Introduction to Algebraic Topology (Part – II)
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Lecture – 01
Introduction

Hello everybody! Welcome to the course on Algebraic Topology part II on the NPTEL portal. I am Anant R. Shastri, retired emeritus fellow, Department of Mathematics IIT-Bombay.

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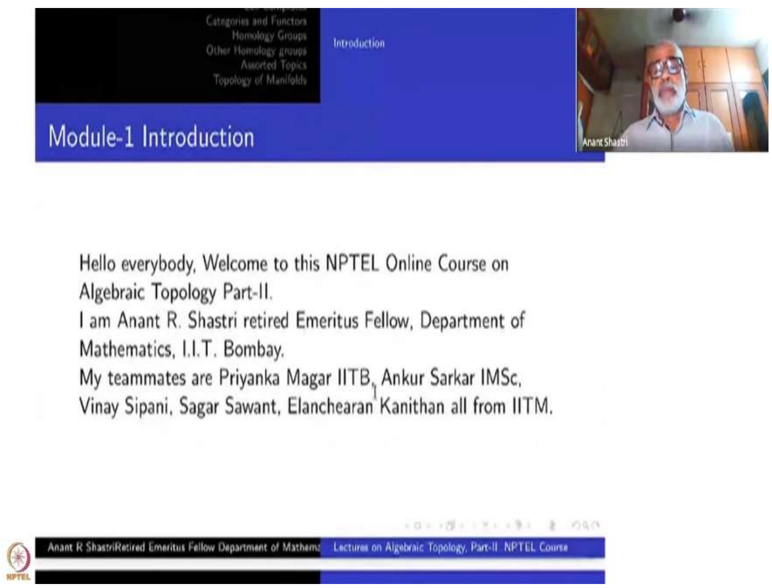

My teammates are Priyanka Magar from IIT Bombay, Ankur Sarkar IMSc, Vinay Sipani, Sagar Sawant, Elanchearan Kanithan, all from IITM.

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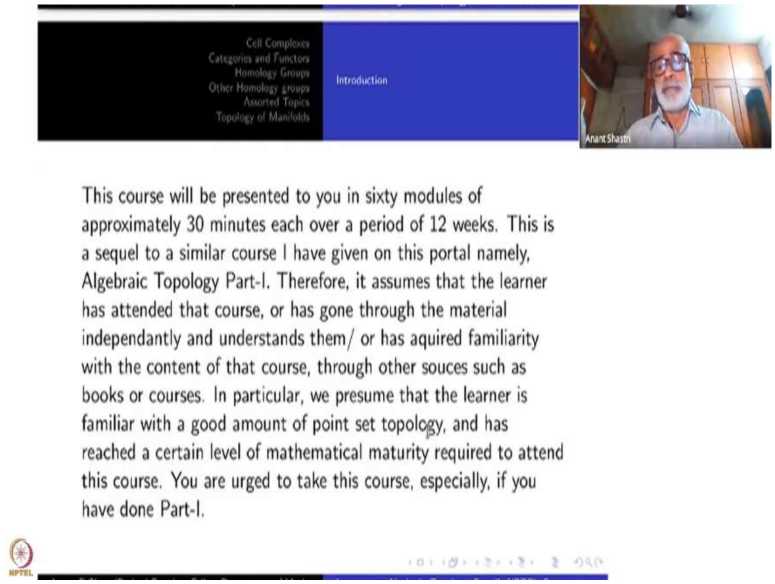

This course will be presented to you in about 60 modules of approximately 30 minutes each over a period of 12 weeks. This is a sequel to a similar course I have given on this very portal, namely, Algebraic Topology Part I. Therefore it is assumed that the learner has attended that course, or has gone through the material independently and understands them or has acquired familiarity with the contents of that course through other books or other courses or something.

In particular, we presume that the learner is familiar with a good amount of points set topology and has reached a certain level of mathematical maturity, so, that she can attend this course comfortably. You are asked to take this course specially if you have done part I. So that you will arrive within Algebraic Topology somewhere.

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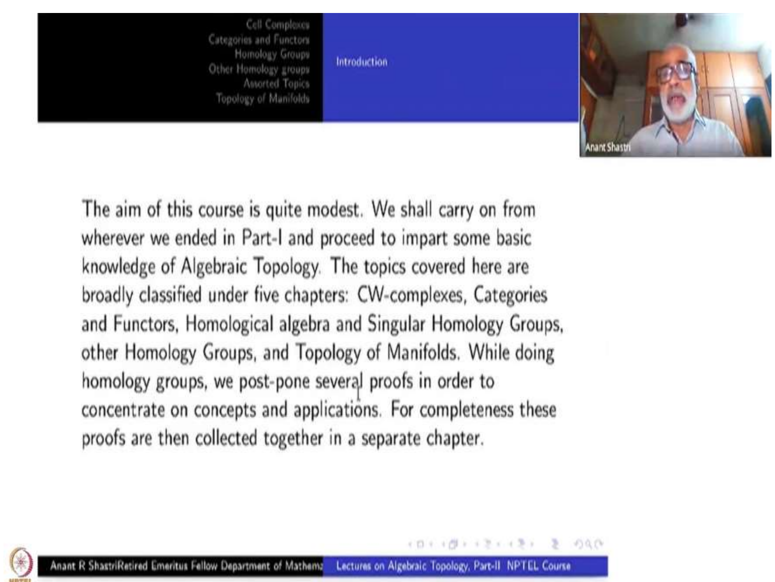

