Introduction to Algebraic Topology (Part-II)
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Lecture – 10
Partition of Unity on CW – Complexes

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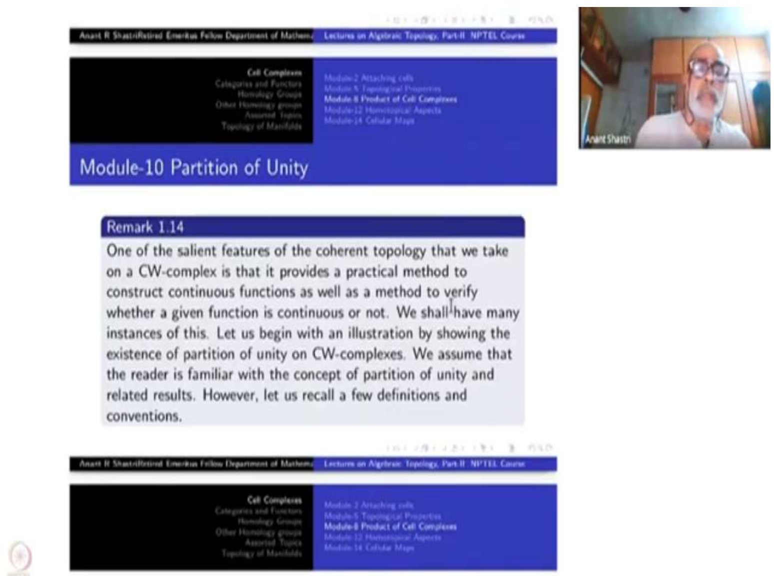

So, today we shall pick up one of the most salient features of the coherent topology which provides us a practical way of constructing continuous functions. As a consequence, we shall prove the existence of partition of unity on CW-complexes. This essentially proves that they are paracompact, because if you have a Hausdorff space, then paracompactness is equivalent to existence of partition of unity. The central theme of constructing the functions by step by step, namely, skeleton by skeleton will occur again and again in other contexts also.

The only thing that we assume is that you are familiar with partition of unity to some extent. Especially, if the space is compact and Hausdorff, then you must be knowing how to construct partition of unity. Nevertheless, whatever is required here, I will recall them. We will recall all the definitions needed.

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You start with a topological space X. By a partition of unity on X, we mean a family of continuous functions (that continuity hypothesis is important to here though it is not in the name), some family indexed by a set Λ, these functions θ_α, are all defined on the whole of X and taking values in the closed interval $[0, 1]$; they will satisfy the following two conditions:

(1) for each $x \in X$, you can find a neighborhood of $x \in X$ such that that neighborhood will intersect support of θ_α for only finitely many alphas.
This is the same thing as saying that only finitely θ_α will be non-zero on that neighbourhood of x. This condition is called local finiteness of the family $\{\theta_\alpha\}$. It makes the second condition below meaningful. The first condition makes the second condition meaningful.

(2) The second condition says that the sum of all $\theta_\alpha(x)$ for any given point in $x \in X$ is equal to 1. The sum makes sense because of condition 1, though the family may be infinite, the sum evaluated at any given point, there are only finitely many non zero terms.

Okay? That is the definition of partition of unity.

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But now it has to have some relations with other things. Namely, starting with an open covering of X, a family $\{\theta_\alpha\}$, is said to be subordinate to the covering, if you have a function on the indexing sets, Λ is the indexing set of $\{\theta_\alpha\}$ and I is the indexing set for the covering, okay, so a function β should be from Λ to $\mathbb{I}$ such such that the support of θ_α, where α is inside Λ, which is by definition, the closure of the set of all the points x wherein $\theta_\alpha(x)$ is not equal to 0, that is a closed subspace X, that must be contained inside $U_{\beta(\alpha)}$. So that function β is called a refinement function. So, each support is contained inside some member of this cover, the given cover.

