Introduction to Algebraic Topology (Part-II)
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Module No # 03
Lecture No # 11
Partition of Unity – Continued

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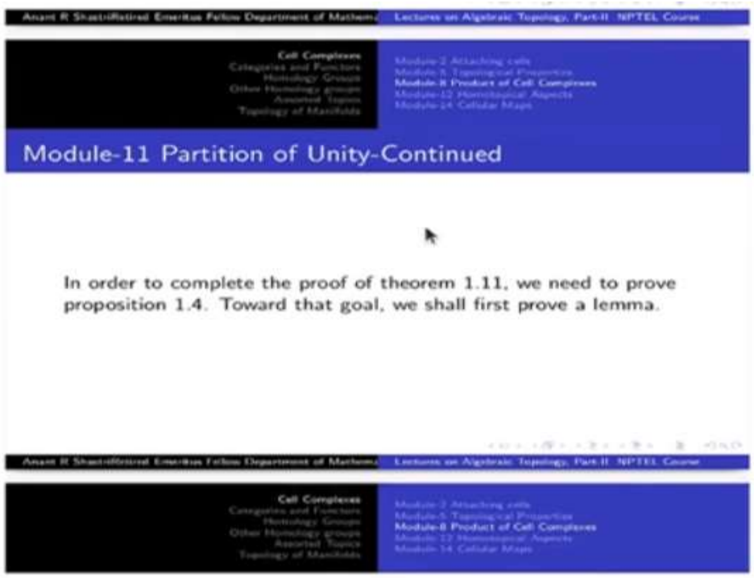

Last time we started the topic of partition of unity on a CW-complex. We proved the major theorem there, namely, the existence of partition of unity subordinate to a given open covering, assuming one of the propositions.

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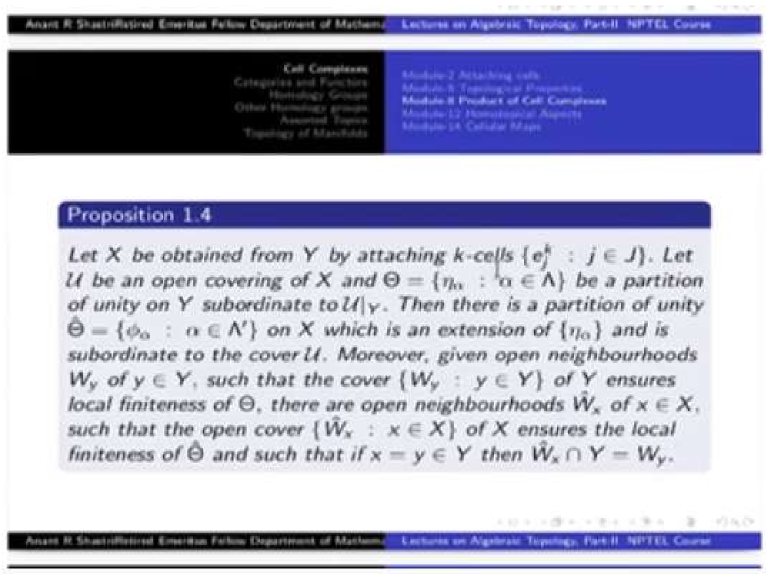

Let me recall the proposition first here. It is an elaborate proposition: Suppose X is obtained by attaching k-cells to Y. Then given a partition of unity on Y, you can extend it to a partition of unity on X. Moreover, the open covering of Y which ensures local finiteness on Y will also get extended to an open covering of X which ensures local finiteness on X, in such a way that the sets themselves get extended, i,e., intersecting with Y, you get the corresponding original open subsets. Before proving this proposition, we will need another step, namely, the case when we are attaching a single n-cell at a time. Notice here that X is obtained by attaching n-cells to Y, may be infinitely many cells. So first we do it for just one k-cell at a time.

A special case is that we start with $Y = \mathbb{S}^{n-1}$, and attach one cell with the attaching map as the inclusion map. So, we start with an open cover for $\mathbb{D}^n$ and a partition of unity subordinate to the restricted cover on $\mathbb{S}^{n-1}$, the boundary. So you must be able to conclude the same thing whatever you have stated in the proposition here, for this particular case. So that is our first lemma that we have to prove.

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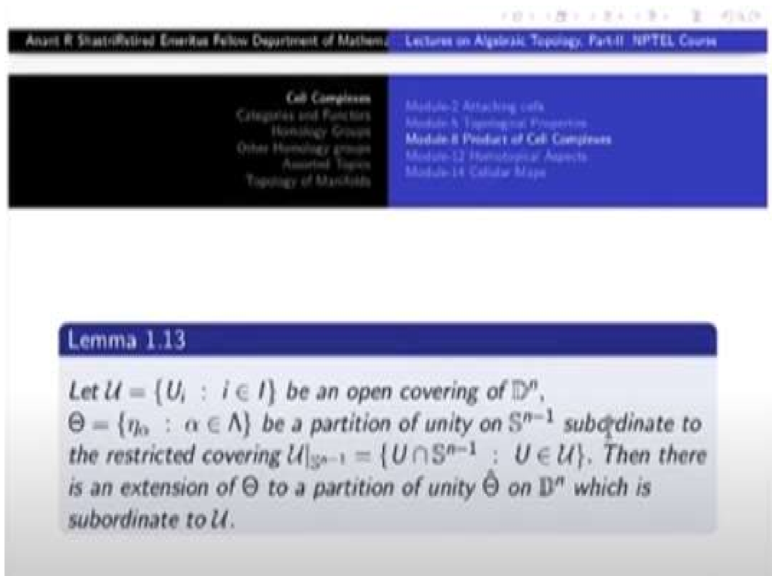

So let us go to that lemma now. You have an open covering $\mathcal{U}$ for $\mathbb{D}^n$, and $\Theta = \{\eta_\alpha, \alpha \in \Lambda\}$, a partition of unity on the boundary $\mathbb{S}^{n-1}$, subordinate to the restricted covering $\mathcal{U}$ restricted to $\mathbb{S}^{n-1}$. Then there is an extension of Θ to a partition of unity $\widehat{\Theta}$ on $\mathbb{D}^n$ which is subordinate to $\mathcal{U}$. The proposition is the same thing with many cells at a time instead of one at a time. The difference is that the subspace is not $\mathbb{S}^{n-1}$ but it is happening inside some arbitrary space Y. That will not cause any problem as seen later.