The aim of this course is quite modest. We shall carry on from wherever we ended in part I and proceed to impart some basic knowledge of algebraic topology. The topics covered here are broadly classified under 5 main chapters: CW-complexes, Categories and Functors, Homological algebra and Singular Homology groups, other Homology Groups and last chapter is on Topology of manifolds.

While doing homology groups, we post-pone a lot of proofs in order to concentrate on concepts and applications for completeness. All these proofs are then collected together in a separate chapter called assorted topics.

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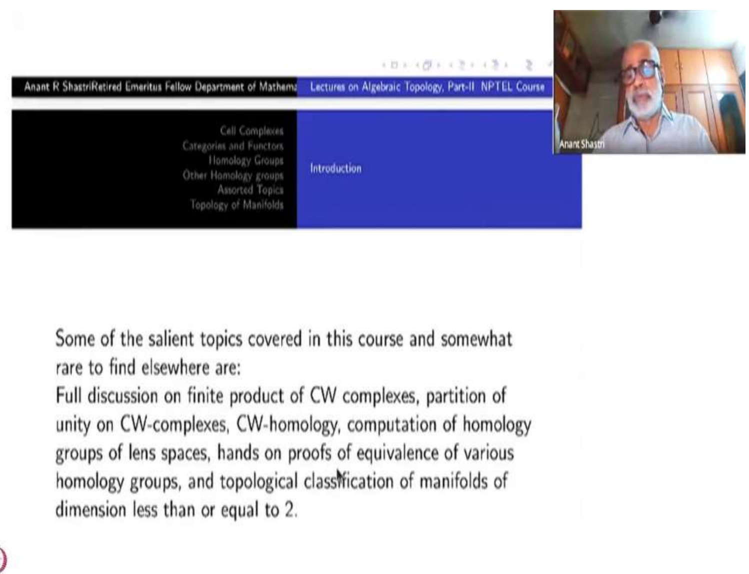

Some of the salient topics covered in this course and somewhat rare to find elsewhere are full discussion on CW-complexes especially product of CW-complexes, partition of unity on CW-complexes, CW-homology computation of homology groups of lens spaces, hands on proof of equivalence of various homology groups and topological classification of manifolds of dimension less than or equal to 2.

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Basic reference is my own book Basic Algebraic Topology, CRC Press, Boca Raton in which you can find other references. For the duration of the course you may stick to the notes.pdf that will be made available to you on the NPTEL website. The notes.pdf will have everything that is covered in these lectures and a full reference for further study. We hope you will find this course useful for you.

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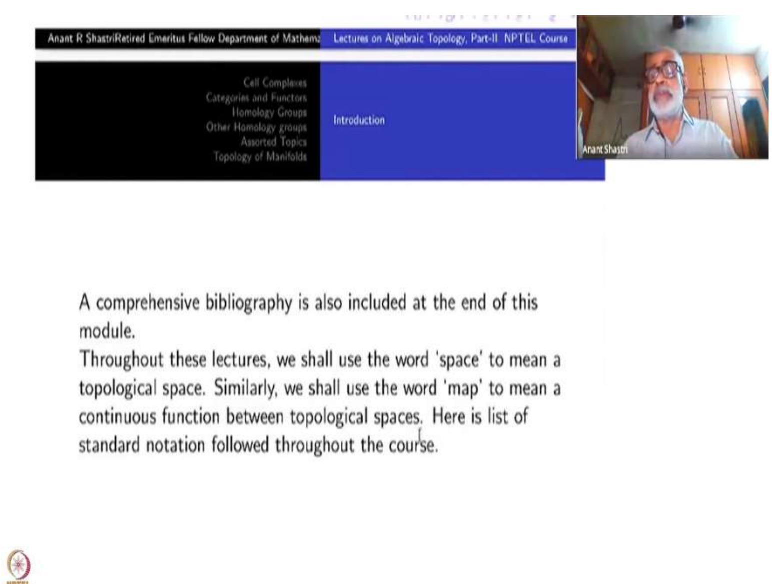

A comprehensive bibliography is also included at the end of this module. For your ready reference. Throughout these lectures we shall use the word `space' to mean a topological space. Similarly we shall use the word map to mean a continuous function between topological spaces. Here is a list of standard notations followed throughout the course.