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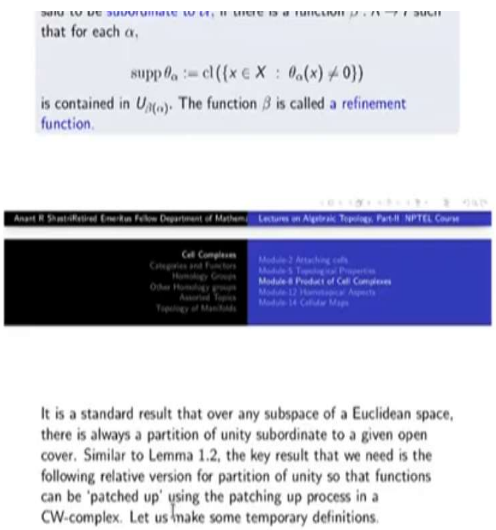

It is a standard result that on any subspace of a Euclidean space, there is always a partition of unity subordinate to any given cover. You see, it's much easier to prove this than proving such a thing on an arbitrary paracompact Hausdorff space, whatever they are. Similar to our earlier Lemma 1.2 of extending neighborhoods and functions, the key result that we need is to use the following relative version of partition of unity.

This will be needed so that functions can be patched up using the patching-up process in a CW-complex. So, what we need is a relative version. This means that if you have some partition of unity defined on subspace, there must be some partition of unity on the whole space extending the functions in the old family. That is what I'm going to define here. These are temporary definitions; you may not find them in literature elsewhere.

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So, start with a family of real valued functions on $X, \Theta = \{\eta_\alpha : \alpha \in \Lambda\}$. An open cover $\mathcal{W}$ of X is said to ensure local finiteness of Θ, (so, we want to bring that local finiteness condition to play a more active role here, by making condition (1) more explicit), so it is supposed to ensure local finiteness of Θ, if for each $W \in \mathcal{W}$, the set F_W of all $\alpha \in \Lambda$ such that W intersection support of α is not equal to emptyset is finite. See, local finiteness ensures some such open cover exists.

So, if $\mathcal{W}$ is such an open cover, then you say that $\mathcal{W}$ ensures the finiteness of Θ. So that is one of the definitions, making condition 1 more elaborate. We can keep talking about such a cover which will ensure the local finiteness of a family.

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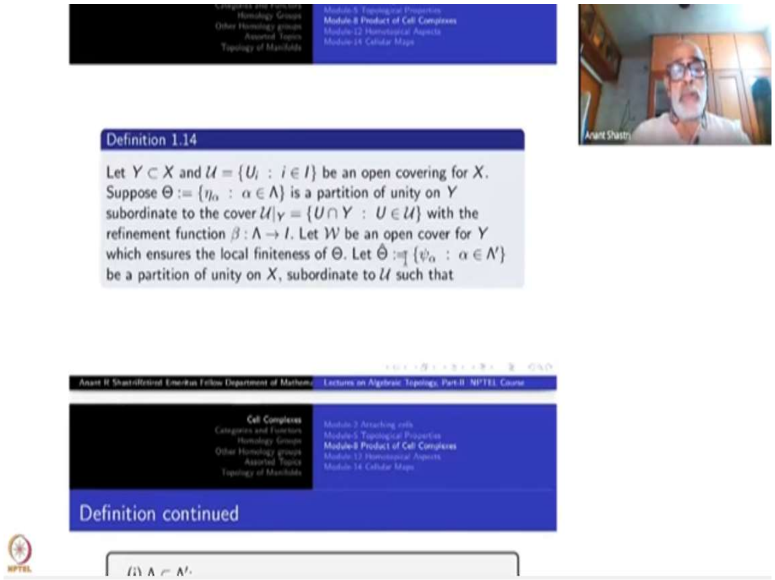

Second definition: Take Y, a subspace of a space X. Now, you have an open cover $\mathcal{U}$ for X and a partition of unity $\Theta = \{\eta_\alpha\}$ on Y, which subordinate to the restricted cover. Restricted cover is defined by taking any number of U of $\mathcal{U}$ and intersecting it with Y, so, collect all of them, that will be obviously an open cover for Y always. The partition of unity on Y should be subordinate to this restricted cover, with the corresponding refinement function β.

Let $\mathcal{W}$ be an open cover for Y of which ensures local finiteness of Θ. So, all these things play an important role in that definition of extension. Now, let us have $\widehat{\Theta}$, okay? $\widehat{\Theta}$ is a collection of ψ_α, where α's are in another indexing set Λ', okay? Suppose this is a partition of unity subordinate to $\mathcal{U}$, such that various conditions are satisfied. (This is going to be a long definition).

