Introduction to Algebraic Topology (Part - II)
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Lecture - 09
Product of Cell Complexes Continued

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Let us continue the study of products of cell complexes. Indeed, last time we already introduced a new definition, viz., the concept of local countability, okay? This includes locally finiteness as well as global countability as special cases. So, what is the condition? Every point is contained in a countable subcomplex and that subcomplex must be a neighbourhood of the point in the whole space okay. `Local' means that it should happen in a neighbourhood, so the word neighbourhood has to come here, okay? Sorry, this was the statement of the lemma.

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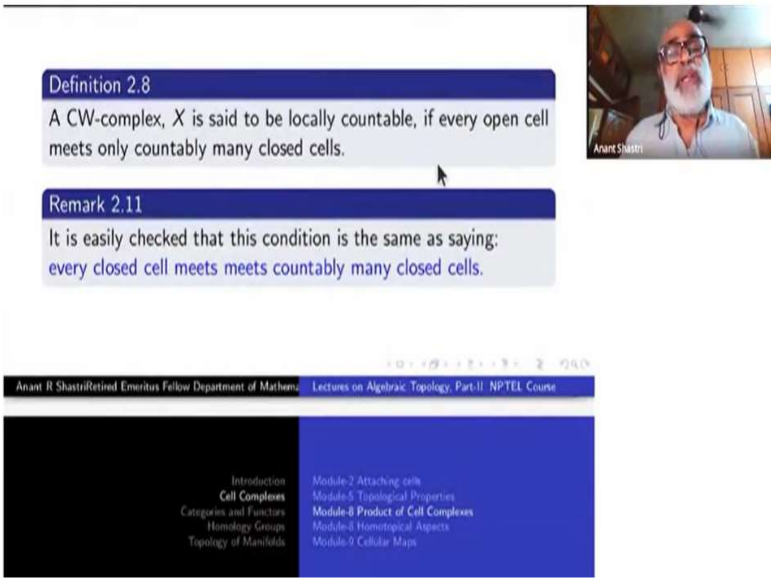

The definition of `locally countable' is that every open cell meets only countably many closed cells. Now, we are going to prove this lemma here okay? This justifies the name of the definition. Obviously, every point will meet only finitely or countably many closed cells, that will be also true okay? So, you could have taken this condition as the definition, that every closed (or open) cell meets countably many closed cells okay? `Every open cell meets countably many open cells' would be wrong, because no open cell meets any other open cell; open cells are mutually disjoint, okay?

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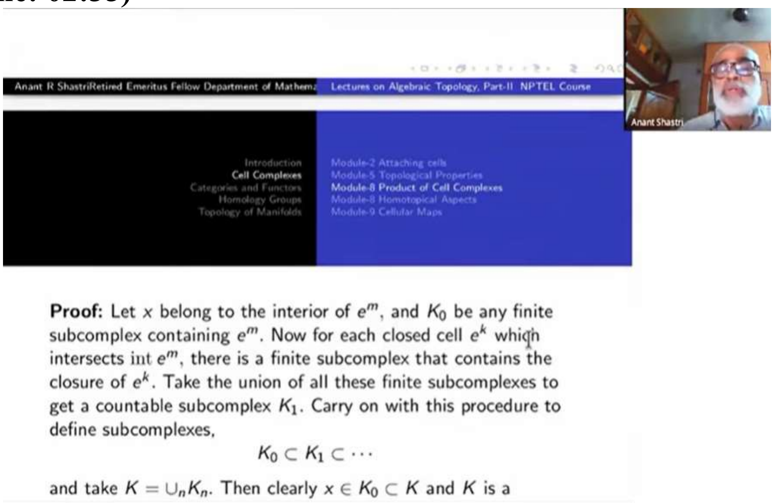

So, let us let us prove this lemma. First of all, each $x \in X$ belongs to the interior of a unique cell, say, interior of e^m. Let K_0 be any finite subcomplex containing e^m. What we have to do? Are you sure that there is a finite subcomplex containing e^m? All that you have to observe is that the closure of e^m, being compact, is covered by finitely many closed cells of dimension less than or equal to $m-1$, okay? Take the closure of all those and so on. You start with a m-

cell e^m, you may have to go down, down, down, you know, whatever cells of lower dimension than m which may intersect one of the previously taken cell.

(Added by the reviewer: In fact every compact subset is contained in a finite subcomplex).
So, there is a finite subcomplex because closure of v is compact okay. Indeed, in this way, we get K_0 to be of dimension equal to m. So, let us fix such a subcomplex K_0. There is no uniqueness here. The problem is that you cannot stop here because K_0 may not be an open set in X, and so, may not be a neighbourhood of x in X.

