbb{J}^n$ minus interior of K_3. Note that K_2 is contained in the interior of K_3, the complement of interior K_3 is completely disjoint from K_2 and hence the distance d between the two closed sets above is positive alright. (We are working within a closed and bounded subset of $\mathbb{R}^n$, everything is within $\mathbb{J}^n$.) Take k such that $1/\sqrt{n}$ times distance is bigger that $1/2^k$. Note that side length of any little cube in L_k^n is equal to $1/2^k$ and therefore the diameter is square of $n/2^k$. That is why we have to divide by $1/\sqrt{n}$ here.

So, if L is the union of all little cubes inside L_k^n which intersect K_2, it follows that K_2 will be covered by L and L itself will be inside K_3. It will not intersect complement of K_3. That is the idea.

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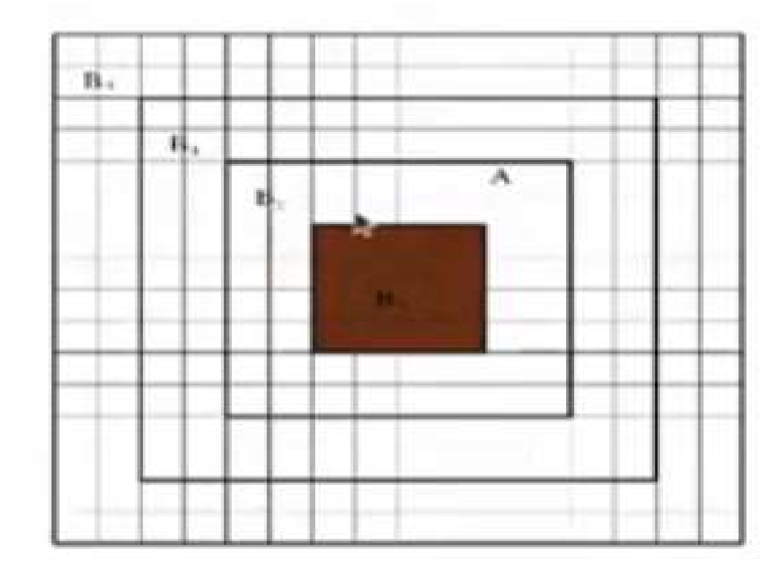

In the picture here, I have drawn these B_i's in the codomain. K_i will be inverse image of these B_i. They have arbitrary shapes subsets not as nice as B_i's. But we have divided the domain so fine that if I take all the little cubes which intersect K_2, then the union will be contained inside K_3, it will not go out of K_3.

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Again as in example 1.5, we can give L a finite simplicial complex structure. After taking a subdivion L' of L if necessary, we get a simplicial approximation $\gamma : L' \to \mathbb{J}^m$ to β.

Again as we have studied before, in the Lattice structure, we can give $\mathbb{J}^n$, a finite simplicial complex structure coming from L_k^n, because $\mathbb{J}^n$ is the union of finitely many n-cubes from L_k^n. So in proving this and some other theorems about CW-complexes we are now using results on simplicial complexes.

So look at the map β restricted to L, which is a continuous function into $\mathbb{J}^m$. We want to approximate it. Whenever you have a function from simplicial complex, another simplicial complex you may have to subdivide the domain to get a simplicial approximation.

So, after taking a subdivision L' of L, if necessary we get a simplicial approximation γ from L' to J^m. So this γ is a simplicial approximation to β. Note that β is defined on a larger space, but I am taking β restricted to L.

Then I am replacing it by a simplicial approximation γ. Now what we know about γ? $|\gamma|$ is a continuous function on $|L|$.$|L|$ is the same as the underline to space of L' which is L. And $|\gamma|$ will be homotopic to β on L. We know much more about it and we are going to use all these properties of simplicial approximation here.

