Introduction to Algebraic Topology (Part-II)
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Module No # 03
Lecture No # 12
Homotopical Aspects

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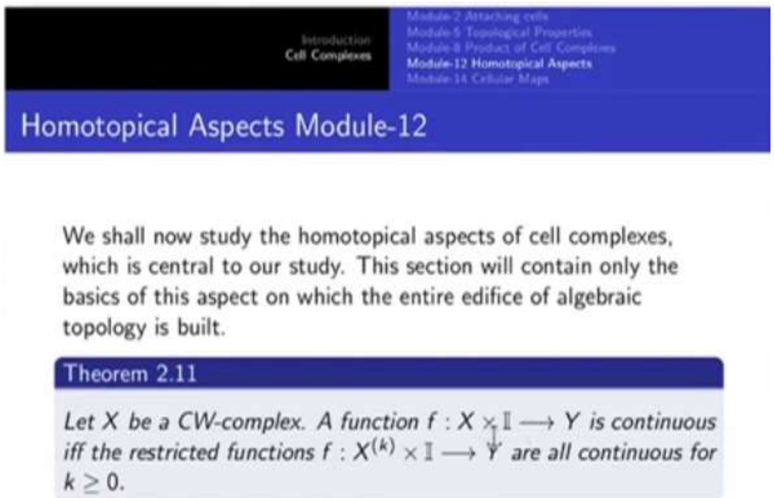

Continuing with the study of cell complexes, now we take up the topic `homotopical aspects'. One of the easy consequences of the study of product that we did is that once we have CW complex X, then $X \times [0,1]$ is automatically a CW complex because the interval $[0,1]$ itself is a CW complex which is compact. Therefore construction of functions on $X \times \mathbb{I}$ which is same thing as homotopies becomes easy. So this is the first theorem that we have here.

Take X be a CW complex. A function f from $X \times \mathbb{I}$ to Y is continuous if and only f restricted each $X^{(k)} \times \mathbb{I}$, all these restrictions are continuous, for each $k \geq 0$. What is this $X^{(k)}$? the k-skeleton of X. All that I am going to use here is that the coinduced topology with respect to this family of closed sets, $X^{(k)} \times \mathbb{I}$ is the same as the product topology on $X \times \mathbb{I}$. We know that co-induced topology $X \times \mathbb{I}$ from the family of compact subsets is equal to the product topology.

Ignore whatever is said in the slide and the lecture here for the proof of this lemma. It can be easily seen as follows.

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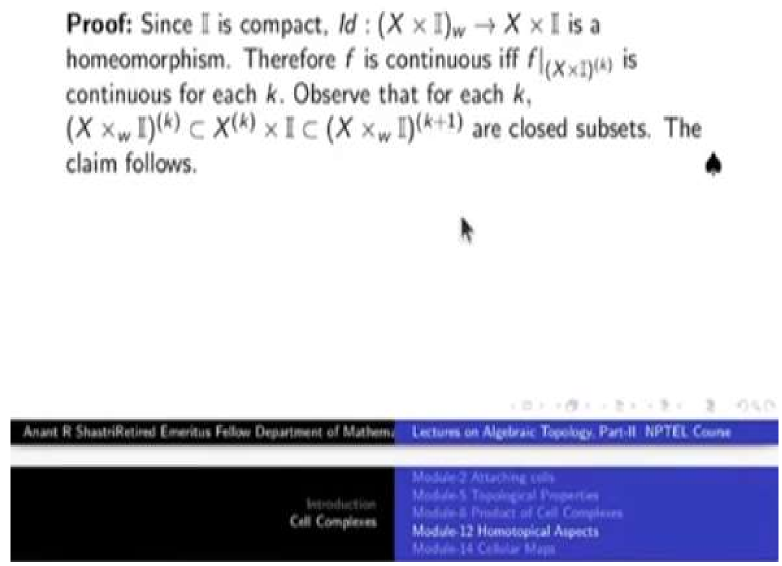

Look at the identity map from $(X \times \mathbb{I})_w$ to $X \times \mathbb{I}$. Since $\mathbb{I}$ is compact, I know that this is homomorphism, because X has compactly generated topology, and $\mathbb{I}$ is compact. Therefore f from $X \times \mathbb{I}$ to Y is continuous iff f restricted to every compact subset K of $X \times \mathbb{I}$ is continuous.