Now for each closed cell e^k which intersects the interior of e^m, there is a finite subcomplex that contains e^k. How many such e^k are there? Countably many, okay? Since there are only countably many such cells, taking the union of all of them, we get a countable subcomplex K_1 containing K_0 and which covers e^m. Now I repeat this process.

I started with a point x which is in the interior of e^m, some m-cell, okay? Then I took a finite subcomplex, which contains this e^m. Okay? Now, some higher dimensional cells may have their attaching maps taking some values in the interior of e^m. okay? So you have to take, for each such cell, a finite subcomplex containing that cell and take the union of all these subcomplexes, where e^k ranges over all those cells whose boundary intersects e^m. Call that K_1, okay?

Now carry on with this procedure. What is K_1? K_1 is a countable subcomplex, okay? For each of its top-dimensional cells in choose another countable subcomplex and take their union to get K_2, containing K_1 and so on. Finally take K to be the union of all these K_n's. It is clear that K is a countable subcomplex which contains the point x. The claim is that this K is an open subset of X and so, it is a neighbourhood of x. Okay? To see that K is a neighbourhood of x, what we will do?

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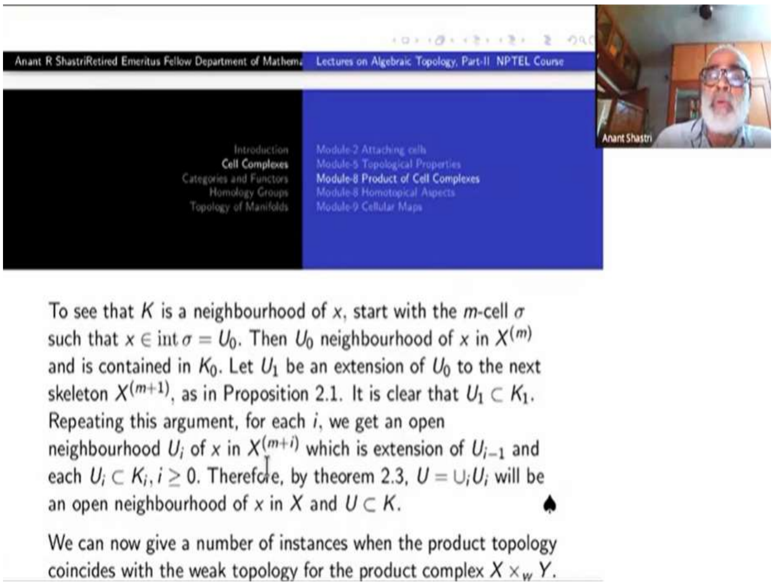

We started with an m-cells σ such that x in the interior of σ, right? Call its interior U_0. It is an open subset inside the subcomplex K_0, okay? K_0 is by choice, an m-dimensional subcomplex which contains the given m-cell σ.

Now, we play the old game: let U_1 be the extension of U_0 to next skeleton $X^{(m+1)}$. We know how to extend open sets from one skeleton to the next skeleton right? As in the proposition 2.1. (So, this is going to be used again and again. Earlier, I have given a lot of emphasis on this one.) Then this U_1 will be contained inside K_1, why? Because U_1 will take only parts of those cells which will intersect U_0 and hence portions which are contained in K_1,. Repeat this argument. What you will get? Each time you will get U_i which is a neighbourhood of x inside $X^{(m+i)}$ and which is an extension of U_{i-1}, and each U_i is contained inside K_i.

I am just using these U_0 inside K_m, U_1 is an extension of U_0, U_1 is contained in K_1 etc., so the inductive step is, okay, for each $i \geq 0$. Therefore, by neighbourhood extension theorem, U which is a union of U_i's is open in X, because its intersection with each $X^{(i)}$ is U_i which is open in $X^{(i)}$. And x belongs to U, and U will be contained in K, because U_i is contained inside K_i. and K the union of all the K_i's, okay? So, many of these ideas must have been clear to you, but you must now practice yourself writing down the proofs correctly.

