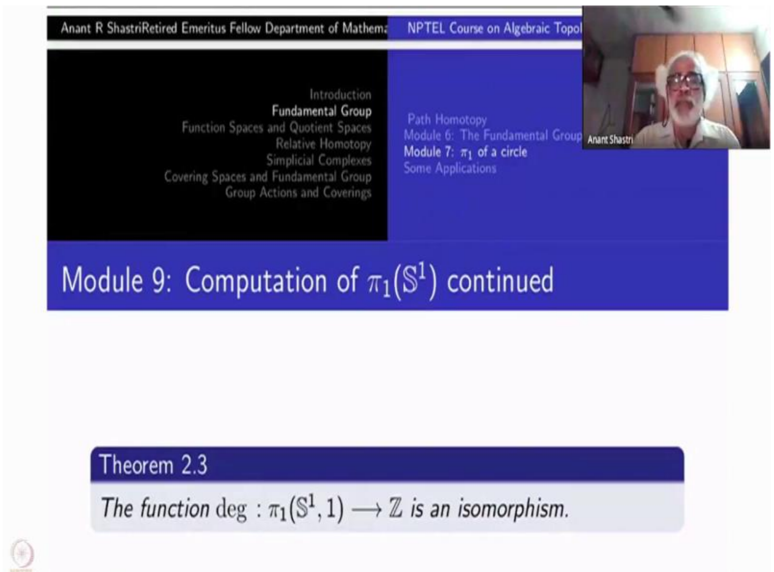
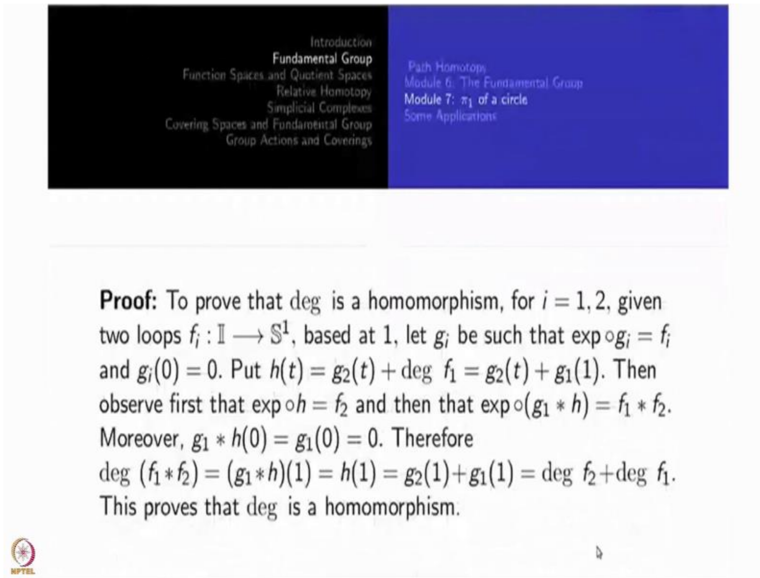
Introduction to Algebraic Topology (Part – I)
Professor Anant R Shastri
Department of Mathematics
Indian Institute of Technology Bombay
Lecture 9
Computation Concluded

(Refer Slide Time: 00:16)



Last time we defined a function from π_1 of $\mathbb{S}^1$ to the set of integers. Taking a loop f, representing an element of π_1 of $\mathbb{S}^1$, we lifted this loop to a map into $\mathbb{R}$, namely exponential composite that map is equal to given loop. In doing this, we fixed the starting point to be 0. The endpoint of this path was the degree of the function the class of f. This function, we want to show now, is a homomorphism. So this theorem is that the function degree from π_1 of $\mathbb{S}^1$ to $\mathbb{Z}$ is an isomorphism; one-one, onto and a homomorphism. One by one, let us prove them.

(Refer Slide Time: 01:35)