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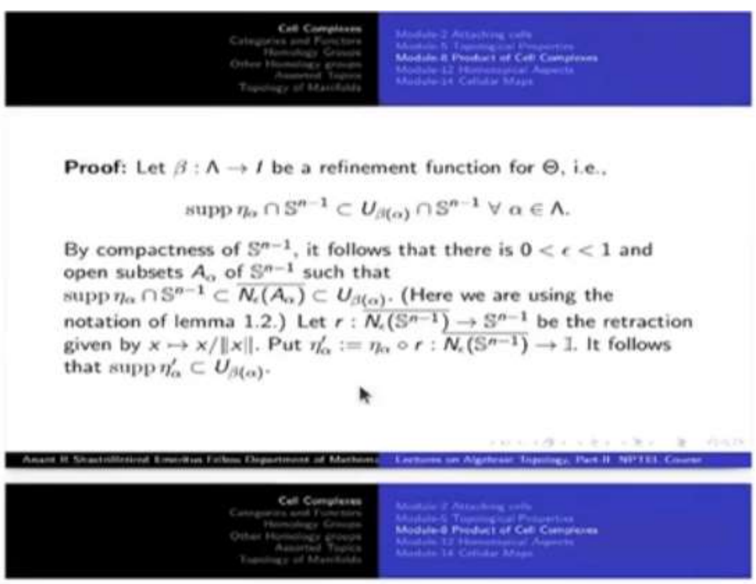

Let us first prove this lemma. This is the major thing that we have to do. So let β from Λ to $\mathbb{I}$ be a refinement function for the family Θ. What does that mean? Support of η_α which is a subset of $\mathbb{S}^{n-1}$, is contained in $U_{\beta(\alpha)}$ intersected with $\mathbb{S}^{n-1}$, for all α. By the compactness of $\mathbb{S}^{n-1}$ and because the entire family Θ is locally finite, we may and do assume that Λ itself is finite. So, it follows that you can find a uniform ϵ between 0 and 1, and for each α, an open subset A_α in $\mathbb{S}^{n-1}$ such that support $\eta_\alpha \cap \mathbb{S}^{n-1}$ is contained inside these ϵ neighbourhoods. Remember what is the definition of N_ϵ of A_α?

The closure of this is contained in $U_{\beta(\alpha)}$. $U_{\beta(\alpha)}$ are given open subsets of $\mathbb{D}^n$, from the open cover $\mathcal{U}$. You take $U_{\beta(\alpha)} \cap \mathbb{S}^{n-1}$, as your A_α; that will be an open subset of $\mathbb{S}^{n-1}$, which will contain the support of η_α. If needed, you can take even smaller open subsets, but there is no need, though I have said `there is exists some A_α'.

So the support of η_α is contained in A_α which is obviously contained in the ϵ nbd. That in turn is contained in its closure. To ensure that this closure is contained in $U_{\beta(\alpha)}$, you may have to choose ϵ sufficiently small. Here we are using the statement and notation of lemma 1.2, which you may remember.

These nbds are like collars of length ϵ, consisting of radial line segments coming out from points of A_α. As $a \in A_\alpha$ varies, the union is the nbd $N_\epsilon(A_\alpha)$. Now let r from $\overline{N_\epsilon(\mathbb{S}^{n-1})}$ to $\mathbb{S}^{n-1}$ be a retraction, given by x mapsto $x/\|x\|$. So, r will push back the vectors to the boundary. So that is a retraction.

Put η'_α equal to η_α composed with r. Note that η_α is defined on $\mathbb{S}^{n-1}$. So now the composed with retraction, it will be defined on $\overline{N_\epsilon(\mathbb{S}^{n-1})}$. So η'_α is defined to be just $\eta_\alpha \circ r$. So I have extended all η_α to a small neighborhood of $\mathbb{S}^{n-1}$. It follows that support of η'_α is contained in $U_{\beta(\alpha)}$.

So η'_α are extended carefully. Now by Tietze's extension theorem, there exists a continuous extension η''_α second of η'_α, denoted by double dashes, defined on the whole of $\mathbb{D}^n$ taking values in the closed interval $[0, 1]$. How do these extensions look like. I do not know, except that Tietze's extension theorem assure the existence. This is not quite good. I want to control the values of these extensions. So I will do a modification now.

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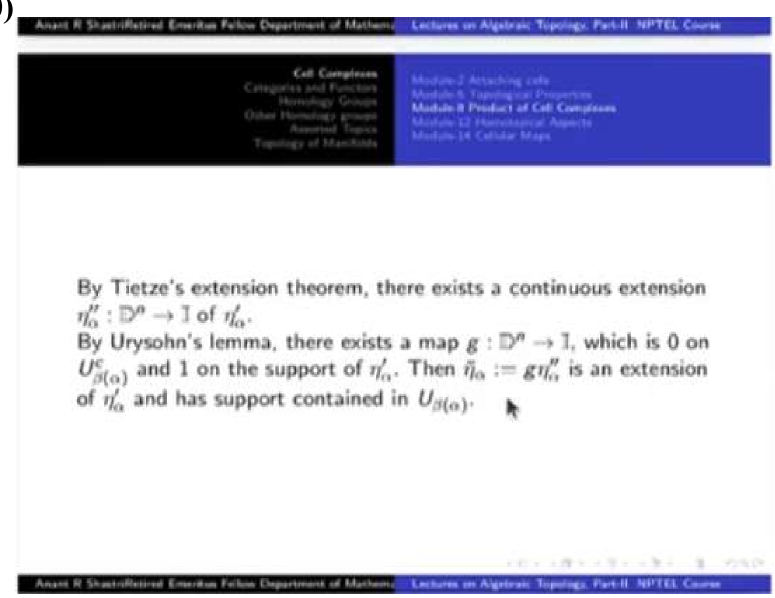