Now given any compact subset K of $X \times \mathbb{I}$, it is easily seen that there is a compact subset L of X such that K is a subset of $L \times \mathbb{I}$. We also know that every compact subset L of the CW complex is contained in some k-skeleton of X. Since f restricted to $X^{(k)} \times \mathbb{I}$ is continuous, it follows that f restricted to K is also continuous. So this theorem will be used now often.

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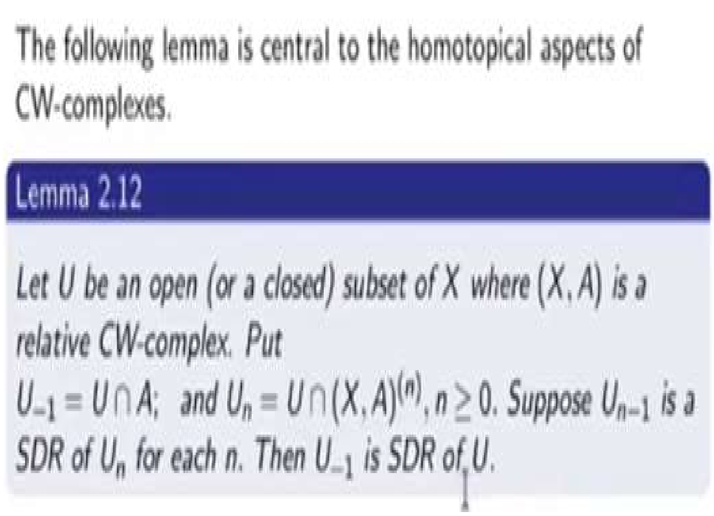

Now we come to the central in the homotopical aspects of CW complexes, for which we have been preparing from the beginning with proposition 2.1. Let U be an open subset (or a closed subset) of X, where (X, A) is a relative CW-complex. Starting with U_{-1} equal to $U \cap A$, put $U_n = U$ intersection the n-skeleton of (X, A) for all integers $n \geq 0$. Suppose that U_{n-1} is a strong deformation retract of U_n for each n. (Anyway, U_{n-1} is contained is U_n because the skeletons form an increasing sequence. U_{-1} is a SDR of U_0 is SDR of U_1 and so on.) Then U_{-1} will be a strong deformation track of the whole set U.

This is the statement. The easy part here is each U_{n-1} is a retract of U_n, then we know that composites of finitely many retractions will give you that each U_k is a retract of U_n for $n > k$. That is easy.

After that we can see that U_{-1} is a retract of U itself as follows: I can define the retraction from U to U_{-1} to be these composites depending upon where my point x in U is? Every x is inside some $X^{(n)}$, okay? So, the point x is in U_n. You define $r(x)$ as the n-fold composite $r_0 \circ \cdots \circ r_{n-1}(x)$.

So that is also easy part. The thing is how to get homotopies? We cannot compose homotopies so easily. Because the domain and codomains are not appropriate. So, let us go through this proof carefully.

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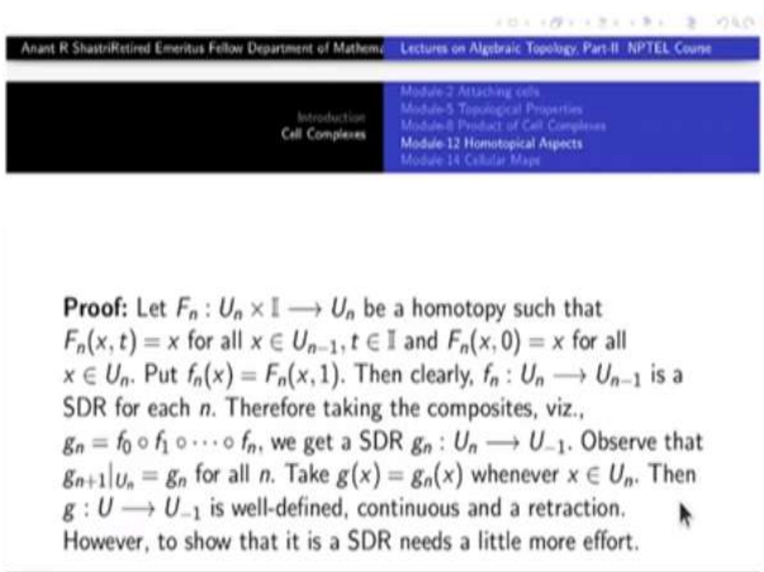