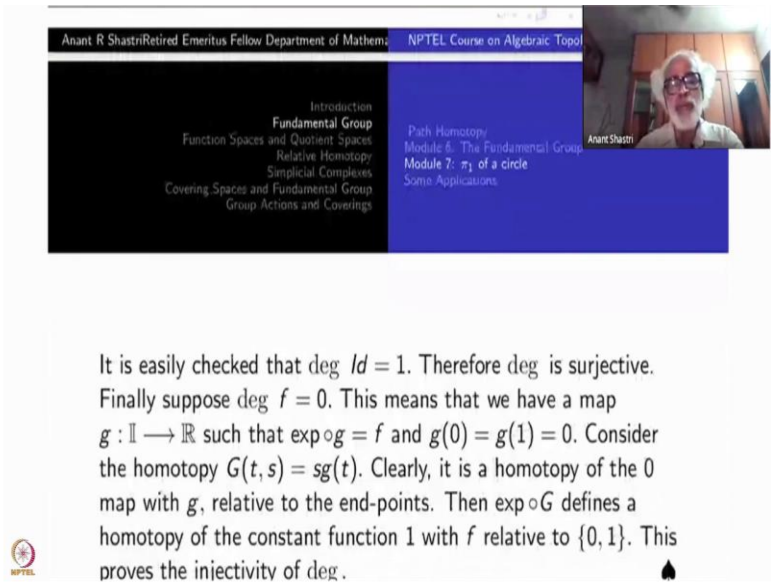
To show that degree is a homomorphism, what we have to do? Let us take two loops f1 and f2 based at 1. Let gi be the path inside $\mathbb{R}$, map into $\mathbb{R}$ such that exponential composed with gi is equal to fi,-- those are the lifts,--- such that their starting point is 0. Then g1 of 1 is degree of f1 and g2 of 1 will be degree of degree of f2. What we have to show is this: the degree of f1 star f2 the composition of paths inside $\mathbb{S}^1$, g_1 one plus g_1 two. So that is what you have to show.

So, after taking the lifts what we want to do is take h of t to be g2 of t plus degree of f1--- add that number, you shift, g2 will be shifted by a number namely degree of f1, by a constant. If you shift a lift by constant integer, it will still be a lift of the same function namely exponential of h of t is the same thing as exponential of g2 of t which is f2 of t. The degree of f1 is an integer ---e raise to 2 pi i times integer is always 1. So f_2 of t will get multiplied by 1. So it does not change.

Degree of f1 is nothing but the endpoint of g1. So this is what my definition of h of t is. Then what I observe is that exponential of h is also equal to f2. Therefore, what we get is exponential of g1 star h will be equal to, (you have to recall how star is defined, and apply exponential function f), it will be exponential of g1 star exponential of h. That will be f1 star f2. So I have got a lift of f1 star f2. Instead of taking a lift of f1 star f2 arbitrarily I have here constructed the lift of f1 star f2 using the lifts of f1 and f2.

All that I have to ensure is its starting point is 0. g1 star h at 0 is same thing as g1 of 0. But g1 of 0, we know is 0. So this is also satisfied. Therefore, if I take the endpoint of g1 star h that must give me the degree of f1 star f2, degree of f1 star f2 is nothing but the endpoint of g1 star h, which is h of 1 now, because end of composition of two paths is the end point of the second one. But h of 1 is g2 or 1 plus g1 of 1 which is degree of f2 plus degree of f1. So the proof   that the degree is a homomorphism is over. Now we shall prove that this function is an isomorphism.

(Refer Slide Time: 6:05)



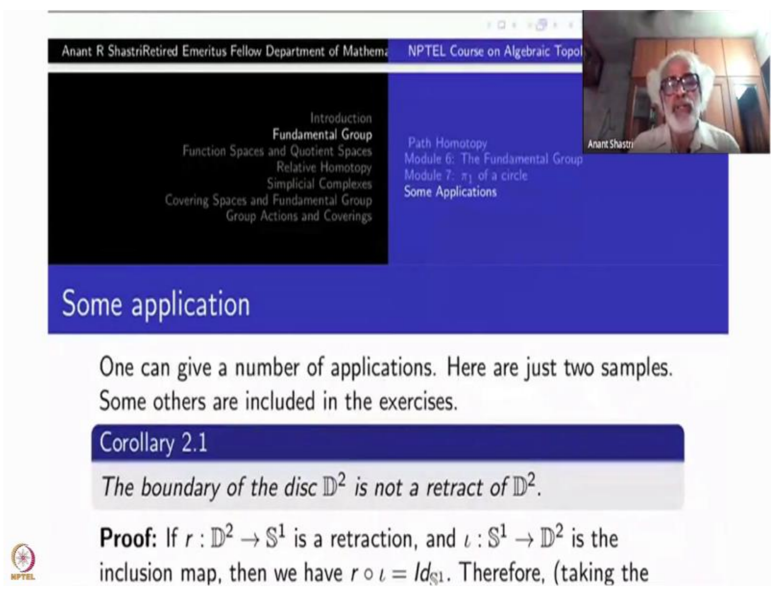
What happens to the identity loop, z going to z on $\mathbb{S}^1$? Namely, if you take as interval, t goes to $e^{2\pi\imath t}$. What is its lift with starting point 0? It will be just t going t. Therefore with t equal to 1, degree will be, that endpoint will be, 1. That means the degree of the identity loop is 1. Any homomorphism into $\mathbb{Z}$ which assumes the value 1 must have the image the whole of $\mathbb{Z}$. It must be surjective because 1 is in the image. So that proves that the degree map is surjective.