By Urysohn's lemma, there exists a map g from $\mathbb{D}^n$ to $[0,1]$, which is 0 on the complement of $U_{\beta(\alpha)}$, ($U_{\beta(\alpha)}$ is an open subset of $\mathbb{D}^n$, so take the complement inside $\mathbb{D}^n$), there it is 0 and it is 1 on the support of η'_α. Remember that support of η'_α is contained in $U_{\beta(\alpha)}$ and so these two sets are disjoint closed subsets.

Now put η'_α equal to g into $\eta'_\alpha{}'$. This is the modification of the extension $\eta'_\alpha{}'$, using a function g, which you may call a cut off function, occurring as a multiplicative factor.

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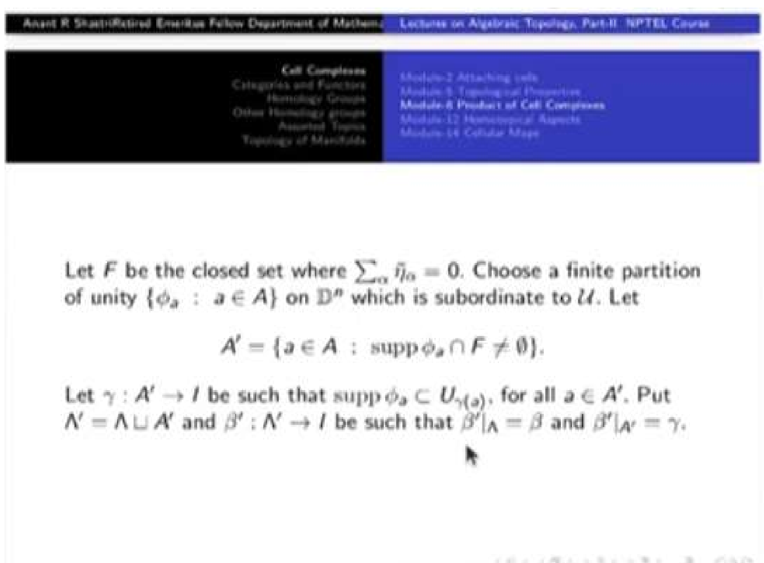

Next, let us look the set F of all those points in $\mathbb{D}^n$, where the sum of all these $\bar{\eta_\alpha}$ is 0. Remember that this is a finite sum here, even if the indexing set is infinite, the number of α's for which η_α is not zero will be finite. Also this summation is definitely equal to 1 on the boundary $\mathbb{S}^{n-1}$; it is the given partition of unity there. Inside it may be zero at some point. If not, the set F will be empty I have no problem in that case. So, we may assume F is non empty. Clearly, F is a closed subset of $\mathbb{D}^n$.

Now we choose a partition of unity $\{\phi_a, a \in A\}$ on $\mathbb{D}^n$ which is subordinate to this open covering $\mathcal{U}$. (Notice that $\{eta\{\bar{alpha}\}\}$ may not form a partition of unity on the whole of $\mathbb{D}^n$.) Put A' equal to the set of all those $a \in A$ such that support of $\phi_a \cap F$ is non-empty. Those are the ones which are important for me, among all these $\phi_a.A$ itself is a finite set here but we do not need to use this fact. So look at all those functions phi which are non-zero on F, i.e., that means that its support should intersect F, I need to keep those functions.

Let γ from A' to I be a set-function such that support of ϕ_a is contained in $U_{\gamma(a)}$. $\gamma(a)$ is a point of the indexing set I. Because I have started with a partition which is subordinate to $\mathcal{U}$, and so such a function exists. Put Λ' equal to Λ union this extra set A'. And let β' equal to β on Λ and γ on $A'.\beta'$ extends this way.

Then this β' is going to be a refinement function for the new family of functions consisting of $\bar{\eta_\alpha}$ and $\phi_a, a \in$ A'. Only thing now left out is to make them into a partition of unity. Their supports cover the whole of $\mathbb{D}^n$ that much I have ensured.

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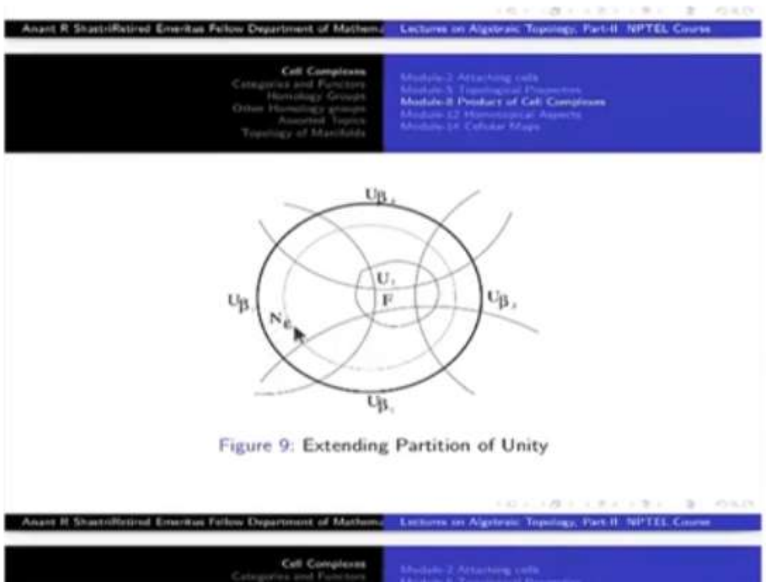

So this is what is shown in this picture. This is my F. There are some U_i's here which cover F. I have put the whole of F inside U_1 does not matter. So F is covered by some this $U_{\gamma(a)}, a \in A'$. There are other open sets which cover the rest. You can study this picture. This N_ϵ are some collar nbds of certain subsets of the boundary here.