Let F_n from $U_n \times \mathbb{I}$ to U_n be a homotopy of the identity map which is relative to U_{n-1} given by the hypotheses that U_{n-1} is SDR of U_n. That is by definition. So these F_n are the given strong deformation retractions. So, F_n takes values inside $U_n, F_n(x,t) = x$ for all $x \in U_{n-1}$ and for all $t \in \mathbb{I}$. Moreover, $F_n(x,0) = x$ for all $x \in U_n$. Here the last map $f_n : U_n \to U_{n-1}$ given by $f_n(x) = F_n(x,1)$ is the retraction onto U_{n-1} of U_n. Therefore if I take $g_n = f_0 \circ f_1 \circ \cdots \circ f_n$, then we get a retraction from U_n to U_{-1}. We do not know at present whether it is a SDR, but we will see that soon.

Observe that the $(n+1)$th function g_{n+1} here restricted to U_n is just g_n, why? Because f_{n+1} restricted to U_n is the identity function, being a retraction of U_{n+1} onto U_n, and g_{n+1} is equal to $g_n \circ f_{n+1}$. So this is the important thing here.

Therefore if I define $g(x) = g_n(x)$ whenever x is in U_n, then this will be a well defined function on all of U. This map will be actually form U to U_{-1} and is a retraction because on U_{-1} all f_k are fixed. So I get a retraction. However to show that this g is strong deformation retraction, we have to work a little harder.

So that step is a new lesson here. How how to get compositions of deformation retractions or infinite `composition' of homotopies in general, okay? So watch it carefully. That lesson itself will be useful else also.

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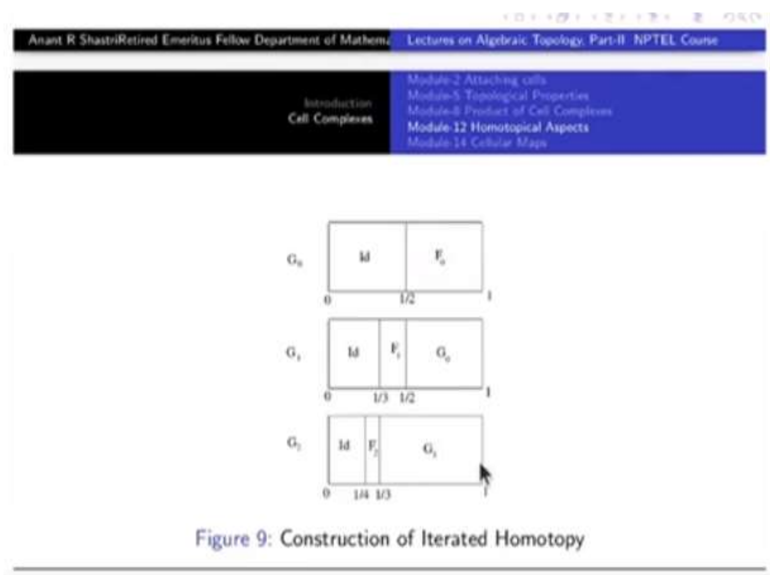

So, look at this picture. Every thing is happening in the closed interval $[0,1]$. I start with the homotopy F_0 and going to define a new homotopy G_0. In the first half of the interval, I do not change any thing, will keep G_0 the identity function $G(x,t) = x$. In the second half I take F_0, properly reparameterised; $G_0(x,t) = F_0(x, 2t)$. That is my G_0. Now it is time to define G_n in a successive way. So in the next step, I define G_1. This time, up the first one-third of the interval the function is Identity. Then from $1/3$ to $1/2$, I will put F_1 and in the rest of the interval I put G_0 . Remember all this F_i are homotopies of the identity map and so on these vertical dividing lines, they will be identities. So they will match up.