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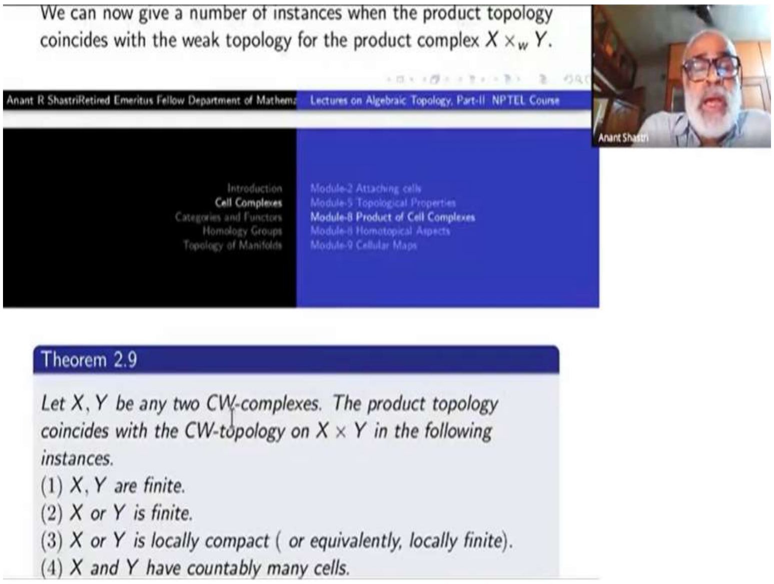

Start with any two CW complexes, X and Y. The product topology on $X \times Y$ coincides with the CW-topology on $X \times Y$ in the following instances (this is not an if and only if theorem, mind you):

(1) X and Y are finite. (We saw this right in the beginning, Indeed, we know this for

(2) X finite or Y finite. Okay? Next,

(3) X or Y is locally compact (or equivalently, locally finite. This has also we have seen earlier.)

(4) X and Y both have countably many cells. This is something new. The last statement is the following.

(5) X and Y are both locally countable.

So, this is the best thing that we could say so far. This covers all the earlier cases. You will see an example also to illustrate that the condition (5) cannot be relaxed further, okay. But there may be other directions in which you may try to ensure the result. But if one of them is not locally countable then the conclusion fails. Some particular cases may be there, in which the product topology but in general this will not be true.

Parts (1), (2) and (3) are already seen. How and when? We have seen that if a complex is finite then it is compact. Product of two compact spaces is compact and hence product topology is compactly generated, and hence coincides with the CW topology. Similarly, X or Y is locally compact and the other is compactly generated then also we have seen that product topology is compactly generated. This takes care of the first three statements. The first thing which we have not yet seen is the case when X and Y have countably many cells, and the next one is when X and Y are locally countable, okay? So, (4) and (5) have to be proved, okay?

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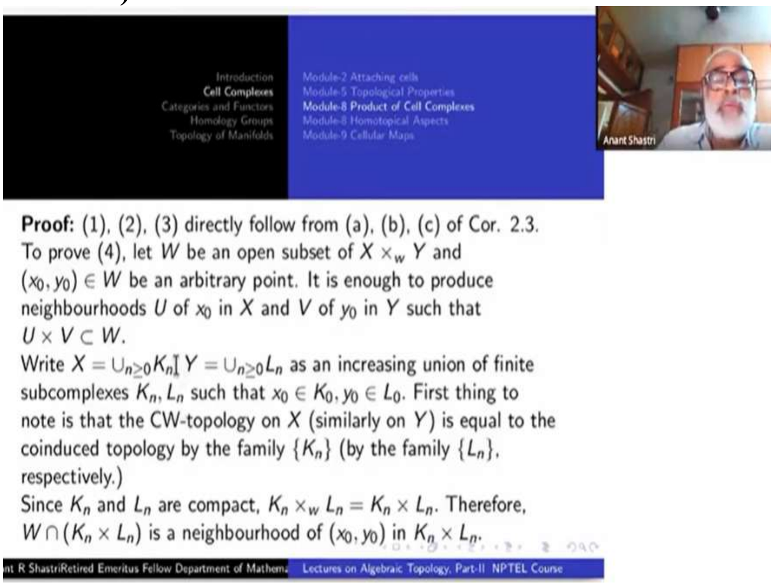

To prove (4), let W be an open subset of this CW complex $X \times_w Y$. For each point (x_0, y_0) inside W, a neighbourhood U of x_0, it is enough to produce a nbd U for x_0 in X and a neighbourhood V of y_0 in Y such that $U \times V$ is contained inside W. If you have done this for every point that would mean that, W is open in the product topology on $X \times Y$. Okay? We started with an open set W inside the CW-topology. Okay?

So, we use the hypothesis in (4) which says X and Y have countably many cells. So, write X as a union of K_n's and Y as a union of L_n's, increasing union of finite subcomplexes. I cannot take skeletons and so on here. Any countable complex, you know, is always a union of finite subcomplexes, how? You enumerate all the cells, okay? Take the first cell it may not be a subcomplex, but it is contained in a finite subcomplex K_1. Now, the union of K_0 and the second cell must be contained in another finite subcomplex. Okay? Take that as K_2. Like this having constructed K_i, take K_{i+1} to a finite subcomplex containing K_i and e_{i+1}. Thus, you can always write a countable complex as an increasing union of finite subcomplexes. So do it for both X and Y. We can further assume that x_0 and y_0 are respectively in K_0 and L_0.