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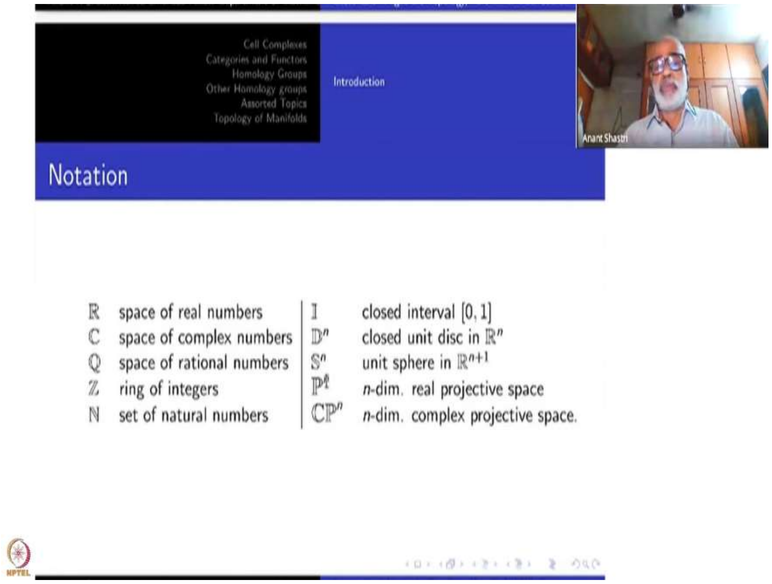

This styled fonts all these styled fonts real numbers, complex numbers, space of rational numbers, integers, set of rational numbers and $\mathbb{I}$ for always closed interval, $\mathbb{D}^n$ closed unit disc in $\mathbb{R}^n$, $\mathbb{S}^n$ in the closed unit sphere in $\mathbb{R}^{n+1}$, $\mathbb{P}^n$ is n-dimensional real projective space in $\mathbb{CP}^n$ is n-dimensional complex projective space.

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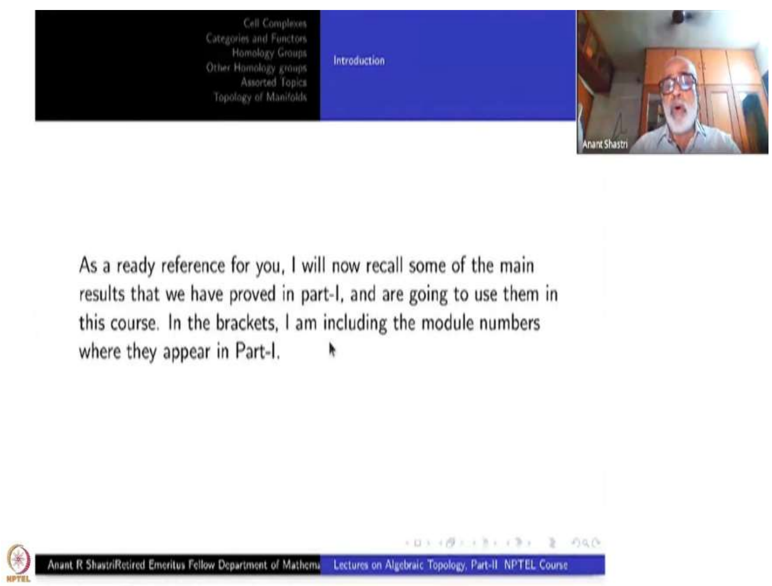

As a ready reference for you, I will now recall some of the main results that we have proved in part I, and are going to use them in this course. In the brackets, I have including the module numbers where they appear in part-I.

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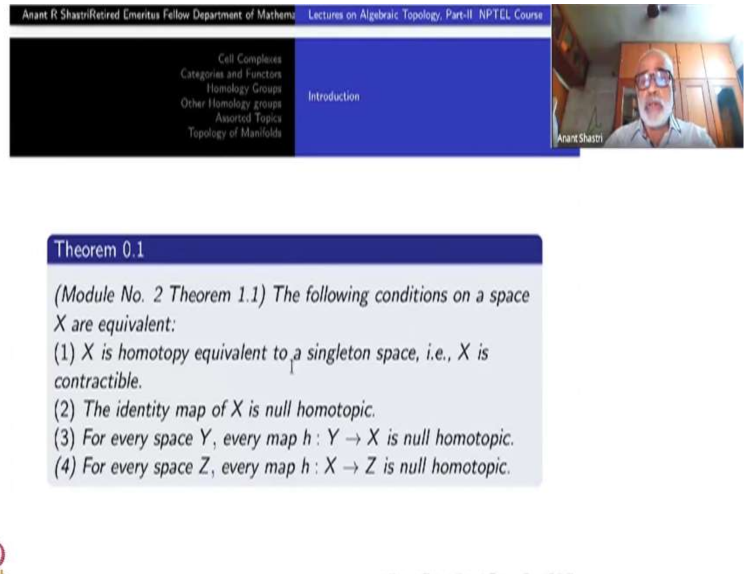

So, module number 2 theorem 1.1 in the part-I is the following theorem. So, I am just recalling them.