Next step I will take Identity on the first $1/4$ of the interval, then F_2 between $1/4$ and $1/3$ and put G_1 in the rest of the interval. Remember that each time you have reperametrise suitably. So in this way, at the n^{th} stage, in the interval from $[0, 1/(n+2)]$, the homotopy G_n will be identity, from $[1/(n+2), 1/(n+1)]$, it will be F_n, and in the remaining interval, we put the earlier homotopy G_{n-1} which we have constructed. I do this is systematically. Now I will have to write down exact formulae.

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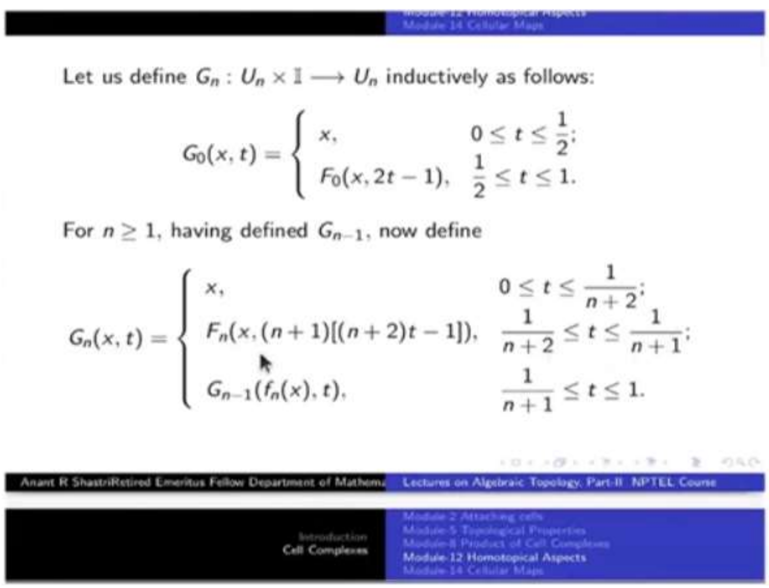

So define G_n, inductively, okay G_n from $U_n \times \mathbb{I}$ to U_n as follows. Induction starts at $n = 1$. First define G_0 as follows:

$G_0(x,t) = x$, for t in the half of the interval and then it is $F_0(x, 2t-1)$ for $1/2 \leq t \leq 1$. I have to see that when $t = 1/2$, there are two definitions here, and they must coincide. When $t = 1/2$, what is F_0? It is $F_0(x, 0) = x$, the identity function.

So inductively, for $n \geq 1$, having defined G_{n-1}, we define G_n as follows: $G_n(x,t) = x$, for $t \in [0, 1/(n+2)]$. (if $n = 1$, this is one third okay?) Next, for $t \in [1/(n+2), 1/(n+1)]$, you put $G_n = F_n$, appropriately reparameterised. You have to write it correctly, as $F_n(x, (n+1)[(n+2)t - 1])$.

For example when $t = 1/n + 2$, what will be the second slot here? $(n+2)$ cancels out and so it will be 0, and we know that $F_n(x, 0) = x$. When $t = 1/(n+1)$ this slot will be $(n+2)$ divided by $(n+1)$ which is equal to $1 + 1/(n+1)$ and so 1 cancel out first and then the $(n+1)$ factor cancels out to give you 1. And so we get F_n(x,1)=f_n(x). In the interval from $[1/(n+1), 1]$, we have to put G_{n-1}, reparameterised appropriately. We choose the second slot to be $[(n+1)t - 1]/n$ so that the interval $[1/n+1, 1]$ is mapped to $[0, 1]$. (In the slide there is an error.) But in the first slot we must put $f_n(x)$ instead of x, for the simple reason that it should match with $G_n(x, 1/(n+1))$ which is already defined. So, this is the difference which was not explained here in this diagram okay?

