Introduction to Algebraic Topology (Part-II)
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Module No # 03
Lecture No # 13
Homotopical Aspects – Continued

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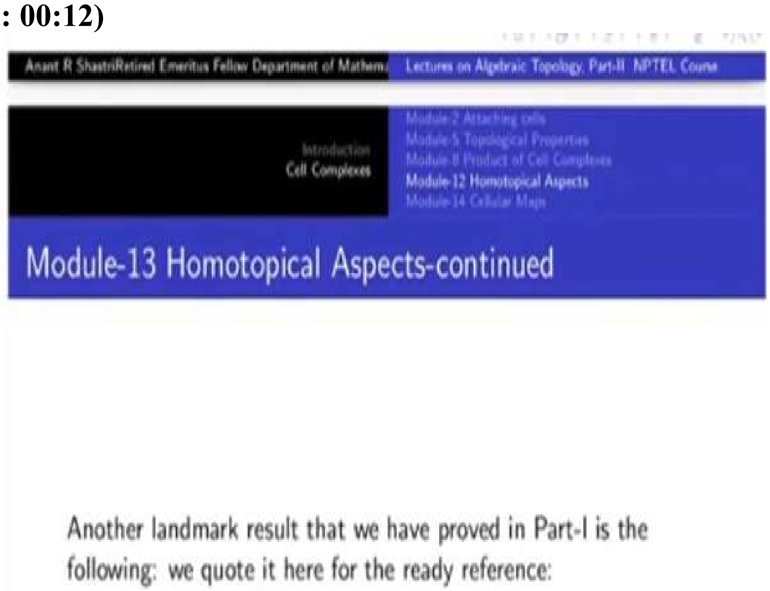

So let me begin with a result that we did in part 1 which is going to be useful again here.

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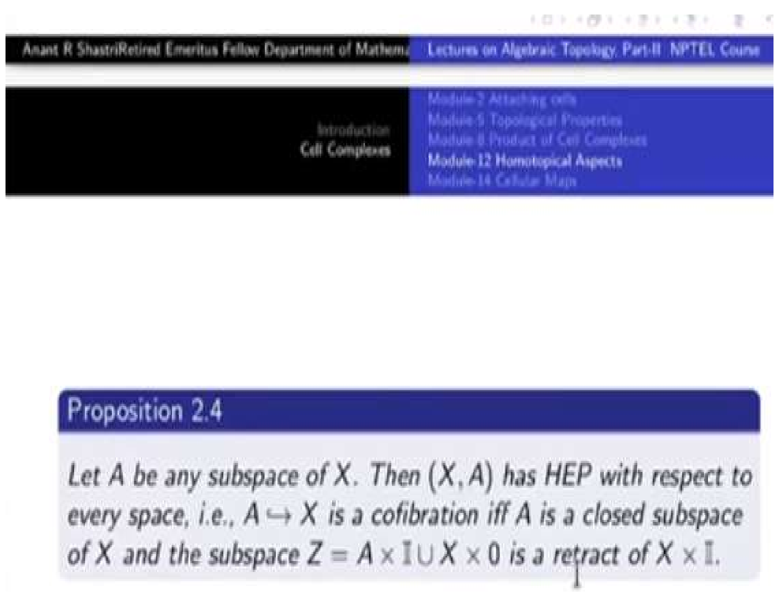

If A is a subspace X, then the pair (X, A) has homotopy extension property with respect to every space, i.e., the inclusion map from A to X is a cofibration if and only if A is a closed subset of X and the subspace $(A \times \mathbb{I}) \cup (X \times 0)$ of $X \times \mathbb{I}$ is a retract of $X \times \mathbb{I}$. This is a more general statement than the previous lemma that we proved last time, where this subspace A was the boundary of the disk and X was the disk. This is a more general statement. So if $(A \times \mathbb{I}) \cup (X \times 0)$ is a retract of $X \times \mathbb{I}$, then the pair X has homotopy extension property and conversely. So this is the just the proposition 1.4 here.