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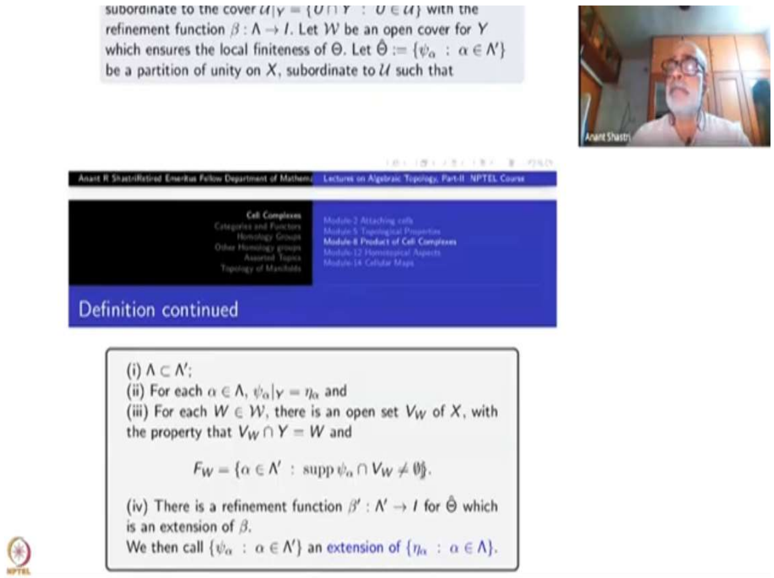

First of all these indexing set here have to be appropriate, the new index set Λ' must be larger than the original indexing set Λ, okay? That is, Λ is contained in Λ'. Equality is also allowed, there is no problem. Next for each α in Λ, the new member ψ_α restricted to Y must be equal to η_α . So, that is part of the meaning of the word `extension' in the definition, okay?

Next, I have to extend the open sets in $\mathcal{W}$ also. For each W inside $\mathcal{W}$, there is an open subset V_W of X, an extension of W which means $V_W \cap Y$ is equal to W. So each W is extended to an open subset of X now. And what is the property of this extension? The set F_W must be equal to the collection of all α belonging to Λ' such that support of $\psi_\alpha \cap V_W$ is non-empty okay.

Note that F_W is already is contained in the right hand side. But I want F_W to be equal to this right hand side, okay? You may adding extra members in new family here, the indexing set Λ' may be larger okay? But support of those extra functions should not enter the open subsets extending members if $\mathcal{W}$ at all, they should be away from them. This is a very strong condition, okay? You have to understand this. So, listen carefully okay?

The fourth condition: there is a refinement function beta prime from Λ' to $\mathbb{I}$, for the family $\widehat{\Theta}$, and this β' must be an extension of β. That means, for an old member, α of Λ, $\beta'(\alpha)$ must be $\beta(\alpha)$ itself.

So, with all these conditions we call the family $\widehat{\Theta}$ is called an extension of Θ. The indexing set is extended, refinement function is extended, okay? And the third condition is that the new functions are somewhat away from the old functions that is something important here. We then call $\{\psi_\alpha\}$ an extension of $\{\eta_\alpha\}$.

This is a technical definition, so that, instead of repeating all these 4 conditions, I can just say `it's an extension'.

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That is, is short, an extended partition portion of unity consists of extensions of all old functions to the larger space together with some extra functions to cover up the rest of the larger space. It is important that we make the technical assumption that the support of these extra functions do not enter into a neighborhood of the smaller space. This is ensured by extending the cover which ensures the local finiteness. Finally, we should also need to extend the refinement function. Such extra caution is needed when we want to build up the partition of unity in the inductive process. All these 4 points here are important to keep in mind.

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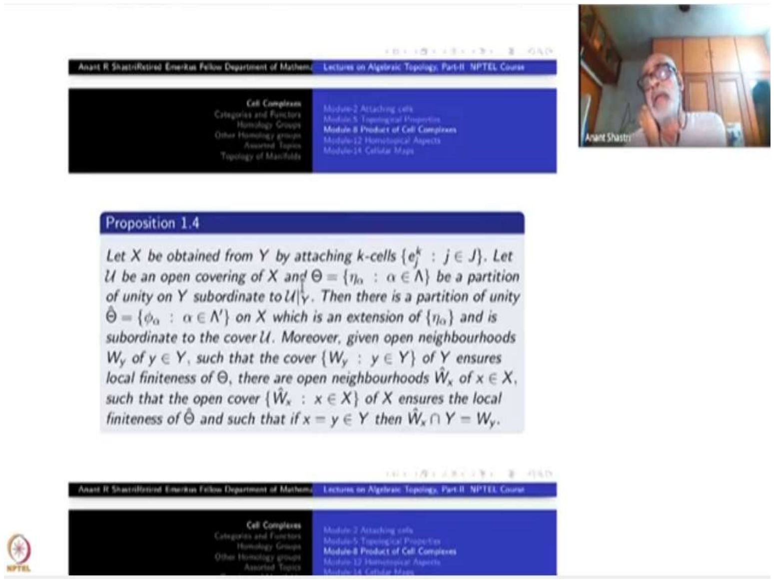