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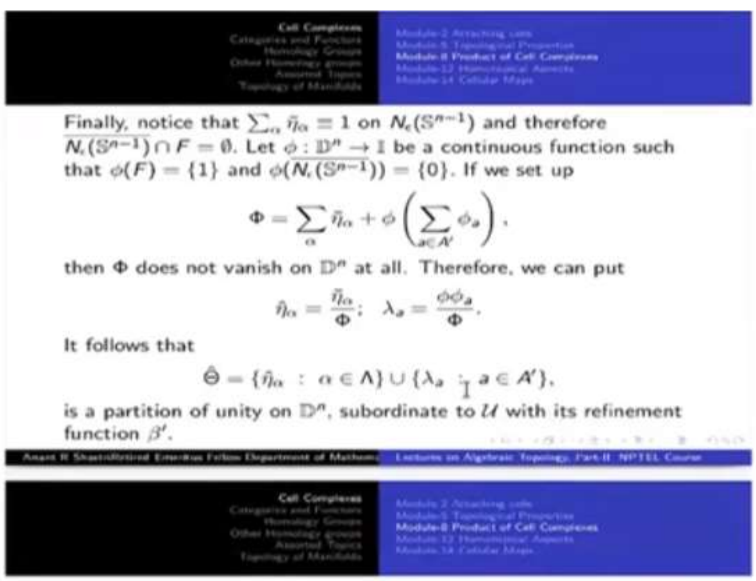

Finally notice that on the collar of this boundary, the sum total is equal to 1, and on F, the sum total is 0. So these two are disjoint closed subsets. So now I will make one more modification. Using Urysohn's lemma, lemma I get a continuous function ϕ from $\mathbb{D}^n$ to the closed interval $[0,1]$, such that $\phi(F)$ is 1 and $\phi(\overline{N_\epsilon(\mathbb{S}^{n-1})})$ is 0. Now the extra functions ϕ_a are all multiplied by this function ϕ, so that they do not enter N_ϵ nbd; this function ϕ is killing them. That is want I meant.

All $\phi_a, a \in A'$ are multiplied by ϕ. Also on F, ϕ is identically 1. So their values on F does not change. So that is all I have here. Therefore what happens is, away from F, one of the $\bar{\eta_\alpha}$ will be non zero. On F, ϕ is identically 1 and the sum of $\phi_a, a \in A'$ is also 1. Therefore this capital Φ which is the sum of all $\eta(\alpha)$'s and ϕ times sum of all ϕ_a's will never vanish on $\mathbb{D}^n$. Clearly it is also continuous. Therefore, we can put $\widehat{\eta_\alpha}$ equal to $\bar{\eta_\alpha}$ divided by Φ, and $\lambda_a = \phi\phi_a/\Phi$.

That is the final modification I am making. I started first with the family $\{\eta_\alpha\}$, then took $\eta'_\alpha, \eta'_\alpha{}'$, and $\bar{\eta_\alpha}$ and finally $\widehat{\eta_\alpha}$ now. Anyway, if you take the sum of all these, the sum in the numerator will be equal to the denominator. Hence this family $\hat{\Theta}$ equal to the family $\{\widehat{\eta_\alpha}\}$ together with $\{\lambda_a, a \in A'\}$ is a partition of unity on $\mathbb{D}^n$. It is subordinate to $\mathcal{U}$, with the refinement function being β' on the indexing set Λ' equal to $\Lambda \cup A'$.

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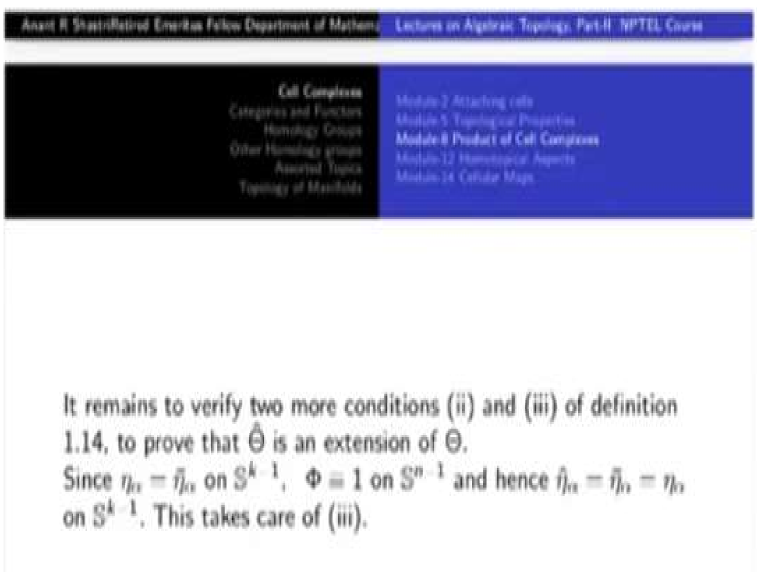

So conditions (i) and (iv) are verified. Condition (ii) and (iii) are yet to be ensured, so that $\hat{\Theta}$ actually becomes an extension of Θ. So start with η_α which is equal to $\bar{\eta_\alpha}$ on $\mathbb{S}^{n-1}$. Therefore the sum of $\bar{\eta_\alpha}$ will be equal to one where as the other summands involve ϕ which is 0 on $\mathbb{S}^{n-1}$. Therefore, this capital Φ is equal to 1. Hence $\widehat{\eta_\alpha}$ which is $\bar{\eta_\alpha}$ is equal to η_α on $\mathbb{S}^{n-1}$. This takes care of (iii).