The CW-topology on X and similarly the CW topology on Y is coinduced from the collections $\{K_i\}$ and $\{L_i\}$ respectively. Therefore to see that a subset is open or closed inside X, I can intersect it with K_n and see whether it is open (or closed, respectively) inside K_n for each n. Since K_n and L_n are compact, $K_n \times L_n$ is compact and hence the CW-topology on it is the same as the product topology, from (1). Since W is open in $X \times_w Y$, its intersection with each $K_n \times L_n$ is an open neighbourhood of $(x_0, y_0) \in K_n \times_w L_n$, and hence in the product topology. We now construct open subsets U and V inductively.

But then each K_n is again contain either a finitely many a covered by finite will be compact since, therefore, it is equal. The first thing to note that CW topology on X and CW topology on Y is equal to the coinduced topology by these families therefore to say something is open or closed inside X_I can intersect it with K_n and see that whether it is open or closed inside K_n since K_n and L_n are compact.

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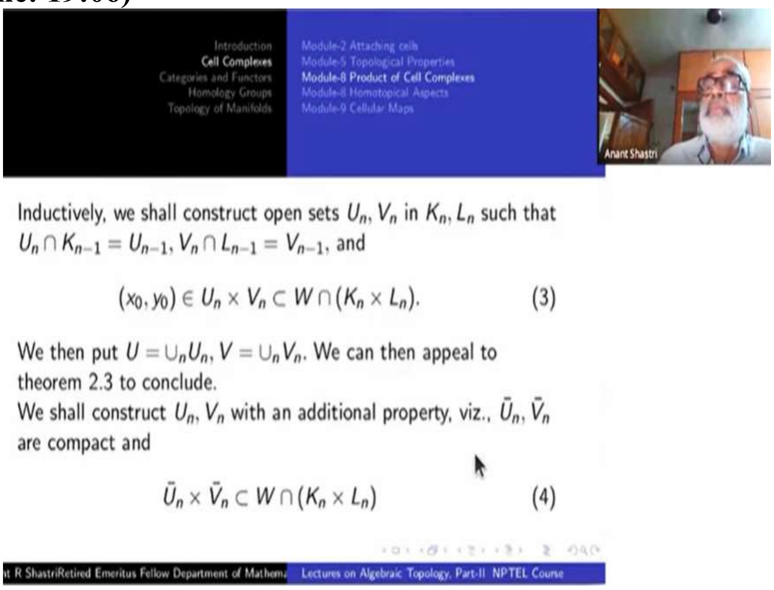

Inductively, we shall construct sets U_n and V_n inside K_n and L_n respectively such that $U_n \cap K_{n-1}$ is U_{n-1}, $V_n \cap L_{n-1}$ is V_{n-1}. Then we put U equal to union of U_n's, V equal to union of V_n's. As before, we can appeal to theorem 2.3 to conclude that U, V are open subsets of X and Y respectively. So, that inductive step has to be done. Okay? For this we have to use one more small topological trick. So, we shall construct U_n, V_n with an additional property and that will help us to carry out the inductive step. What is this additional property? viz., $\bar{U_n}$ and $\bar{V_n}$ are compact. So, in the construction we are going to make certain U_n and V_n have their closures compact.

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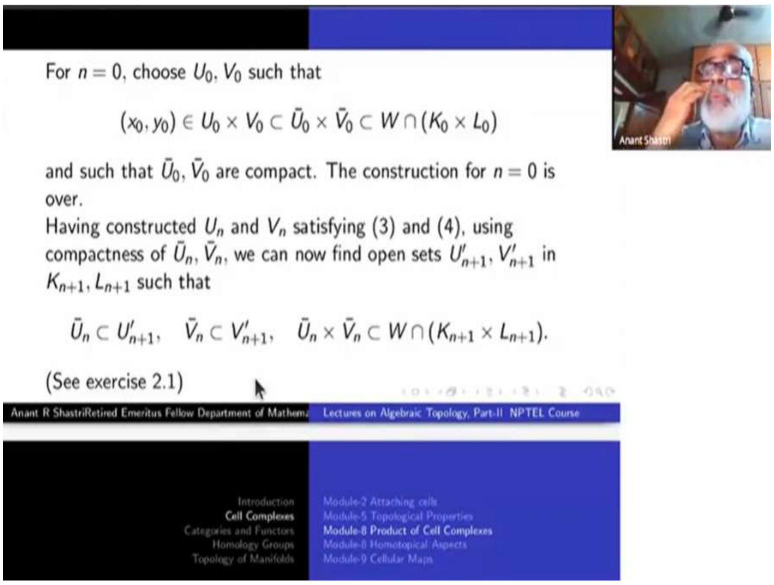