So, here is an elaborate proposition which will ensure the step by step extensions. Let X be obtained from Y by attaching k-cells, k is fixed, okay? So, you can mention an index in set for these k-cells say e_j, $j \in J$. Let $\mathcal{U}$ be an open covering for X and Θ be a partition of unity indexed by Λ on Y and subordinate to this restricted cover $\mathcal{U}_Y$. Then there is a partition of unity $\widehat{\Theta}$ which indexed by Λ' on X, which is an extension of Θ and is subordinate to the cover $\mathcal{U}$. Okay?

The statement of the proposition could be over here. But we want to control what is happening to an open covering which ensures local finiteness of the partitions. So we add: Moreover, given open neighborhoods, W_y, y belonging to Y, such that the open cover $\{W_y, y \in Y\}$ ensures local finiteness of Θ, there are open neighborhoods $\widehat{W_x}$ of $x \in X$, such that the open cover $\{\hat{W}_x, x \in X\}$ ensures the local finiteness for $\widehat{\Theta}$, and these $\hat{W}_x$ are nothing but whenever this $x = y$ is inside Y, then $\hat{W}_x$ intersection with Y is equal to W_y.

So, while extending functions, we want to keep track of this point. Now why you need such a thing? That may be more curious to you than the proof of this proposition, which I will post-pone and let me go to the proof of the partition of unity using this proposition. Then you will know the role of these extra demands we have made. Later on we can figure out how to prove this proposition, okay?

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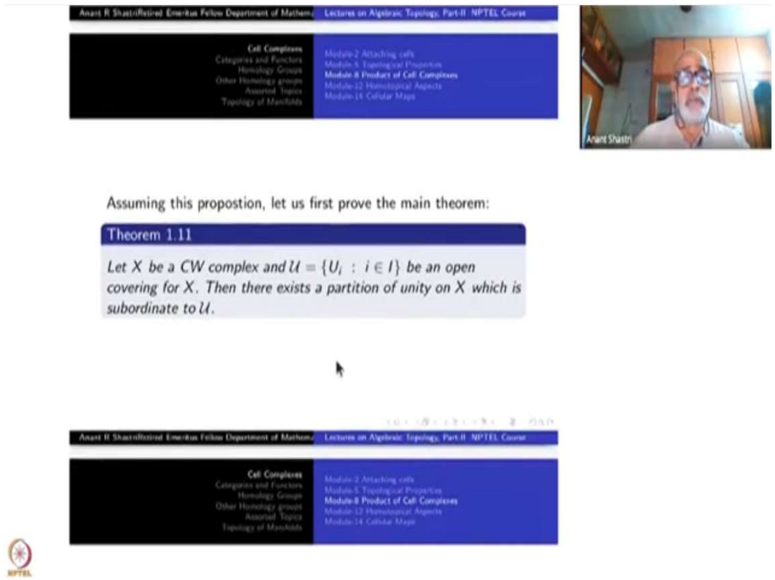

So, assuming this proposition, let us first prove the main theorem, okay? So, what is the main theorem? Start with any open covering of a CW-complex X. There is a partition of unity on X which is subordinate to this cover. Theorems are always neatly stated as briefly as possible, okay? With all the hypotheses included, that is actually the standard style of the statement of a final theorem, so that it is quotable without all the paraphernalia of the notations that you might have introduced in the preamble to the theorem. So, taking open covering for a CW complex there is always a partition of unity subordinate to that.