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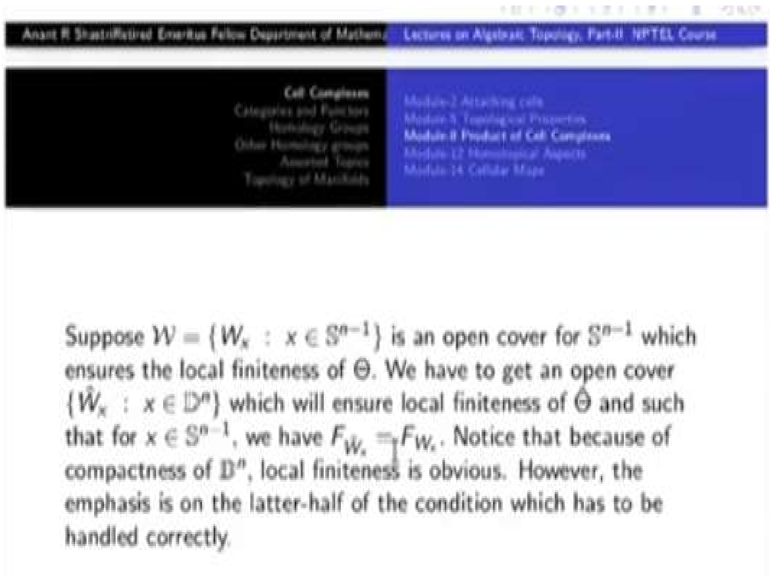

Finally suppose $\mathcal{W} = \{W_x : x \in \mathbb{S}^{n-1}\}$ is an open cover for $\mathbb{S}^{n-1}$ which ensures local finiteness of Θ. We have to get an open cover $\{W_x : x \in \mathbb{D}^n\}$ for $\mathbb{D}^n$, which ensures local finiteness of $\hat{\Theta}$, such that for$x \in \mathbb{S}^{n-1}$, $F_{\hat{W}_x}$ is same thing as F_{W_x}, where F is set of all indexes α wherein the corresponding functions are non zero.

Notice that it is not enough to prove local finiteness of $\hat{\Theta}$. We have to see that the open covering ensuring the local finiteness get extended. That is important, which has to be handled correctly. So that has to be verified.

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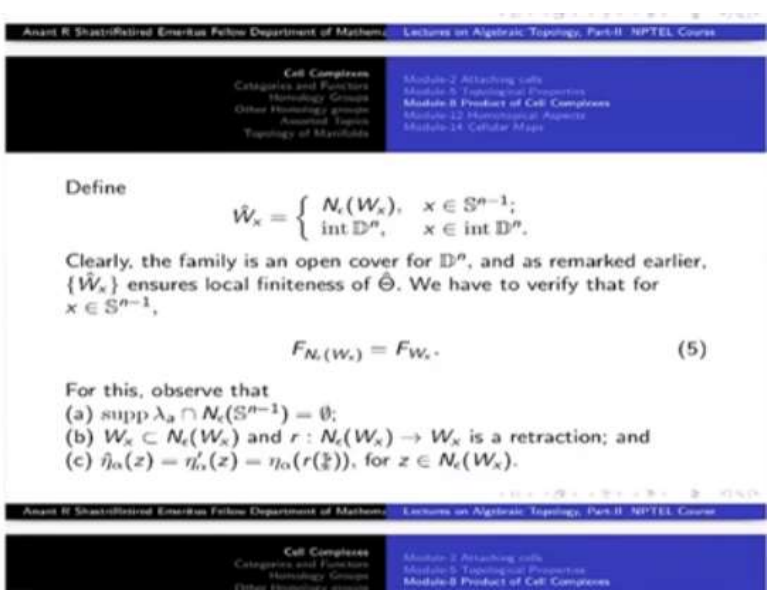

So for that, we have to do some work, namely, define $\hat{W}_x$ to be equal to $N_\epsilon(W_x)$. Remember what N_ϵ means. They are all subsets of $N_\epsilon(\mathbb{S}^{n-1})$. So put $\hat{W}_x$ equal to $N_\epsilon(W_x)$ if $x \in \mathbb{S}^{n-1}$, and equal to interior of $\mathbb{D}^n$ if x is in interior of $\mathbb{D}^n$. If $x \in \mathbb{S}^{n-1}$, then it follows that $\hat{W}_x \cap \mathbb{S}^{n-1}$ is equal to W_x. So that part is fine. Clearly this family is an open cover for $\mathbb{D}^n$ which ensures local finiteness of $\hat{\Theta}$.

We have to verify that for $x \in \mathbb{S}^{n-1}$, $F_{N_\epsilon(W_x)}$ is equal to F_{W_x}. That there are no extra indexes coming here, this is what we have to show here. For this, we observe three things:

(a) if you take $a \in A'$, support of $\lambda_a \cap N_\epsilon(\mathbb{S}^{n-1})$ is empty.
(b) $N_\epsilon(W_x)$ contains W_x, and r is a retraction. And
(c) $\hat{\eta_\alpha}(z)$ is $\eta'_\alpha(z)$ which is equal to $\eta_\alpha(r(z))$ for all $z \in N_\epsilon(\mathbb{S}^{n-1})$.

For, every point z in this neighborhood, you are pushing it to the boundary and then taking η_α. From (b) and (c), it follows that both sides of this equation, are contained inside Λ. What is Λ'? It consist of Λ and A', the members of A' do not come here. This is what I have said. Support of $\lambda_a \cap N_\epsilon(\mathbb{S}^{n-1})$ is empty, that is part (a). So these two sets on either side are inside Λ.

But why they are equal? Equal because of this (b). F_{W_x} is what? Set of all indexes $\alpha \in \Lambda$ such that η_α is not 0, right? It is contained in $F_{N_\epsilon(W_x)}$ because W_x is contained in $N_\epsilon(W_x)$.