So, for $n = 0$, what are U_0, V_0? They must be inside K_0 and L_0, right? x_0, y_0 belong to U_0, V_0 respectively, $\bar{U_0} \times \bar{V_0}$ contained in $W \cap (K_0 \times L_0)$ and such that such that $\bar{U_0}$ and $\bar{V_0}$ are compact. How can you ensure this?

The points x_0 and y_0 are in some open cells, right? The closure of cells are compact right? So, this is happening inside K_0 and L_0 which are finite CW complexes. So, that is why this is possible, in other words, all that I am using here is that compact Hausdorff spaces are locally compact also. So, I can do this one to be compact construction for $n = 0$.

So, now, assume that this construction for some n okay? Then, you will see that compactness of $\bar{U_n}$ and $\bar{V_n}$ allows you to find open sets say U'_n and V'_n in K_{n+1}, and L_{n+1} respectively, so that U'_n is contained here U'_{n+1} and V'_n containing V'_{n+1}, $U'_{n+1} \times V'_{n+1}$ is contained in $W \cap K_{n+1}$. So this is a version of Wallman's theorem for compact subsets of the product space. I have stated it as an exercise, okay?

I will just tell you what this exercise is. It is shared below as 1.6. Suppose you have a neighbourhood W of $K \times L$ where K and L are compact subsets of X and Y, Okay? Then you can get open subsets U and V inside X and Y respectively, such that K is contained inside U, L is contained inside V, $U \times V$ itself is contained in the given neighbourhood W. I repeat. Suppose, you start with the product space $X \times Y$, arbitrary product space okay? (If you want, you assume that they are Hausdorff spaces.) K and L compact subsets of X and Y respectively, $K \times L$ is contained in an open set W of the product space. Then you can choose open subsets U, V containing K and L respectively such that $U \times V$ itself is contained inside W, okay? So, this is what we have used here. U'_n contained in U'_{n+1} and V'_{n+1} will mean K and L_{n+l}, Okay.

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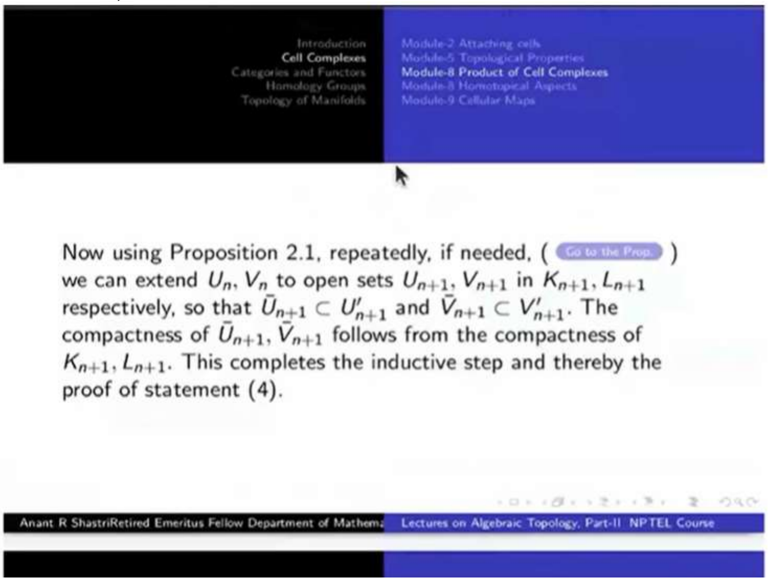

Now use 2.1 repeatedly. You can extend U_n, V_n to open subsets U_{n+1}, V_{n+1} in K_{n+1}, L_{n+1} respectively, such that $\overline{U_{n+1}}$ is inside U'_{n+1} and $\overline{V_{n+1}}$ is in V'_{n+1}. The point here in choosing, U_{n+1} is that $U'_{n+1} \cap K_n$ may not be equal to U_n, but now U_{n+1} will have that property. So, this was the gist of 2.1 right? You can extend... that means intersection with this one must be equal to U_n. So, U'_{n+1} and V'_{n+1}, you know, are some arbitrary open subsets, larger than the original ones, and so when intersected with the old one they maybe larger than the old ones, that will cause problem. So, these extension proposition is needed. Now your problem is over.