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p_0Now, how do we begin the proof? We start with the 0-skeleton. The 0-skeleton is a discrete set okay? That discrete set itself will be used as the indexing set for the partition of unity unity now, okay? So, Λ^0 is going to be the indexing set which we take equal to $X^{(0)}$, namely, the set of all the vertices of X. Define these functions, now doubly indexed, the first 0 indicating, the 0-level, the second index x corresponds to the set $X^{(0)}$, the function $p_{0,x}$ is defined on $X^{(0)}$ to $\mathbb{I}$ by the formula $p_{0,x}(y)$ is equal to 1 if $y = x$ and 0 otherwise. These are indicator functions, they are delta-functions; if $x = y$ it is 1 otherwise it is 0. So, at each point you take the function identically 1 at that that point and 0 elsewhere. That completes the first stage construction.

Obviously, when you take the sum of all these what happens? At any given point you get only one function takes value 1 and all other functions take the value 0. Therefore the family is automatically a partition of unity on $X^{(0)}$. And since $X^{(0)}$ is discrete, each singleton is an open subset here. It is closed also. So, support of $p_{0,x}$ is equal to $\{x\}$. Each point x is inside some number of $\mathcal{U}$, because $\mathcal{U}$ is a covering for the whole of X. So, this family $\{p_{0,x}\}$ is subordinate to $\mathcal{U}$. We shall fix up a subordinated function here. Since $X^{(0)}$ is discrete, it is clear that $\mathcal{W}^0$ equal to $\{\{x\} : x \in X\}$ is an open covering ensuring the local finiteness.

So, let β^0 from Λ^0 to $\mathbb{I}$ be a refinement function, that is, I have to choose, for each point x belonging to $X^{(0)}$, some member $\beta(x) \in \mathbb{I}$ such that x is in $U_{\beta(x)}$. You can choose any such function, you have some freedom here.

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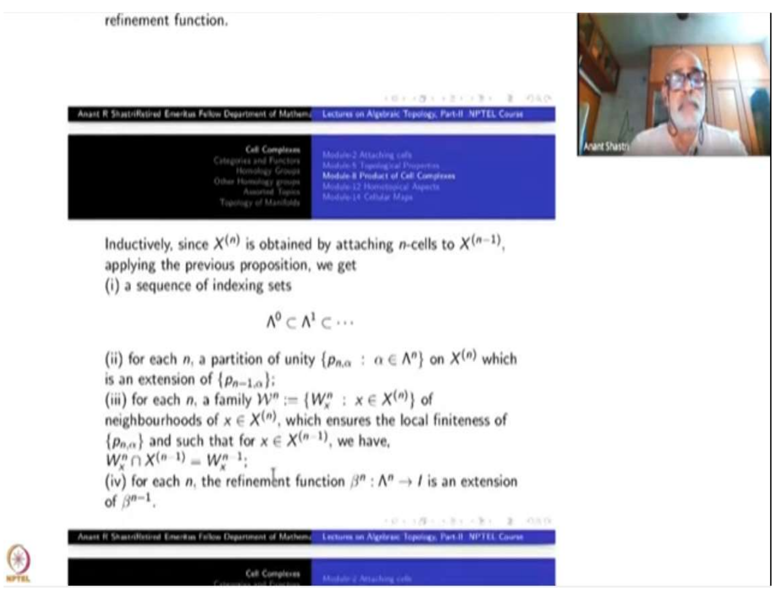

Immediately, we can take up the induction step and the previous proposition comes into picture. For each n, $X^{(n+1)}$ is obtained by attaching $(n+1)$ cells to $X^{(n)}$, right? Applying the previous proposition, we get:

(i) a sequence of indexing sets Λ^0 subset of Λ^1 etc.,

(ii) partition of unity $\{p_{n,\alpha}\}$ on $X^{(n)}$ which is an extension of $\{p_{n-1,\alpha}\}$; doubly indexed, where α ranges over Λ^n. Whenever you take an α inside Λ^{n-1}, then $\{p_{n,\alpha}\}$ restricted to $X^{(n-1)}$ is equal $\{p_{n-1,\alpha}\}$;

(iii) A family $\mathcal{W}^n$ of open sets which ensures local finiteness of $\{p_{n,\alpha}\}$. We shall members of $\mathcal{W}^n$ by W_x^n, x ranging over $X^{(n)}$, each W_x^n being a neighborhoods x in $X^{(n)}$.

They are such that whenever $x \in X^{(n-1)}$, $W_x^n \cap X^{(n-1)}$ equals W_x^{n-1}. So, this was the part of the proposition. So, we have all these things here. The fourth condition is that for each n, the refinement function is also an extension. So, I'm just repeating the previous proposition here. Okay.