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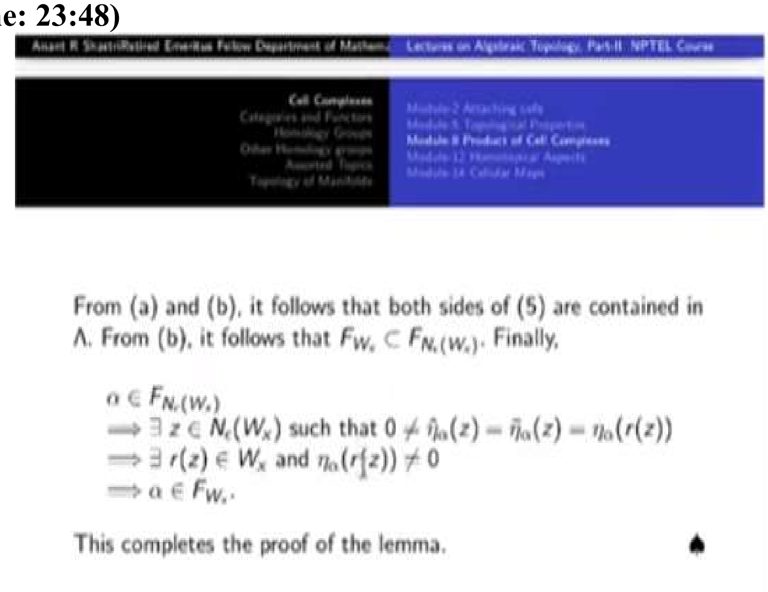

Finally, take alpha belonging to $F_{N_\epsilon(W_x)}$. What is the meaning of this? There exists $z \in N_\epsilon(W_x)$ such that 0 is not equal to $\hat{\eta_\alpha}(z)$. But $\hat{\eta_\alpha}(z)$ is equal to $\bar{\eta_\alpha}(z)$, z which is $\eta_\alpha(r(z))$. But then I have got a point $z' = r(z) \in W_x$ such that $\eta_\alpha(z') \neq 0$.

That means this alpha is inside F_{W_x}. So every point of F of W_x is there which is not actually we need to have that one. So this was the hardest part of the entire story of partition of unity.

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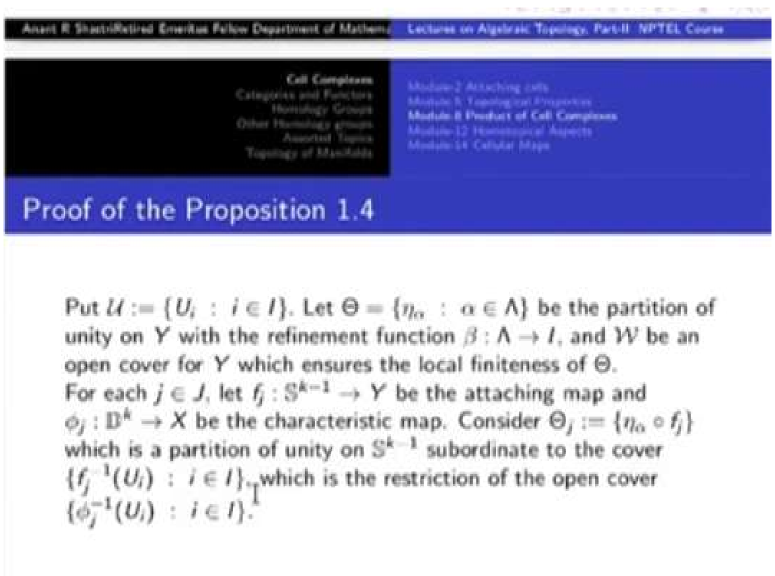

Now we can complete the proof of the proposition 1.4 very easily. This is needed this is very important. If you assumed that the CW-complex is locally finite, then this part of them lemma was not neeeded. It is needed in the proof of the proposition, without the assumption of local finiteness, on attaching maps.

To start with, we have an open cover $\mathcal{U}$, indexed by the set I for the space X. Let $\Theta = \{\eta_\alpha : \alpha \in \Lambda\}$ be a partition of unity on Y, with β from Λ to I be a refinement function, and $\mathcal{W}$ be an open covering which ensures the local finiteness of Θ on Y. For $j \in J$, let f_j from $\mathbb{S}^{k-1}$ to Y be the attaching map of a k-cell in obtaining X from Y. Let ϕ_j from $\mathbb{D}^k$ to X be characteristic map of the j^{th}-cell.

For each fixed $j \in J$, consider the family $\Theta_j = \{\eta_\alpha \circ f_j : \alpha \in \Lambda\}$. So, I am pulling back the partition of unity defined on Y onto $\mathbb{S}^{k-1}$, I do not want the whole of Y at a time. I want to concentrate on $\mathbb{S}^{k-1}$. I have the map f_j, a continuous function from $\mathbb{S}^{k-1}$ to Y. So $\{\eta_\alpha \circ f_j\}$ is a partition of unity on $\mathbb{S}^{k-1}$, which is subordinate to the open cover $f_j^{-1}(\mathcal{W}) = \{f_j^{-1}(U_i) : i \in I\}$ which is the restriction to $\mathbb{S}^{k-1}$ of the open cover $\{\phi_j^{-1}(U_i) : i \in I\}$. Each $\phi_j^{-1}(U_i)$ is an open subset of $\mathbb{D}^k$.