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Let $\eta : \mathbb{J}^n \to \mathbb{I}$ be a continuous function such that $\eta(K_1) = \{1\}$ and $\eta(\mathbb{J}^n \setminus \text{int } K_2) = \{0\}$. Put

$$h(x,t) = t\eta(x)|\gamma|(x) + (1 - t\eta(x))\beta(x), \quad (x,t) \in L \times \mathbb{I}.$$

Then for $x \in L \setminus \text{int } K_2$, we have $h(x,t) = \beta(x)$. Therefore h can be extended to a map $\text{int } K_4 \times \mathbb{I} \to \text{int } \mathbb{J}^m$ by putting it equal to $\beta(x)$ on $\text{int } K_4 \setminus \text{int } K_2$.

So here is some technicality of analysis that we have to use, namely, let η from $\mathbb{J}^n$ to the closed interval $\mathbb{I}$ be a continuous function such that on K_1, it is identically 1 and outside the interior of K_2, it is identically 0. This is ensured by Urysohn's lemma, because these two closed sets K_1 and $\mathbb{J}^n$ minus interior of K_2 are disjoint. Then I would I would prefer to take $h(x,t)$ to be simply the linear combination of $|\gamma|$ and β but that will not work. I need to do a bit of circus here.

I take this $|\gamma|(x)$ but then I multiply it by $\eta(x)$. Then I try taking the linear combination of gamma $\eta(x)$ and $\beta(x)$. Even that will not do. So have to do one more somersault a final one, namely, take $h(x,t) = t\eta(x)|\gamma|(x) + (1 - t\eta(x))\beta(x)$, for all (x,t) belonging to $L \times \mathbb{I}$.

Now suppose x is in L minus interior of K_2. Then $\eta(x) = 0$ and hence $h(x,t) = \beta(x)$, because the first term is zero and the second term will $\beta(x)$. That is one of the condition we wanted note that.

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We shall construct a homotopy $h : \text{int}\, K_4 \times \mathbb{I} \to \text{int}\, \mathbb{J}^m$ such that $h(x,t) = \beta(x)$, if $t = 0$, or if $x \in \text{int}\, K_4 \setminus \text{int}\, K_2$ and such that $\beta_1 = h(x,1)$ has the property that

$$\text{int}\, \mathbb{J}^m \setminus \beta_1(\text{int}\, K_4) \neq \emptyset.$$

Remember that outside the interior of K_2, h is identically $\beta(x)$. To begin with we have defined h on $L \times \mathbb{I}$, but then extend it by $\beta(x)$ on the whole of $\mathbb{J}^n \times \mathbb{I}$. So that was one of the conditions.

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Let $\eta : \mathbb{J}^n \to \mathbb{I}$ be a continuous function such that $\eta(K_1) = \{1\}$ and $\eta(\mathbb{J}^n \setminus \text{int}\, K_2) = \{0\}$. Put

$$h(x,t) = t\eta(x)|\gamma|(x) + (1 - t\eta(x))\beta(x), \quad (x,t) \in L \times \mathbb{I}.$$

Then for $x \in L \setminus \text{int}\, K_2$, we have $h(x,t) = \beta(x)$. Therefore h can be extended to a map $\text{int}\, K_4 \times \mathbb{I} \to \text{int}\, \mathbb{J}^m$ by putting it equal to $\beta(x)$ on $\text{int}\, K_4 \setminus \text{int}\, K_2$.

Now come to therefore h can be extended to a map that is what I have told you. It is only defined on $L \times \mathbb{I}$. But now we can defined it on interior of $K_4 \times I$ by just putting equal to $\beta(x)$ ignore t and $h(x,t) = \beta(x)$ for all points outside interior of K_4 minus interior of K_2 i.e. outside interior of K_2.