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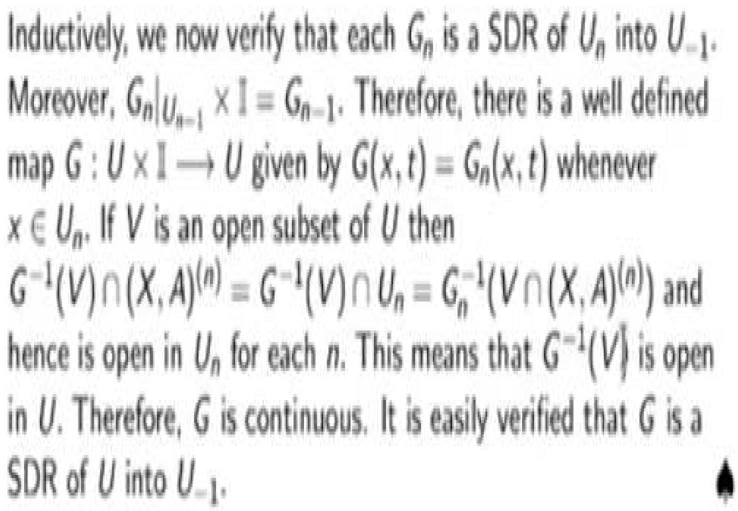

Inductively, we now verify that each G_n is a SDR of U_n into U_{-1}. Moreover, $G_n|_{U_{n-1}} \times \mathbb{I} = G_{n-1}$. Therefore, there is a well defined map $G : U \times \mathbb{I} \longrightarrow U$ given by $G(x,t) = G_n(x,t)$ whenever $x \in U_n$. If V is an open subset of U then $G^{-1}(V) \cap (X,A)^{(n)} = G^{-1}(V) \cap U_n = G_n^{-1}(V \cap (X,A)^{(n)})$ and hence is open in U_n for each n. This means that $G^{-1}(V)$ is open in U. Therefore, G is continuous. It is easily verified that G is a SDR of U into U_{-1}. ♠

So the definition of G_n is clear. Now, inductively, we can easily verify that each G_n is a strong deformation of retract of U_n into U_{-1}. Moreover G_n restricted $U_{n-1} \times \mathbb{I}$ is equal to G_{n-1}.

Therefore the map G from $U \times \mathbb{I}$ to U, given by $G(x,t) = G_n(x,t)$, whenever $x \in U_n$ is well defined.

I have to show that G is continuous. But this is more or less obvious from the property of the coinduced topology. For any open subset V of U, what is $G^{-1}(V)$ intersection with the n-skeleton? That is equal to $G^{-1}(V) \cap U_n$. But that is $G^{-1}(V) \cap U_n$ is what? It is the same as $G_n^{-1}(V)$. And G_n is a continuous function so this is open subset of U_n for each n. This means $G^{-1}(V)$ is open inside U. The rest of the verification that G is a strong deformation retract of U into U_{-1} is direct. Because, if $x \in U_{-1}, G(x,t) = x$ for all t, no matter where t lies. So that is why it is a strong deformation retraction of U onto U_{-1}.

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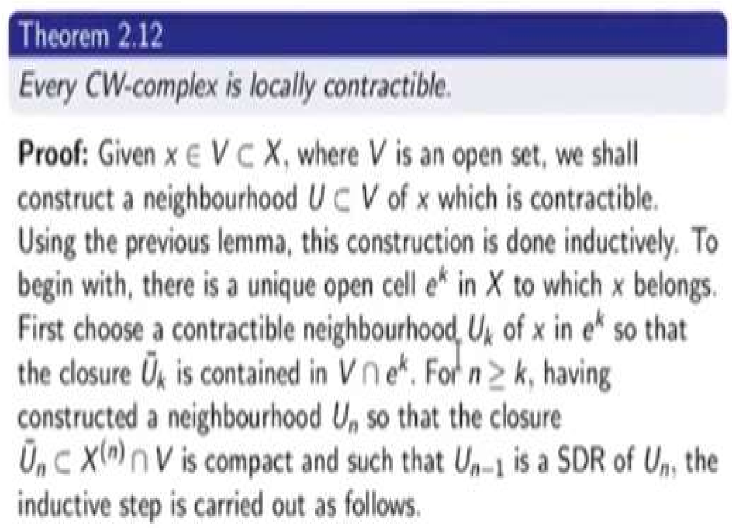

Now we can have a nice theorem like this one which we want to remember forever. You may not be able to remember immediately how this is constructed. You may just forget the actual formulae but you should remember this picture here. Then this formulae can be redone. Nobody remembers them. You can work it out by yourself so that things match up correctly because you want functions make sense. When you want to patch of two continuous functions defined on two closed sets, all you need to ensure that they agree on the intersection, that is all. Now what is the theorem? Theorem is that every CW complex is locally contractible.