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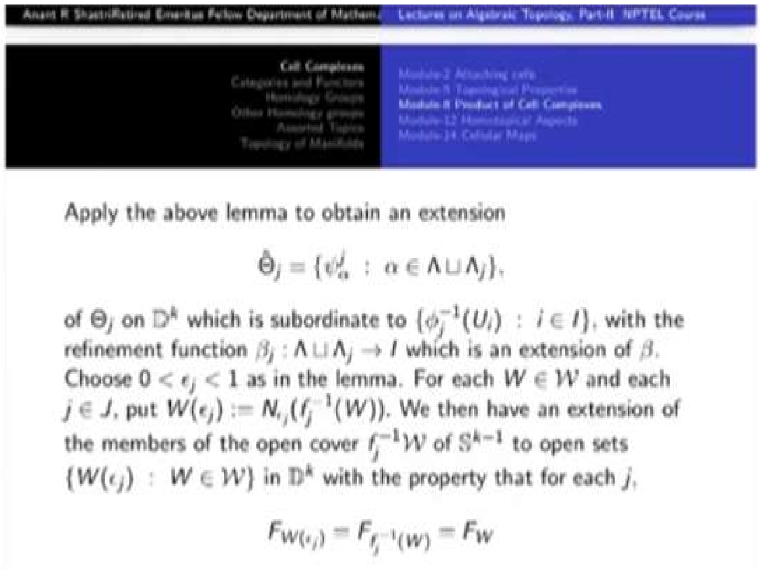

Now I apply the previous lemma to this situation of one single $\mathbb{D}^k$., one single cell. By applying above lemma, you get an extension $\hat{\Theta}_j$. Let us denote these functions by ψ^j_α, where α belongs to $\Lambda \cup \Lambda_j$. There in the lemma, I had $\Lambda \cup \Lambda'$, but now the Λ' depends on j and so we change the notation. That is all.

The extension $\hat{\Theta}_j$ will be subordinate to this open cover $\{\phi_j^{-1}(U_i) : i \in I\}$, with the refinement function β_j from $\Lambda \cup \Lambda_j$ to I, which is an extension of β. The lemma guarantees all this to me. Also, for each j, I have an ϵ_j between 0 and 1. For each $W \in \mathcal{W}$, and for each $j \in J$, consider, $N_{\epsilon_j}(f_j^{-1}(W))$ in $\mathbb{D}^k$.

If you drop out this j here, all this is part of the lemma only. Let us denote this set by a simpler notation $W(\epsilon_j)$. Then the family $\{W(\epsilon_j); W \in \mathcal{W}\}$ is an extension of the open cover $f_j^{-1}(\mathcal{W})$. Also $F_{W(\epsilon_j)}$ is the same as $F_{f^{-1}(W)}$, which is precisely equal to F_W, (because I have taken the partition functions to be $\eta_\alpha \circ f_j$; whatever happens inside Y, the same thing is happening for the pull-back).

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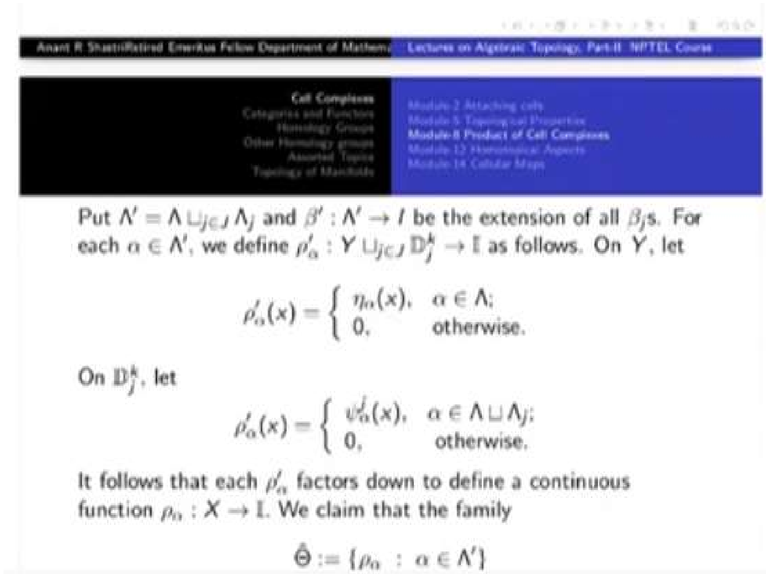

Put $\Lambda' = \Lambda \cup_{j\in J} \Lambda_j$. Let β' be the extension of these all β_j's which on Λ is equal to β and on Λ_j, it is β_j. For each $\alpha \in \Lambda'$, we define ρ'_α from the disjoint union of Y with all the $\mathbb{D}^k_j$'s (one copy taken for each $j \in J$,) to the interval $[0,1]$ as follows:

On Y, if α is in Λ, then it must be equal to η_α; no changed there and equal to 0 otherwise. That is what we want any way; you do not want new indexes to enter Y. so on Y. On each $\mathbb{D}^k_j$, if α is in Λ or in Λ_j, then take it to be ψ^i_α. Otherwise take it to be zero.

So since the union is a disjoint union, the functions ρ'_α are well defined. But when you come to X, by attaching the cells via the f_j's along the boundary, what happens to these functions? They will agree with the corresponding original functions on Y here. So for each α, all these functions together define a family of continuous functions from X to I. For each α, there will be a ρ_α, (I am dropping that prime) ρ_α from X to I. If q is the quotient map, every point in X is $q(x)$ where x belongs to Y or a unique $\mathbb{D}^k_j$. Then $\rho_\alpha(q(x))$ is nothing but $\rho'_\alpha(x)$.

We claim that this family is the partition of unity which extends the partition of unity on Y. Clearly, it is an extension, for $q(x)$ is x itself. So, let us verify these properties one by one.