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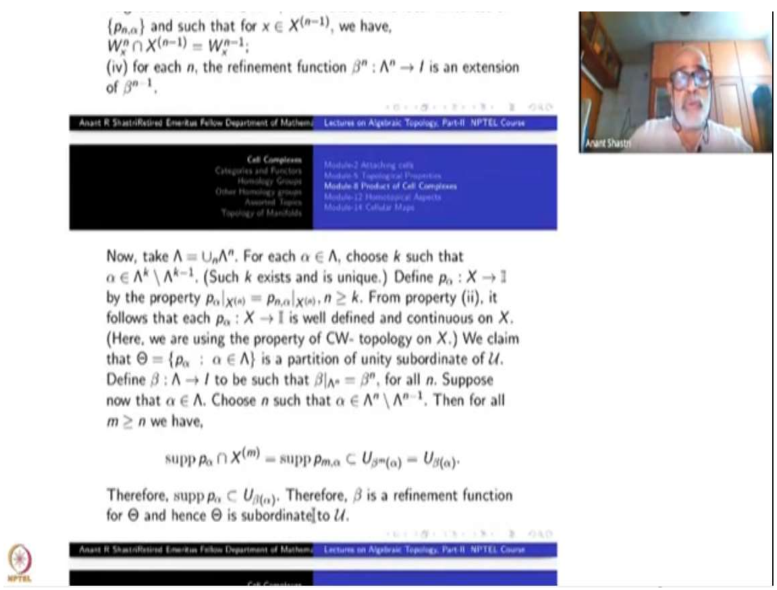

Now, what do we do? Put Λ equal to the union of all Λ^n.

Given $\alpha \in \Lambda$, I can choose say k such that α first time appears in Λ^k. That means it is not in Λ^{k-1} or any of the previous ones, first time it appears in Λ^k, okay? Now define p_α from X to the unit interval $\mathbb{I}$, by the property that p_α restricted to any n-skeleton is equal to $p_{n,\alpha}$ on $X^{(n)}$, for all $n \geq k$.

We know that these restriction functions are continuous first of all, and secondly, that they agree when you restrict them further to lower skeletons. Okay? that they are extensions, right? So, once alpha is inside Λ^k, that is the first time the function $p_{k,\alpha}$ appears and after that it gets extended at each step. So this makes sense.

From property (ii) it follows p_α is well defined and continuous on X. The continuity follows from the fundamental topological property of a CW-complex. Okay? We claim that this family $\Theta = \{p_\alpha\}$ is a partition of unity on the whole of X, subordinate to the cover $\mathcal{U}$. That will complete the proof, okay? I have defined a function β here, which is a refinement function β from Λ to $\mathbb{I}$ which extends all the β^n's, i.e., restricted to Λ^n it is β^n. This makes sense because β^n restricted to Λ^{n-1} is equal to β^{n-1}.

Suppose now, α is in Λ. Choose n such that α is in Λ^n for the first time. So, α is in Λ^n but not in Λ^{n-1}. Then for all $m > n$, what we have? Support of $p_\alpha \cap X^{(m)}$ is equal to support of $p_{m,\alpha}$ because p_α restricted to $X^{(m)}$ is equal to $p_{m,\alpha}$.

That is contained in $U_{\beta^m(\alpha)}$, by the very definition of β^m. But $\beta^m(\alpha) = \beta(\alpha)$, because α is in Λ^m, since α is in Λ^n and $m \geq n$. Since this is true for all $m \geq n$, that gives the support of this p_α is contained in $U_{\beta(\alpha)}$. Therefore, β is a refinement function for the family Θ and hence Θ is subordinate to $\mathcal{U}$. This is very important. Okay?

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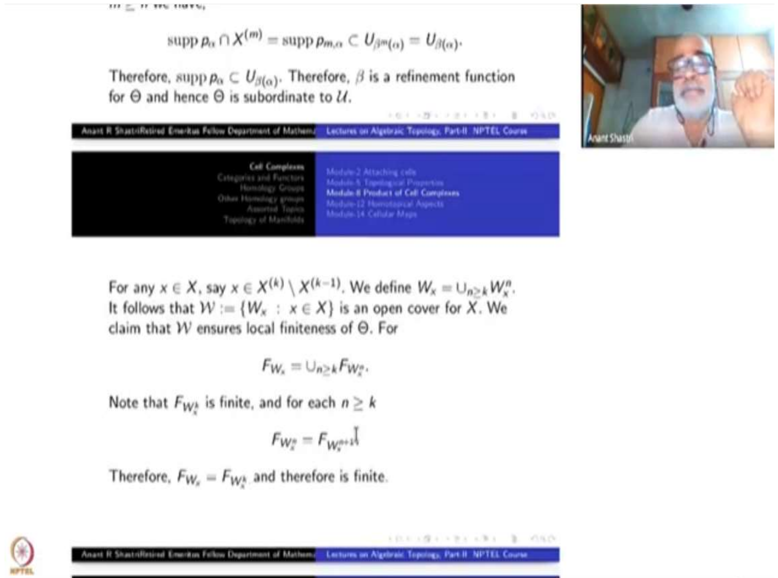