Right in the beginning we told you that CW-complexes share lots of topological properties with the Euclidean spaces. Every open subset is locally contractible in a Euclidean space. That property is here with CW complexes also.

Later on when we define manifolds. They are also locally contractible, because they are actually locally Euclidean space. However, a CW complex may not be locally Euclidean, but still is locally contractible. A very important property.

How to prove this one? What is the meaning of locally contractible first of all? Given a point $x \in X$ and an open subset V containing that point, we have to have an open subset U which contains x and is contained in V and which is contractible. (There are also some slightly weaker versions of this definition but we are not satisfied with any weaker versions. We are going to prove this strong version.) So every open nbd of a point contains an open nbd of that point which is contractible. That is what we have to prove alright.

So using the previous lemma this homotopy lemma, we want to construct the contraction inductively, skeletonwise. So this construction is done inductively. To begin with there is a unique cell e^k in X so that x belong to the interior of e^k. (So this fact is used again and again, namely, a CW complex is the disjoint union of its open cells.) First choose a contractible neighborhood U_k of $x \in e^k$. Because interior e^k is homeomorphic to interior of $\mathbb{D}^k$, right? Interior $e^k \cap V$ will be open nbd of x, inside $\mathbb{D}^k$. Actually, we can and will choose a neighborhood U_k of x such that it is contractible and its closure is contained $V \cap e^k$.

Now for $n \geq k$ having constructed a neighborhood U_n so that the $\overline{U_n}$ is contained in the intersection V and n-skeleton of X, we want to construct U_{n+1} with the same property and with an additional property that $U_n \subset U_{n+1}$ is a strong deformation retract of U_{n+1}. So this inductive is carried out as follows.

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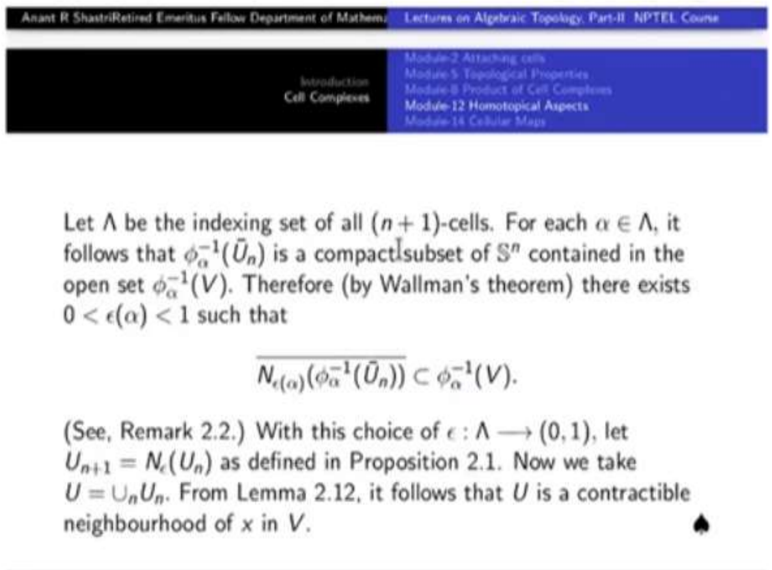

(Once this done, the conclusion will follow from the lemma. You see I actually, U_k now takes the place of U_{-1} in the lemma. I would like to do this inductive step now.)

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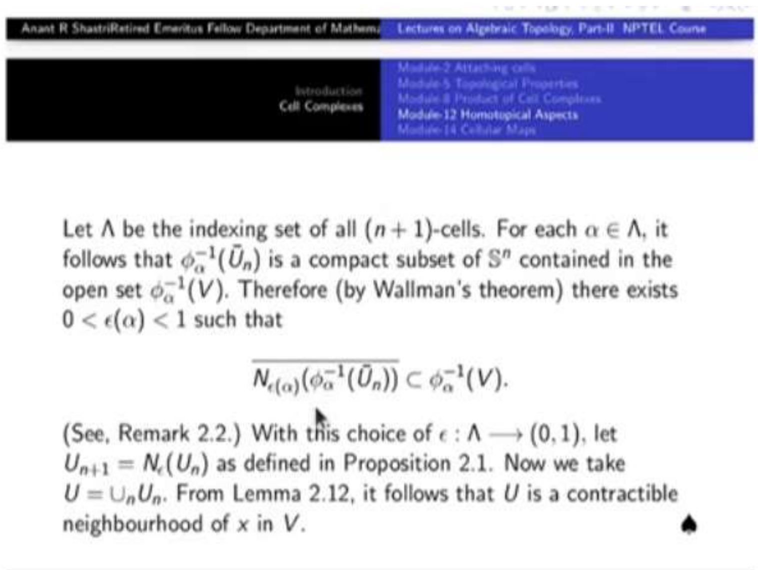