We are going to use it here. There is no point in recalling the proof and so on you better read it on your own from the notes on Part 1.

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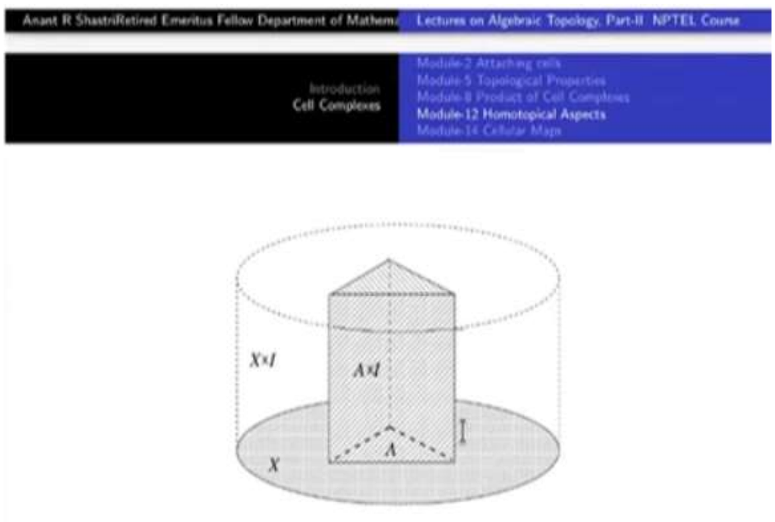

Figure 10: A retract and homotopy extension property

So this was the picture. This is X, the subspace A is shown by this triangle $A \times \mathbb{I}$ is this prism, $X \times \mathbb{I}$ is this solid cylinder. The solid cylinder can be slowly deformed leaving this portion here all the way at the bottom and change this entire thing collapse into this part. So this is the meaning of saying that $X \times \mathbb{I}$ can be strongly deformed into $(X \times 0) \cup (A \times \mathbb{I})$. This happens every time the inclusion map is a cofibration and conversely. So right now you can take this as the definition for cofibration, if you do not know what the homotopy extension property is.

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Theorem 2.13

If (X, A) is a relative CW-complex, then $A \hookrightarrow X$ is a cofibration.

Proof: By Proposition 2.4, it is enough to see that $X \times 0 \cup A \times \mathbb{I}$ is a retract of $X \times \mathbb{I}$. Applying the above lemma successively to various skeletons of X, we get retractions $r_k : X^{(k)} \times \mathbb{I} \longrightarrow X^{(k)} \times 0 \cup X^{(k-1)} \times \mathbb{I}$. Now define $r : X \times \mathbb{I} \longrightarrow X \times 0 \cup A \times \mathbb{I}$ by the formula $r|_{X^{(k)} \times \mathbb{I}} = r_0 \circ r_1 \circ \cdots \circ r_k$. Verify that r is well defined continuous and is identity on $X \times 0 \cup A \times \mathbb{I}$. ♠

Now, what we are we going to do? If (X, A) is a relative CW-complex then the inclusion map is a cofibration. This is the theorem. All that I am going to prove is that $(X \times 0) \cup (A \times \mathbb{I})$ is a strong deformation retract of $X \times \mathbb{I}$, sorry not a SDR but just a retraction. That is enough. (Actually, in practice, whenever you have show that something is a strong deformation retract it is enough to show that it a retract.) In the previous lemma we have it when X is obtained from Y by attaching k-cells. Then $(X \times 0) \cup (A \times \mathbb{I})$ is a strong deformation retract of $X \times \mathbb{I}$. You do not need strong deformation retract, but just a retract is good enough.

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Lemma 2.13

Let X be obtained from Y by attaching k-cells, then $X \times 0 \cup Y \times \mathbb{I}$ is a strong deformation retract of $X \times \mathbb{I}$.