What we have done? I will just recall what we have done statement (4). If you have countable CW-complexes X and Y, then the product topology coincides with the CW-topology. The 5th one is easier, if you use the 4th one, okay?

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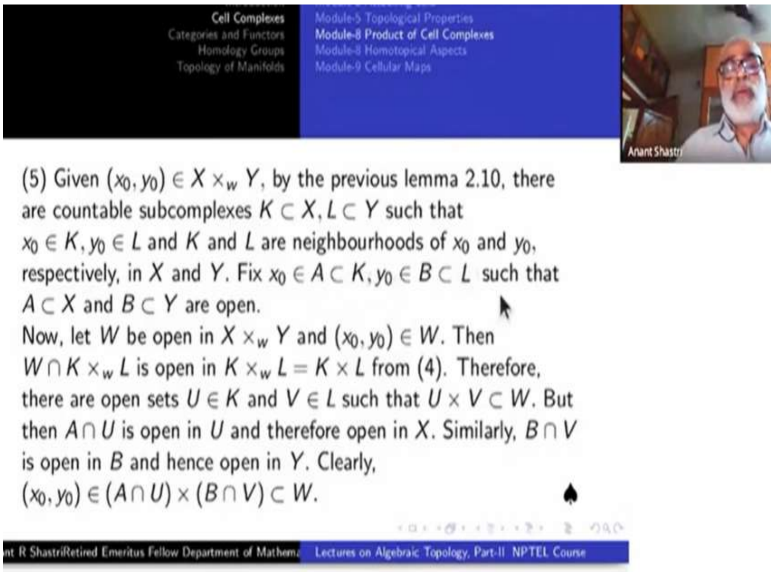

So, given a point $(x_0, y_0) \in X \times Y$, by the previous lemma, which we started with today, there are countable subcomplexes K, L respectively in X and Y right? such that K and L are neighbourhoods of x_0 and y_0 in X and Y respectively. Fix x_0 belonging to A contained in K, and y_0 belonging to B contained in Y such that A and B are open subsets. That is possible because K and L are neighbourhoods. Okay?

Now, essentially what we are trying to do is to reduce this problem (5) to problem (4), viz., to the case when both of K and L are countable case, by replacing X by K and Y by L. But, we still do not have that picture completely, because the final conclusion has to be for the whole of $X \times Y$, not for just $K \times L$, right? So, this needs some argument.

So, let W be open in $X \times_w Y$, the CW topology. Take (x_0, y_0) belonging to W, okay? Then $W \cap (K \times_w L)$ will be open in $K \times_w L$ which is equal to $K \times L$ by (4). Therefore, there are open nbds U and V of x_0 and y_0 respectively in K and L such that $U \times V$ is contained in W. Then $A \cap U$ is open U, okay? Therefore, it is open inside X. Similarly $B \cap V$ is open in Y. Now (x_0, y_0) belongs to $(A \cap U) \times (B \cap V)$ contained in W, okay. So, (4) actually helps to solve this problem (5).

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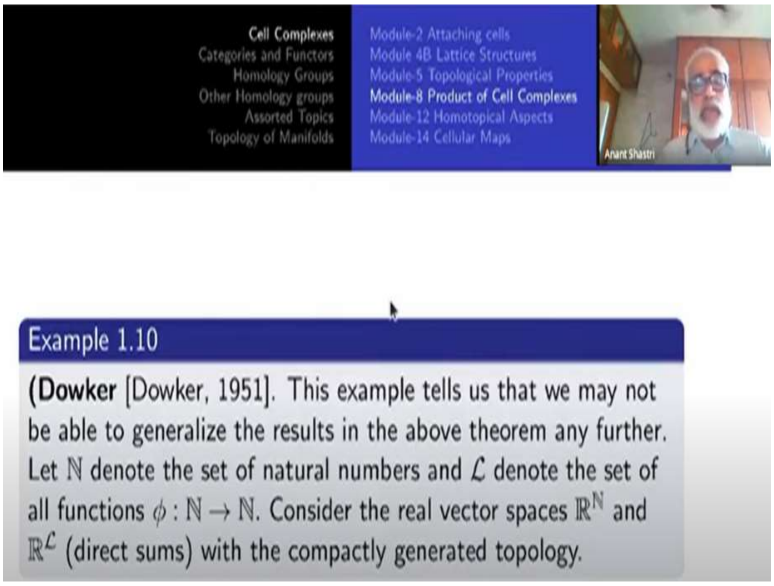