The following conditions on a space X are equivalent:

(1) X is homotopy equivalent to a singleton space.

(2) X is contractible. (By definition of contractility, identity map of X is null homotopic. But this can also be taken as definition because they are all equivalent).

(3) For every space Y, every map h from Y to X is null homotopic.

(4) For every space Z, every map h from X to Z is also null homotopic.

These 4 conditions are very useful. So, whichever way you can use them when X is contractible that is the point.

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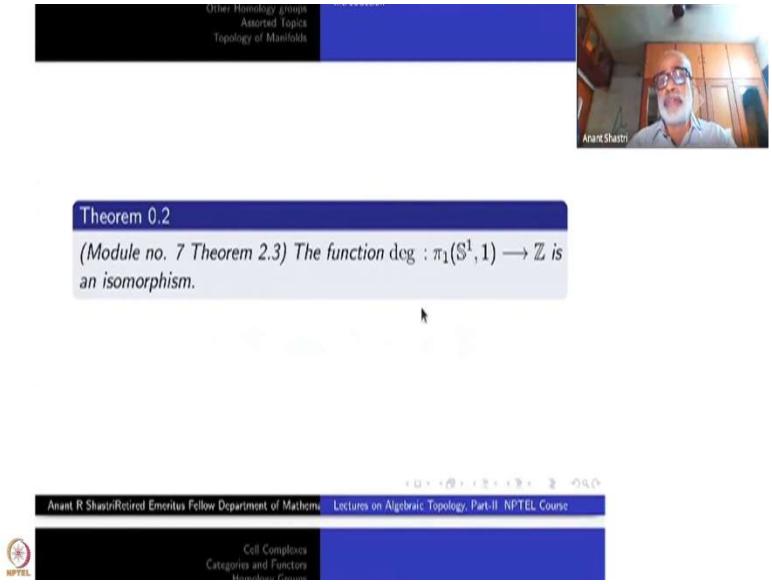

Then we have this degree of a map from $\mathbb{S}^1$ to $\mathbb{S}^1$ which classifies the fundamental group of $\mathbb{S}^1$. The function degree from $\pi_1(\mathbb{S}^1, 1)$ to $\mathbb{Z}$ is an isomorphism.

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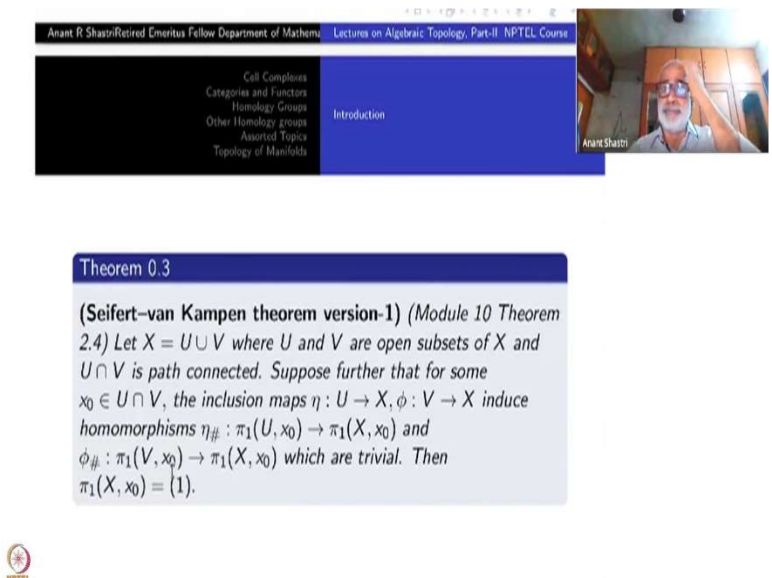

Next, there is the simplest version of van Kampen theorem module 10 theorem 2.4 in part-I.

Let X be union of 2 open subsets U and V and $U \cap V$ path connected. Suppose further for some x_0 that is the base point in $U \cap V$ the inclusion maps η from U to X, ϕ from V to X both induced homomorphisms $\eta_\#$ and $\phi_\#$ are trivial. Then $\pi_1(X, x_0)$ itself is trivial.

So, the emphasis is here is that the induced homomorphisms are trivial homomorphisms. This will happen if $\pi_1(U, x_0)$ itself is trivial and $\pi_1(V, x_0)$ is trivial. That is a special case.