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Put $\beta_1(x) = h(x, 1)$. We claim that $\beta_1(\text{int } K_4)$ misses some points in B_1. This will complete the proof of the lemma.
Suppose $x \in K_1$. Then $\beta_1(x) = |\gamma|(x)$. Since γ is a simplicial approximation to β and $\beta(x) \in B_1$, it follows that $|\gamma|(x) \in B_1$. Therefore

$$|\gamma|(K_1) \subset B_1.$$

Since γ is a simplicial map, its image is contained in the n^{th}-skeleton of B_1. But B_1 is of dimension $m > n$. Therefore,

$$U := B_1 \setminus \beta_1(K_1) = B_1 \setminus |\gamma|(K_1) \neq \emptyset.$$

Now looked at the last map here, i.e., $\beta_1(x) = h(x, 1)$. We claim that this β_1 of interior of K_4 (make sense because we have extended the h on the whole $K_4 \times \mathbb{I}$) will miss some points in B_1 itself. This will complete the proof of the lemma. The whole of B_1 is not covered by the image of β_1. This is a claim.

Suppose x is K_1. Let us see where it has to go under β_1. Note that $\beta_1(x)$ now equal to $|\gamma|(x)$, because, for $x \in K_1, \eta(x)$ is equal to 1, and we have $t = 1$. Now γ is a simplicial approximation to β, and $\beta(x)$ is in B_1 by definition because x is in K_1 and this K_1 is nothing but $\beta^{-1}(B_1)$. It follows that $|\gamma|(x)$ is also in B_1 because B_1 is a subcomplex. Therefore, $|\gamma|(K_1)$ is contained inside B_1.

(So this observation is not the end but it helps us finally in what we want to say that the whole of B_1 is not covered by β_1.) First of all, note that γ is a simplicial map defined on an n-dimensional simplicial complex. Therefore, its image is contained in the n-th skeleton of B_1. But B_1 is of dimension $m > n$, it's n-th skeleton is a very small close subset, the complement of it is open dense subset. Therefore, $U = B_1 \setminus \beta_1(K_1)$ which is equal to $B_1 \setminus |\gamma|(K_1)$ is non-empty open set.

So let us see what happens to other points in K_4. Just $\beta_1(K_1)$ cannot cover B_1, may be there are point outside $|\gamma|(K_1)$. Now, let x belongs to $L \setminus K_1$. By the definition of K_1, this means $\beta(x)$ is outside B_1 because K_1 is is the inverse image of B_1. Say x is one of the little cubes A of L_2^m but not contained in B_1.

So in this picture now I have come outside here it may be here or here, does not matter I do not know but it is not in the shaded part. The image of x is somewhere here in one of this little cubes so that is the part here now so it is one of the little cubes A of L_2^m not contained in B_1. Note that this implies that A does not intersect the interior of B_1. Now since each A is a sub-complex of J^m, since we have assumed that $\beta(x)$ is in A, it follows that $|\gamma|(x)$ is also inside this sub-complex A. A property of simplicial approximation is used again. It follows that the entire line segment between $\beta(x)$ and $|\gamma|(x)$ lies inside A, because A is a convex subset. Therefore this convex combination $\eta(x)|\gamma|(x) + (1 - \eta(x))\beta(x)$ is inside A. This implies that $\beta_1(x)$ which is also in the line segment belongs to A. But A does not intersect interior of B_1. Therefore, the entire of $L \setminus K_1$ is going out of interior of K_1.

Finally now suppose x is interior of $K_4 \setminus L$. First we took point is K_1, then in $L \setminus K_1$. Now we are taking interior of $K_4 \setminus L$, which is contained in the interior of $K_4 \setminus K_2$. Then by definition, $\beta_1(x) = \beta(x)$ and hence is not in B_2. So, again, β_1(interior K_4 minus L) does not intersect interior of B_1.

It follows that U which is non-empty is completely contained inside $J^m \setminus \beta_1(J^n)$. So this comes completes the proof of the lemma. Next time we shall complete the proof of CW approximation theorem, which is much simpler than the proof of this lemma, okay? Thank you.