You look at all the $(n+1)$-cells in X, indexed by Λ. For each $\alpha \in \Lambda$, we have ϕ_α be the characteristic function of the $(n+1)$-cell. Inverse image of $\overline{U_n}$ under ϕ_α, $\phi_\alpha^{-1}(\overline{U_n})$ is a close subset of $\mathbb{S}^n$, so it is a compact subset of $\mathbb{S}^n$. (compactness of $\overline{U_n}$ is not necessary and it is wrong also as in the slides). It is contained in the open set $\phi_\alpha^{-1}(V)$ which is an open subset of e_α^{n+1}. Therefore, you can find an $\epsilon(\alpha)$ which depend upon α between 0 and 1 such that (remember the lemma before 2.1) the closure of $N_\epsilon(\phi_\alpha^{-1}(\overline{U_n})$ is containing $\phi_\alpha^{-1}(V)$. For each α, given we have an $\epsilon(\alpha)$, so I have function ϵ from Λ to the open interval $(0,1)$.

You define U_{n+1} to be $N_\epsilon(U_n)$ as in Proposition 1.1. This is the union of all these $N_{\epsilon(\alpha)}(U_n)$ along with U_n itself. That is definition as in proposition 1.1. Now we take U = union of all U_n's. From lemma 2.12 it follows that U is contractible neighborhood of x. So basically it is the lemma 2.12 which is employed here, this proposition 2.1.

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Lemma 2.13

Let X be obtained from Y by attaching k-cells, then $X \times 0 \cup Y \times \mathbb{I}$ is a strong deformation retract of $X \times \mathbb{I}$.

Proof: Let $\{f_\alpha : \alpha \in \Lambda\}$ be the family of attaching maps. One of the standard result that we have proved in Part-I is that there is a strong deformation retraction

$$h : \mathbb{D}^k \times \mathbb{I} \to \mathbb{D}^k \times 0 \cup \mathbb{S}^{k-1} \times \mathbb{I}$$

Let

$$H : \sqcup_\alpha D^k_\alpha \times \mathbb{I} \to \sqcup_\alpha (\mathbb{D}^k_\alpha \times 0 \cup \mathbb{S}^{k-1}_\alpha \times \mathbb{I})$$

be just the disjoint union of copies of h.

The next theorem is about co-fibrations and so on so let me just begin this one with a lemma and then we will stop and take a break. So X be obtained from Y by attaching k-cells. See it seems that all lemmas start with this one the inductive step for CW complexes, so that we can carry on with the inductive steps and inductive constructions.

So X is obtained from Y by attaching k-cells. Then look at $(X \times 0) \cup (Y \times \mathbb{I})$. This is a strong deformation retraction of $X \times \mathbb{I}$. Let me go through the proof of this one and then stop. First understand the case special case when we are attaching just one cell.

We have proved, in part 1 of this course that there is strong deformation retraction from $\mathbb{D}^k \times \mathbb{I}$ onto $(\mathbb{D}^k \times 0) \cup (\mathbb{S}^{k-1} \times \mathbb{I})$. So this space is like a tub it is like a cylinder with no top and but with the bottom closed. That cylinder is a strong deformation retract of the entire solid cylinder, the full cylinder. So this is what we have. This was one of the central results in part 1.