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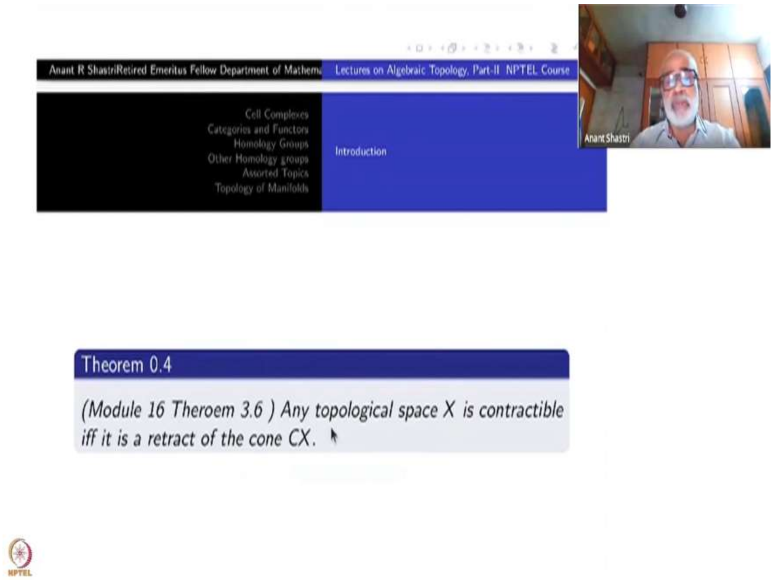

So, the next theorem is in module 16 theorem 3.6: Any topological space X is contractible if and only if it is a retract of the cone on itself.

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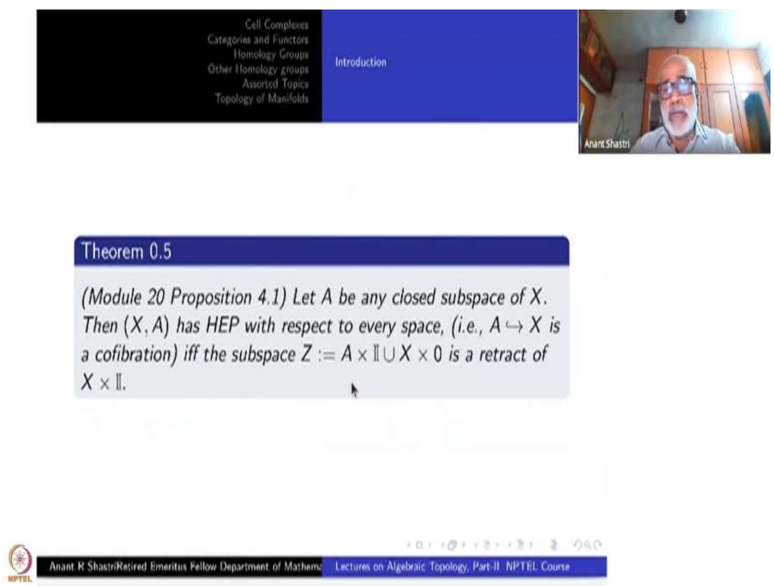

Module 20 proposition 4.1. Let A be any closed subspace of X. Then (X, A) has homotopy extension property with respect to every space, that is, the inclusion map A to X is a cofibration, (this is the definition of cofibration), if and only if the subspace $Z = A \times \mathbb{I} \cup X \times 0$ of $X \times \mathbb{I}$ is a retract of $X \times \mathbb{I}$.

This is a very useful result.

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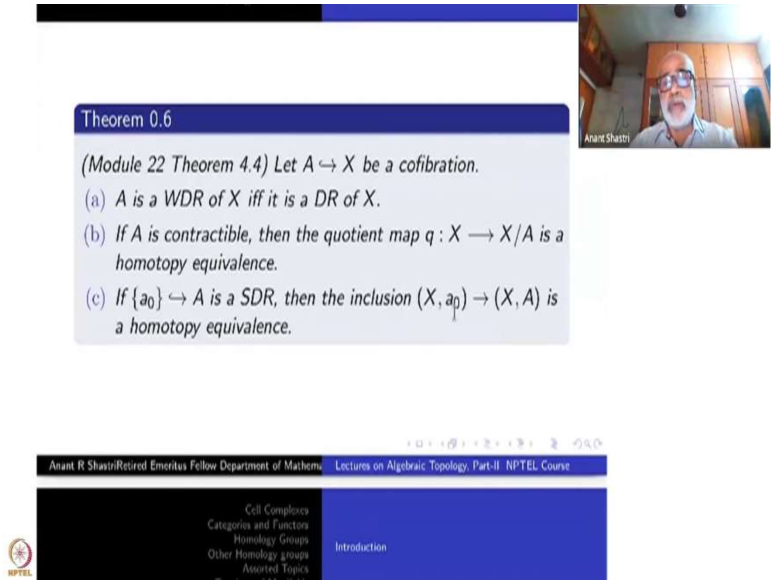

Module 22 theorem 4.4. Now let A to X be a cofibration defined here for the cofibration, that is it has homotopy properties with respect to every space to include a map here. Then there are 3 statements here:

(i) A is weak deformation retract of X, if and only if it is a deformation retract of X.