Here is an example, due to Dowker, which tells you that if both X and Y are not locally countable then product topology need not coincide with the CW-topology, okay? So, for this we start with the following notation:

Let Euler font $\mathbb{N}$ denote the set of all natural numbers and $\mathcal{L}$ denote the set of all functions ϕ from $\mathbb{N}$ to $\mathbb{N}$. Consider the real vector space $\mathbb{R}^{\mathbb{N}}$ which is the direct sum of copies of $\mathbb{R}$, as many copies as natural numbers. Let $\mathbb{R}^{\mathcal{L}}$ denote the direct sum of as many copies of $\mathbb{R}$ as this $\mathcal{L}$, which is indexed by functions from $\mathbb{N}$ to $\mathbb{N}$. On both these spaces, both, you give the compactly generated topology, okay? We are now going to construct a subspace of this $\mathbb{R}^{\mathbb{N}}$ that will be called X and another subspace of $\mathbb{R}^{\mathcal{L}}$ called Y, with the subspace topologies, which will be automatically compactly generated. Both will be 1-dimensional CW-complexes. When we take the product of X with Y, the product topology will not coincide with the CW-topology. So that is the idea, okay?

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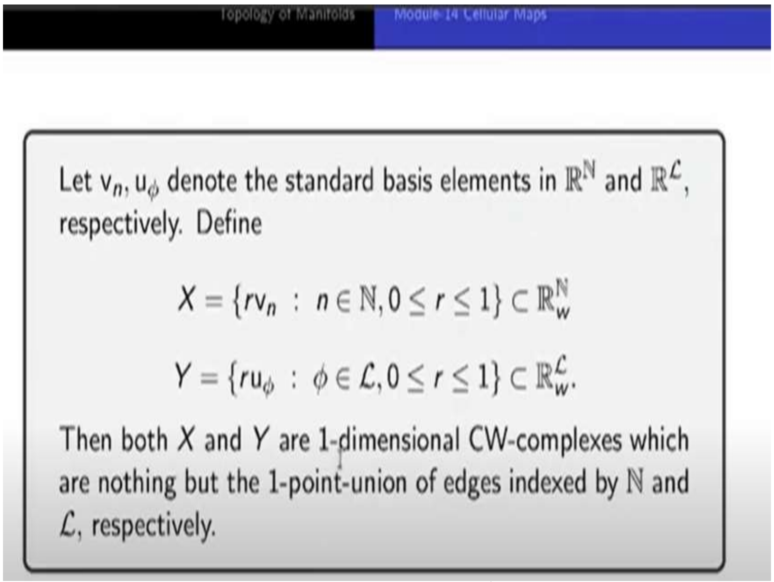

So, choose the standard basis elements for $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathcal{L}}$, namely, $\{v_n\}$ and $\{u_\phi\}$, respectively indexed by natural number and functions ϕ from $\mathbb{N}$ to $\mathbb{N}$. Now put X equal to all the line segments emanating from 0 and ending at the point v_n. So, you can write them as rv_n, where r is real number between 0 and 1. So, look at all of them. They will be incident at the origin of $\mathbb{R}^{\mathbb{N}}$, okay? and the other endpoint will be the vector v_n, okay? This is a 1-dimensional CW-complex, okay, consisting of just edges, all of them having single point in common and indexed by these vectors v_n themselves. Similarly, Y will consist of all points ru_ϕ, where $0 \le r \le 1$ and ϕ ranges over all functions from $\mathbb{N}$ to $\mathbb{N}$. So, it is exact similar to X, only the number of edges here is very huge. Then both X and Y are CW complexes, and 1-dimensional, Okay? And you can think of them as one point union of a number of edges. Okay?

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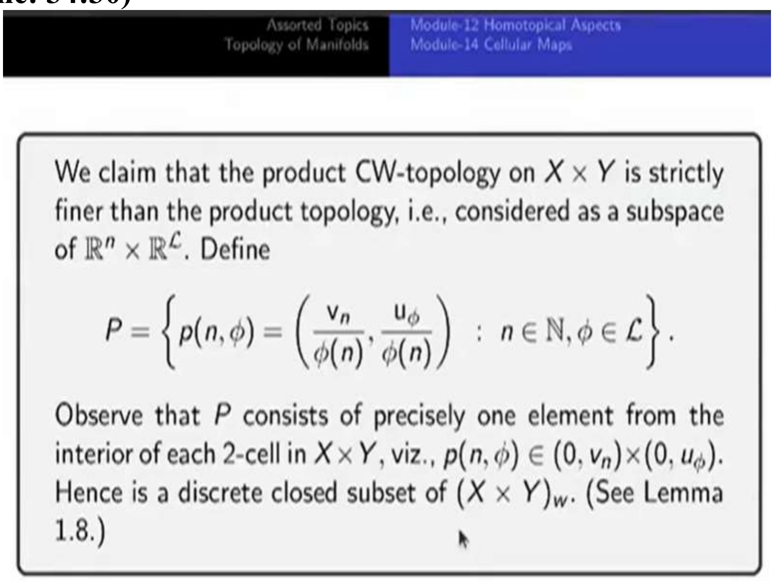