Proof: Let $\{f_\alpha : \alpha \in \Lambda\}$ be the family of attaching maps. One of the standard result that we have proved in Part-I is that there is a strong deformation retraction

$$h : \mathbb{D}^k \times \mathbb{I} \to \mathbb{D}^k \times 0 \cup \mathbb{S}^{k-1} \times \mathbb{I}$$

Let

$$H : \sqcup_\alpha D^k_\alpha \times \mathbb{I} \to \sqcup_\alpha (\mathbb{D}^k_\alpha \times 0 \cup \mathbb{S}^{k-1}_\alpha \times \mathbb{I})$$

be just the disjoint union of copies of h.

We are going to apply the above lemma successively for each skeleton. You get a map r_k from $X^{(k)} \times \mathbb{I}$ to $(X^{(k)} \times 0) \cup (X^{(k-1)\times\mathbb{I}})$, which is identity on $(X^{(k)} \times 0) \cup (X^{(k-1)} \times \mathbb{I})$. because $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching k-cells.

Now you define r from $X \times \mathbb{I}$ to $(X \times 0) \cup (A \times \mathbb{I})$ by the formula r restricted to $X^{(k)} \times \mathbb{I}$ to be $r_0 \circ r_1 \circ \cdots \circ r_k$. When you take $r_k(x,t)$, where $x \in X^{(k)}$, you will get a point of $X^{(0)} \times \mathbb{I}$, or $X^{(k-1)} \times \mathbb{I}$. Each time if the second coordinate is zero, then you keep the point as it is, and if the second coordinate is >0, then apply the next retraction. Finally, when you take r_0, you will either hit $(X \times 0) \cup (A \times \mathbb{I})$. Thus, during this compositions, point in $X \times 0$ are never disturbed, Similarly, points in $A \times \mathbb{I}$ are also never disturbed. That is why r also has this property. Why r is continuous? Because it is continuous restricted to each skeleton cross $\mathbb{I}$.

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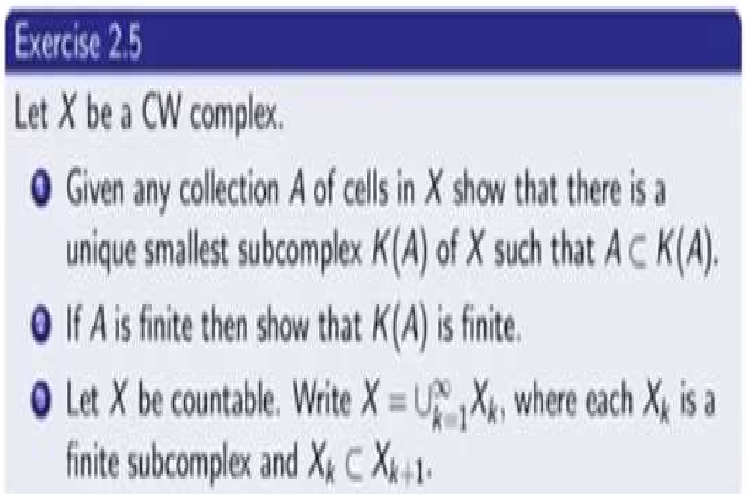

So here is an exercise. In fact I have already explained and used this one. But now I will like you to have some practice of writing down the proof:

(1) Take any collection of cells in X, and let A be the subspace which is the union of these cells. Show that there is a unique smallest subcomplex K_A of X such that A is contained inside K_A. There are many subcomplexes K containing A, where A is just the union of some collection of cells. The smallest one is unique.

(2) If A is finite then K_A is finite. So this is an important result. So when you take any subcomplex K containing A, it may not be finite. The smallest one is finite. So, you have to do some thinking.

(3) Let X be countable. Then write X as a union of an increasing sequence of finite subcomplexes X_k. Here K_k is not necessarily the k-th skeleton by the way. Each X_k has to be a finite sub complex the countable of complex.

The first two parts above can be used to prove part 3 here. Part 3 is the one which I have used, actually and I have indicated its proof also. But now you have to write a rigorous proof. So that is the exercise for today. I would not like to start with another theme here. So we will come to that one next time. Thank you.