(ii) If A is contractible then the quotient map q from X to X/A, (where A is collapsed to a single point that quotient map q from X to X/A), is a homotopy equivalence.

(iii) If a_0 to A is an inclusion map (that is single point) is a strong deformation retract (that means not only that A is contractible, a_0 is a strong deformation retract is a stronger thing), then the inclusion map (X, a_0) to (X, A) of the pairs is a homotopy equivalence as pairs.

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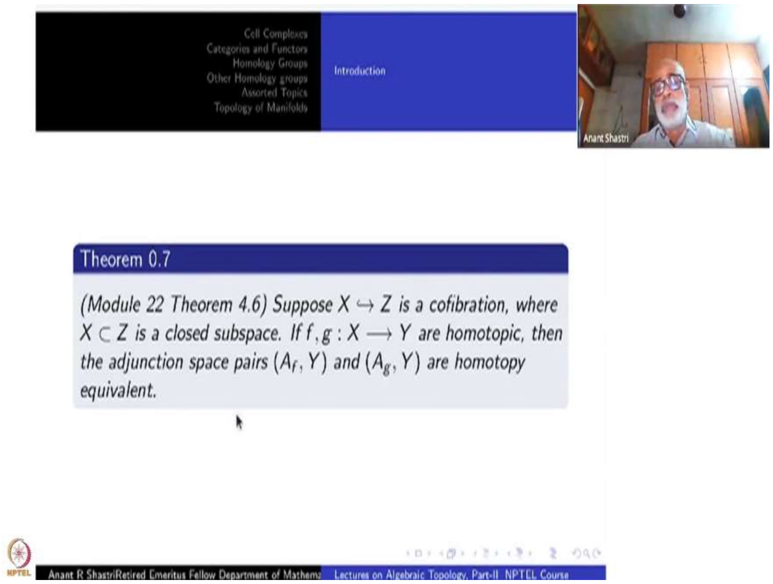

Module 22 and theorem 4.6. Suppose X to Z is a cofibration where X is a closed subspace. If we have 2 functions f and g from X to Y which are homotopic, then the adjunction pairs (Y is a subspace of both $A_f, A_g (A_f, Y)$ and (A_g, Y) are homotopic equivalent. In particular, this tells you that if you have the inclusion map of a smaller subspace X to a larger space is equivalent to a cofibration and if you perform adjunction space constructions with maps f, g from X to Y, the adjunction space is A_f and A_g are homotopy equivalent provided f and g are homotopic to each other.

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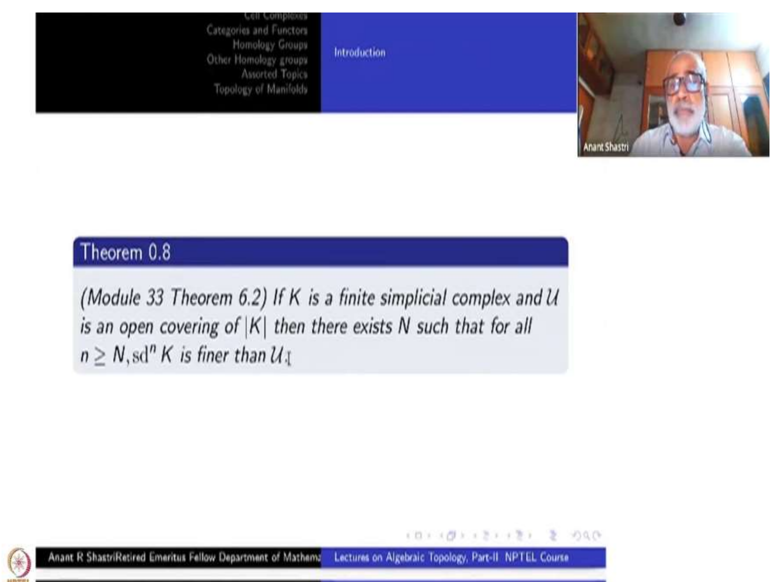

Module 33: Here we will come to simplicial complexes. If K is a finite simplicial complex and $\mathcal{U}$ is an open covering of X, then there exists N such that for all $n > N$, $sd^n K$ is finer that $\mathcal{U}$.