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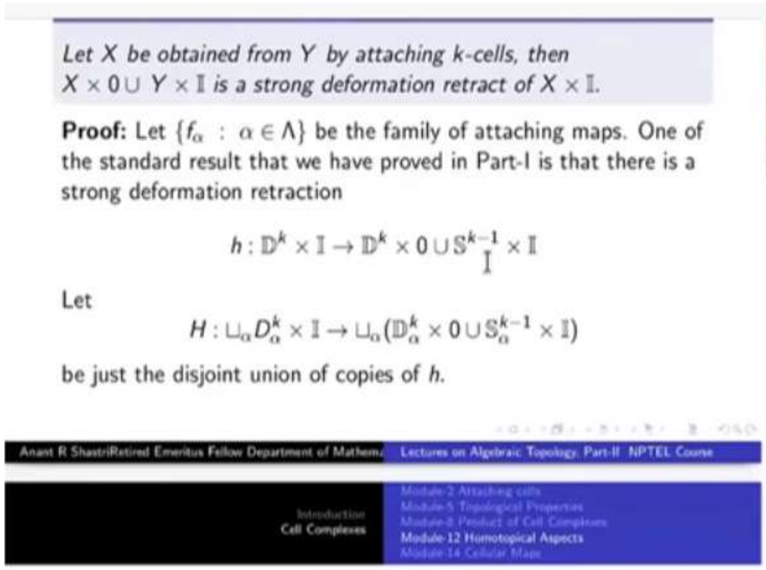

Let now $\{f_\alpha : \mathbb{S}^{k-1} to Y : \alpha \in \Lambda\}$ be a family of attaching maps indexed over Λ. Let H from the disjoint union of $\mathbb{D}^k_\alpha \times \mathbb{I}$ to disjoint union of these things on the RHS, that means I am taking copies of h for each α. It is just the disjoint union of copies of the same function. Let f from the disjoint union of $\mathbb{S}^{k-1}_\alpha \times \mathbb{I}$ to $Y \times I$ be defined by using the attaching maps on the first coordinate viz., $f_\alpha(x)$, and the second coordinate is just t. So, $\{f_\alpha\}$ are the attaching maps for all k-cells. Once you have these maps, what is the space $X \times \mathbb{I}$? It is the quotient of $Y \times \mathbb{I}$ disjoint union with all $\mathbb{D}^k_\alpha \times \mathbb{I}$, because X is the quotient of Y disjoint union all the $\mathbb{D}^k_\alpha$. So $X \times \mathbb{I}$ is a quotient of this one.

This needs a proof. By the way, in general, this is not true that if q from X to Y is a quotient map then $Id \times q$ from $Z \times X$ to $Z \times Y$ is a quotient map. So you must have some hypothesis on Z, namely locally compactness. So, here $Z = \mathbb{I}$ is a compact space, so that this is okay here.

We have, f from disjoint union $\mathbb{S}^{k-1}_\alpha \times \mathbb{I}$ to $Y \times \mathbb{I}$ is the map given by $(x, t) \mapsto (f_\alpha(x), t)$, where $x \in \mathbb{S}^{k-1}_\alpha$. It follows that $X \times \mathbb{I}$ can be though of as the quotient space of $Y \times \mathbb{I}$ disjoint union of all $\mathbb{D}^k_\alpha$'s by the single relation (x, t) equivalent to $f(x)$, for all x in the disjoint union of all $\mathbb{S}^{k-1}_\alpha$ and for all t.

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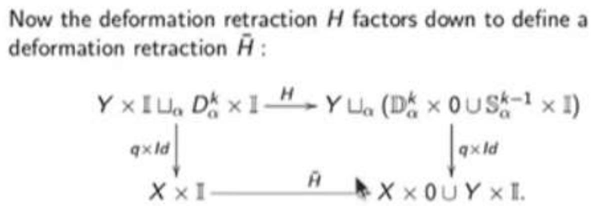

I want to define a map here below and I have a map already on the top stage. On each part α here, it is a strong deformation retraction, and on the part $Y \times I$, it is just the identity map, alright? So, if follows that H factors down through the quotient maps. That is all. Why? Because on the boundary times $\mathbb{I}$, as well as on $Y \times \mathbb{I}$, the entire homotopy is just the identity map. So you have got a map here at the bottom. To verify that this is a SDR, is the same as verifying it at the top.

So let us stop here and continue next time. We will come back to this important result and how we are going to use this one. Thank you.