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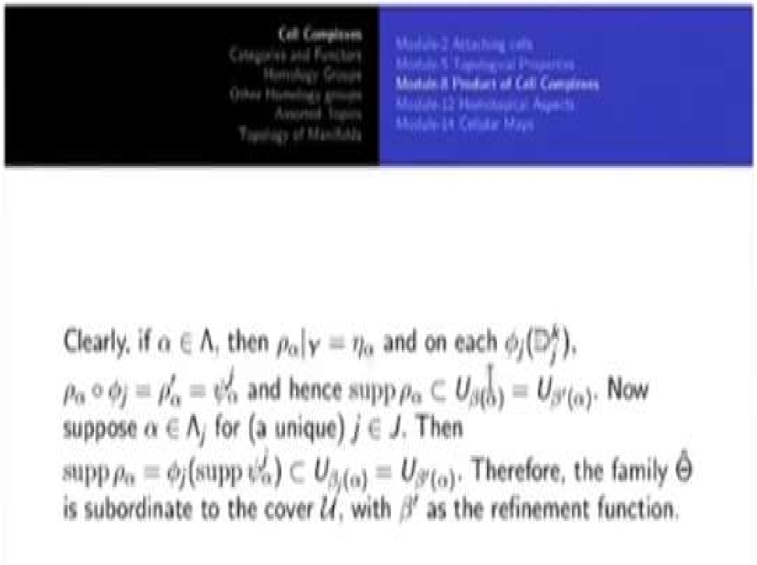

If α is in Λ, then ρ_α restricted to Y is η_α on each $\phi_j(\mathbb{D}^k)$. So $\rho_\alpha \circ \phi_j$ is equal to ρ'_α which is equal to ψ^j_α. Therefore, support of this one is contained to $U_{\beta_j(\alpha)} = U_{\beta'(\alpha)}$. If α is in some Λ_j(such an index j is unique right?), then support of ρ_α is ϕ_j of support of ψ^j_α and hence again is contained in $U_{\beta_j(\alpha)} = U_{\beta'(\alpha)}$, by definition of β'. Therefore the family $\hat{\Theta}$ is subordinate to the open cover $\mathcal{U}$, with β' as the refinement.

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Now, let us take a look at covering ensuring local finiteness. For each $W \in \mathcal{W}$, put $W(\epsilon)$ equal to q of the disjoint union of $W(\epsilon_j)$'s, where q is the quotient map onto X. Automatically this $W(\epsilon)$ is an open subset in X, because its inverse image under q is precisely the disjoint union which when intersected with each $\mathbb{D}^k_j$ is open and also intersected with Y, it is open. Also for each $j \in J$, let W_j denote q of the interior of $\mathbb{D}^k_j$. Then the collection $\{W_j, j \in J\}$ together with the collection $\{W(\epsilon) : W \in \mathcal{W}\}$ forms an open cover for X. We claim that this cover will ensure local finiteness of $\hat{\Theta}$.

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For each $j \in J$, note that F_{W_j} is the set of all $\alpha \in \Lambda'$ such that support of $\rho_\alpha \cap W_j$ is non empty. As soon as you write j here this will be happening in $\Lambda \cup \Lambda_j$ and hence this set is contained in $\Lambda \cup \Lambda_j$. Now by the compactness of $\mathbb{D}^k$ and local finiteness of Θ_j, it follows that only finite members of Θ_j's are non-zero on $\mathbb{D}^k_j$. Therefore it follows that this F_{W_j} is finite.

Similarly, for $W \in \mathcal{W}$, first observe that $F_{W(\epsilon)}$ is the set of all $\alpha \in \Lambda'$ such that the support of $\rho_\alpha \cap W(\epsilon)$ is non-empty. This set is contained in Λ. There many be infinitely many $W_{\epsilon(j)}$'s involved. But the whole point is none of them will contribute any extra element. This is the important point. It is subset of Λ itself. In particular, it follows that this set is equal to F_W.

So, we come to this summation. This step is similar to the proof in the final theorem. Given $x \in X$, suppose this $x \in Y$. Then $\rho(x)$ is 0, if α is not in Λ. So only the summands for which α is inside Λ will contribute. But then this is a partition of unity on Y, and therefore sum is equal to 1. Otherwise, x is in inside the interior one of the k-cells, say $\mathbb{D}^k_j$. Such a j is unique. Also, then $x = \phi_j(x')$ for some x' inside $\mathbb{D}^k$. In that case, $\rho_\alpha(x)$ is 0 for all α not in $\Lambda_{j'}$, where j' is not equal to j. Only when α is in Λ_j, or Λ you will get something. So, this total is being taken over $\Lambda_j \cup \Lambda$.

But then it is equal to the sum of $\psi_\alpha^j(x')$, the sum of the extended partition of unity, and therefore, is equal to 1.

So this completes the proof that $\hat{\Theta}$ is an extension of Θ. So this part of the proof is more or less a repetition of the proof of the final theorem that we have proved. I have told you already that the existence of partition of unity along with Hausdorffnesss of the space X, actually ensures that the space X is paracompact. If you do not know what paracompactness is, you do not have to worry about that right now. Essentially in practice. Paracomactness is to ensure the existence of partition of unity. That is the key result required in analysis everywhere and in algebraic topology or defferential geometry etc.

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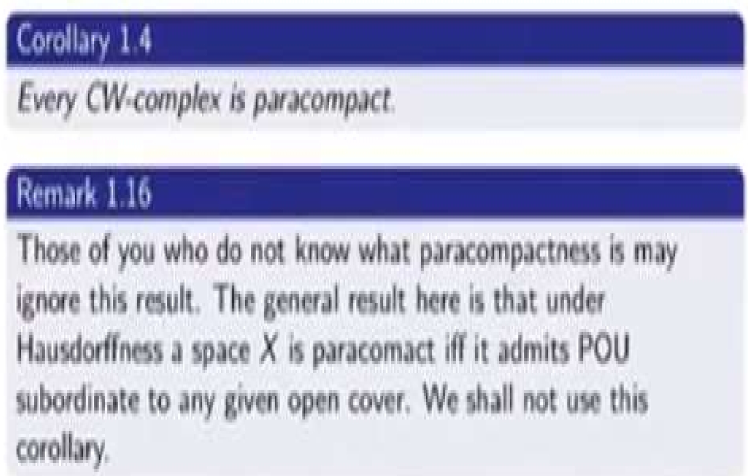

So that was the theorem that I wanted to tell you now as a corollary. Every CW complex is paracompact.

I think this where we stop today thank you.