Here, sd^n denotes the barycentric subdivision iterated n times $sd \circ sd \circ \cdots \circ sd$($n$ times), the Barycentric subdivision iterate n times of K that will be finer than $\mathcal{U}$. So, this is possible in the case when K a finite simplicial complex.

You should begin with an open covering $\mathcal{U}$ of $|K|$. Let me recall. A subdivision K' is finer that $\mathcal{U}$ means that if you take any vertex and its open star. If you take all vertices that open cover that open covering for $|K|$; that is finer than $\mathcal{U}$ means that each open star is contained in some member of $\mathcal{U}$.

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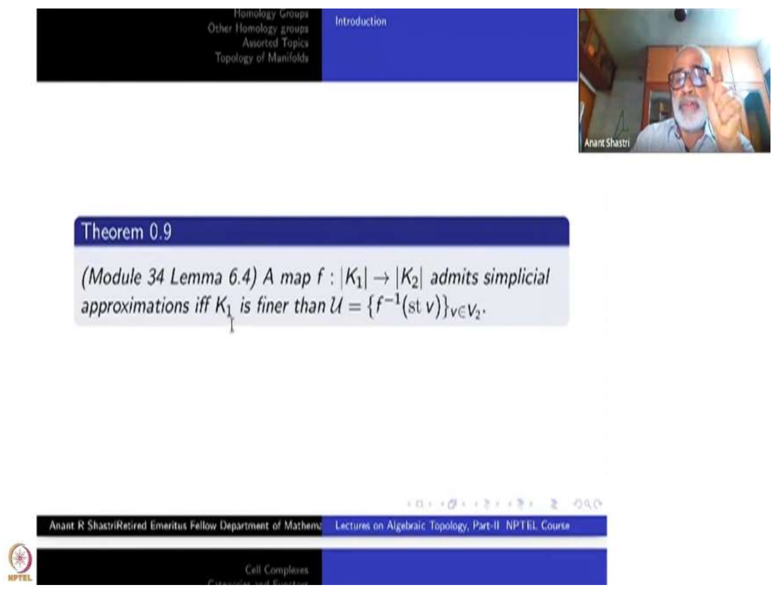

Once again, given a map f from $|K_1|$ to $|K_2|$, a continuous function, map means continuous function remember that, there will be simplicial approximations to f if the simplicial complex K_1 is finer than the open covering $f^{-1}(st\ v)$ where v ranges over vertices of K_2.

$st\ v$ is an open cover for $|K_2|$, f inverse of that is an open cover of $|K_1|$ and K_1 is finer than that covering means star of u for any vertex u must be contained inside $f^{-1}(st\ v)$ for some vertex v in K_2. If that happens then there will be simplicial approximation to f.

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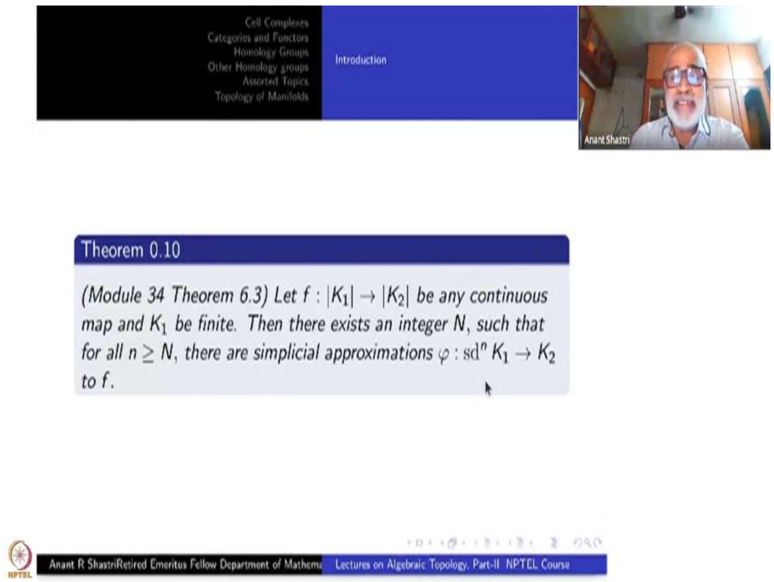

Suppose that f from $|K_1|$ to $|K_2|$ is any map and K_1 is finite. Then there exists an integer N such that for all $n \geq N$, there are simplicial approximations ϕ from the barycentric subdivision iterated n times of K_1 to K_2 to the function f. (This n should be sufficiently large, N depends on f.)