The claim, as I have told you, is that the CW topology on $X \times Y$ is strictly finer than the product topology. So, in order to prove that, we will display a subset P which is discrete closed subset inside the CW-topology, but it will have the origin 0 as a limit point in the product topology.

That subset P is precisely consists of points $p(n, \phi)$, doubly indexed by n as well as ϕ. What are they? The first coordinate is $v_n/\phi(n)$ and the second coordinate is $u_\phi/\phi(n)$, for all $n \in \mathbb{N}$ and for all functions ϕ. So, this will be one of the points on the edge cross edge, okay? Each $p(n, \phi)$ is a point in the 2-dimensional cell in the product space, a 1-cell here cross a 1-cell there, okay? As you take n and ϕ different, okay, say, n' and ϕ', that will be in a different 2-cell. So, all these points are in the interior of exactly one 2-cell, $[0, u_n] \times [0, \phi]$, okay? Therefore, by our old lemma, the set P is a discrete closed subset of the CW-topology on $X \times Y$.

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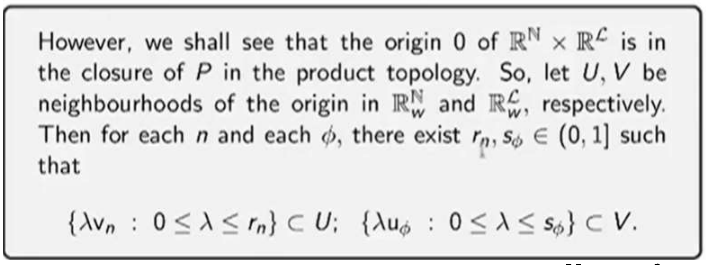

However, we are now going to show that the origin 0 in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathcal{L}}$ is the closure of P, with respect to the product knowledge. What is the meaning of that? Take any open subset around 0 and show that it intersects the set P, okay? So, an open subset around 0 in the product topology will contain a smaller open subset of the form $U \times V$, where U is a neighbourhood of 0 in $\mathbb{R}^{\mathbb{N}}$ and V is a neighbourhood of 0 in $\mathbb{R}^{\mathcal{L}}$, okay? Automatically, for each n and ϕ, we will have some r_n and s_ϕ in $(0, 1]$, numbers depending upon n and ϕ such that the entire line segment $\lambda r_n v_n$ where λ varies from 0 to 1, is contained inside U, and similarly the segment $\lambda s_\phi u_\phi$ will be contained in V. This will be true for every n and every ϕ.

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Consider $\psi : \mathbb{N} \to \mathbb{N}$ given by

$$\psi(n) = \max\left\{n, \left\lfloor \frac{1}{r_n} \right\rfloor\right\} + 1.$$

Clearly $\psi(n) \to \infty$ as $n \to \infty$ and hence we can choose m such that $\psi(m) > \frac{1}{s_\psi}$. It follows that $(\frac{1}{\psi(m)}\mathrm{v}_m, \frac{1}{\psi(m)}\mathrm{u}_\psi) \in P \cap (U \times V)$. This proves that P is not closed in the product topology $\mathbb{R}^{\mathbb{N}}_w \times \mathbb{R}^{\mathcal{L}}_w$.

Therefore, now, I can give you a sequence of points which converge to the origin. So, consider ψ from $\mathbb{N}$ to $\mathbb{N}$ given by $\psi(n)$ equal to the maximum of the two numbers plus 1, where the two numbers are n and the integral part of $1/r_n$. So, these two are some natural numbers now, okay? Take the maximum and add one. Then $\psi(n)$ tends to infinity as n tends to infinity because $\psi(n)$ is bigger than n. Hence, you can choose little m such that $\psi(m)$ is bigger than $1/s_\psi$, as well as $1/r_n$. Automatically, it follows that $1/\psi(m)$ is less than equal to r_n and s_ϕ. Therefore, the point $(v_m/\psi(m), u_\phi/\psi(m))$ belongs to $P \cap (U \times V)$.

So, that completes the proof of the counter example. Thank you.