Now for any $x \in X$, say again $x \in X^{(k)} \setminus X^{(k-1)}$. (There is a unique such k.) Define W_x to be union of all W_x^n, where $n \geq k$. Starting with $X^{(k)}$, you get W_x^k that's the first time you get then keep extending this open sets. It follows that $\{W_x, x \in X\}$ is an open cover for X. Why each W_x is open in X? Because intersected with any $X^{(n)}$, it will be W_x^n which is open in $X^{(n)}$. All right. Of course for $n < k$, it's empty. That is fine. So, this is an open cover core for X, Okay?

Now, we claim that this open cover ensures local finiteness for Theta. Okay? After that taking the sum makes sense and showing that it is equal to 1 is very easy. So, let us see how the local finiteness comes. For F_{W_x} is the union of all the $F_{W_x^n}$ where $n \geq k$.

Because these are inside the indexing Λ, each member has to be in one of the sets Λ^n's. k is the starting point because x is not in $X^{(k-1)}$ but in $X^{(k)}$. And for $n \geq k$, what happens? $F_{W_x^n}$ is equal to the next one, $F_{W_x^{n+1}}$. The conditions in the proposition ensure that.

So the new entries from $\Lambda^{n+1}, \Lambda^{n+2} \ldots$, they don't come inside. That's why the this is equality. F_{W_x} is equal to $F_{W_x^k}$. In particular, it is finite. Okay?, That's means the cover $\mathcal{W}$ ensures the local finiteness of Θ.

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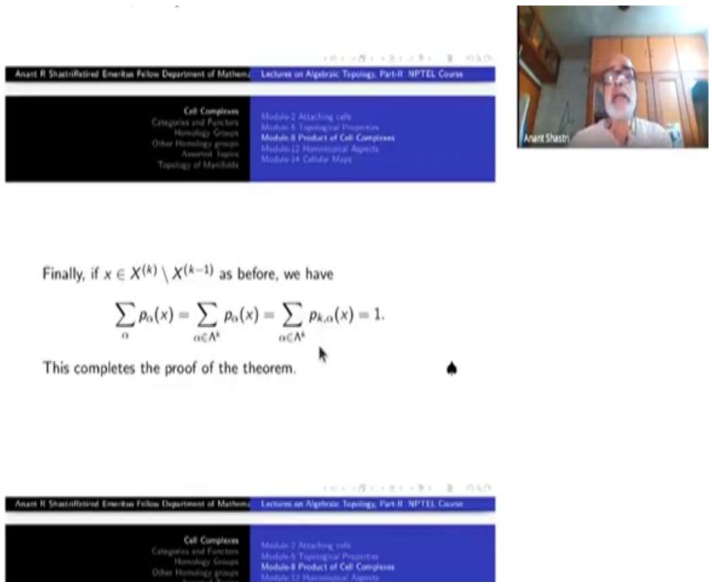

Finally, again assume that x is inside $X^{(k)} \setminus X^{(k-1)}$ as before. After all, there is a unique such k. The summation $p_\alpha(x)$ for all $\alpha \in \Lambda$ is equal to summation taken over α in only Λ^k. Because if α is not in Λ^k, then $p_\alpha(x)$ is actually zero. So, this summation is only for those α inside Λ^k.

But then this is the same as the summation of $p_{k,\alpha}(x)$. Sinc x is in $X^{(k)}$, each $p_\alpha(x)$ is equal to $p_{k,\alpha}(x)$. Hence the sum total is equal to 1, because on $X^{(k)}$, this is a partition of unity. So, that completes the proof of the theorem. So, next time we will take care of this proposition okay? This proposition will be proved in two steps. Okay? Thank you.