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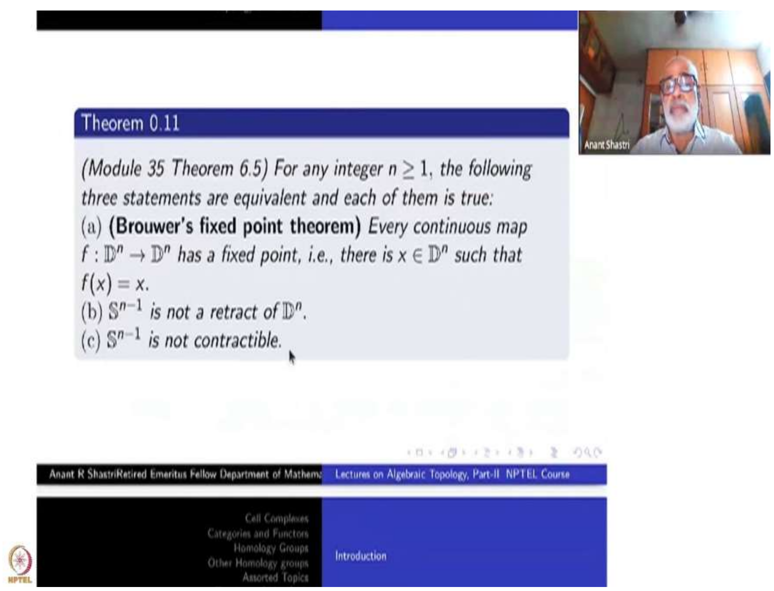

Using these simplicial approximations, we have produced this famous theorem namely, Brouwer's fixed point theorem.

For any integer $n \geq 1$, the following 3 statements are equivalent to each other. And each of them is true. The first statement is Brouwer's fixed point theorem:

(a) every continuous mapping f from $\mathbb{D}^n$ to $\mathbb{D}^n$ has a fixed point. (Fixed point means what? That is a point $x \in \mathbb{D}^n$ such that $f(x) = x$.)

The second statement says:
(b) the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ is not a retract of $\mathbb{D}^n$.

Third statement is:
(c) $\mathbb{S}^{n-1}$ is not contractible.

These 3 statements are equivalent. In the proof first we show the equivalence of these statements using simpicial approximation and then we just prove that $\mathbb{S}^{n-1}$ is not contractible. Therefore all these 3 things are true. before proving this when we use simplicial approximation.

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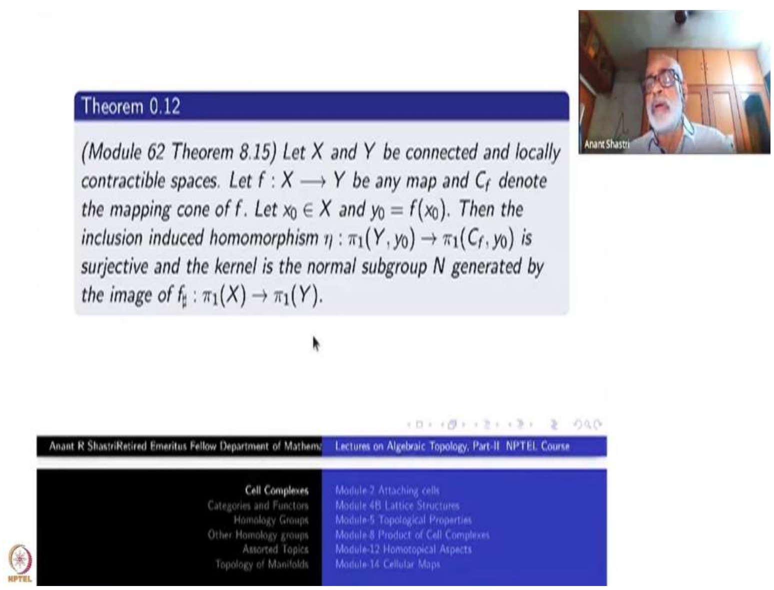

In module 62, there is a lot of covering space theory which I will recall only when I need them. But right now this is one theorem to which we have to appeal again and again. So that just indicates how far we have gone in covering space theory and fundamental group.

Let X and Y be connected and locally contractible spaces, let f from X to Y be any map and C_f denote the mapping cone of f.

Let x_0 belong to X and y_0 equal to $f(x_0)$. (So, your chosen base point for X that is x_0 goes to y_0.) Then the inclusion induced homomorphism η from $\pi_1(Y)$ to $\pi_1(C_f)$ which is the mapping cone Y, this homomorphism is surjective and its kernel is the normal subgroup N generated by the image of $f_\#$ from $\pi_1(X)$ to $\pi_1(Y)$.

Note that f is from X to Y, take the normal subgroup generated by the image of $f_\#$. Image is a subgroup, but may not be a normal subgroup. So, take the normal subgroup, go modulo that, that will guve you a group isomorphic to $\pi_1(C_f)$. So, this is the statement of this theorem.

So that is all for today, hope you will enjoy this course, We, myself and my teammates are there for you to help out with all your troubles, all your queries. Use the portals that NPTEL has provided to you carefully and usefully. And you know you can learn a lot from this experience. Thank you.