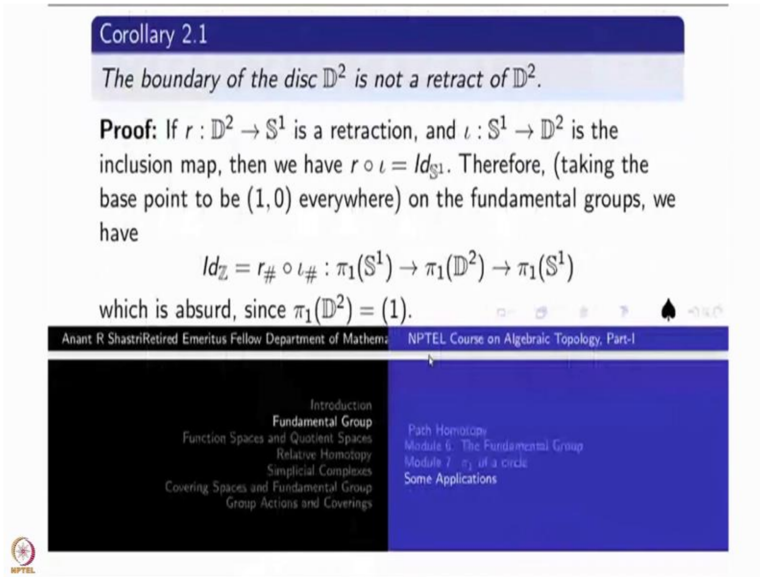
Now finally suppose degree of f is 0 what does that mean? The lift of f, namely let us call it g. Starting point of g is 0, and it must have endpoint also 0. That is the meaning of degree of f is 0. Degree of f is g of 1. So I have a loop, I have a loop here in $\mathbb{R}$. $\mathbb{I}$ to $\mathbb{R}$ you have a function; both the endpoints are 0. g 1 and g 0 are both equal to same point 0 (need not be 0,   if g has same point is enough). We have seen earlier, that such a map is null-homotopic as a loop, as a path homotopy, to a constant path. For this you take s times g of t plus one minus s times g of 0. That is the way it looks. We have seen that one several times now. We are using it now. Therefore there is a homotopy of g to the constant path. Namely, G of t comma s is s times g of t. (Because both end points are 0, I do not have to add up.) So this is a homotopy of the constant map 0 with g relative to the endpoints, that is a path homotopy.

If you take exponential of this homotopy that will give you the homotopy of the constant function 1 with the original function f, the loop f. Because the function G is a relative homotopy namely fixes is 0 and 1, exponential of that also fix 0 and 1. So this is a path homotopy. Therefore the class of f what we have taken, must be the 0 element of the group, I mean additive group whatever, it is the identity element of the group. We showed degree of f is 0, the class f must be also the identity element of the group pi one. So this proves that (degree being already homomorphism,) it is injective now. The kernel of a homomorphism is 0 means it is injective.

So this proves that the degree map is an isomorphism and completes the computation of π_1 of $\mathbb{S}^1$. So, π_1 of $\mathbb{S}^1$ is an infinite cyclic group. So that is the conclusion. So the first non-trivial fundamental group that we have computed is infinite cyclic group. For all other spaces, (we have contractible spaces, convex subsets and so on,) the fundamental group was trivial.

(Refer Slide Time: 10:43)





Let us give just a few examples---applications here. These would keep coming again and again, will have lot of applications. But for that we have developed other things. Right now we can give one very important application, namely, Brouwer fixed point theorem for 2-disks, two dimensional disks, not in general. General case will take more time. So the boundary of the disk $\mathbb{D}^2$ is not a retract of $\mathbb{D}^2$. That is what we will first prove. The boundary of $\mathbb{D}^2$ is not retract of $\mathbb{D}^2$. What is the meaning of a retract? That there is no function r from $\mathbb{D}^2$ to $\mathbb{S}^1$ which is identity on $\mathbb{S}^1$. $\mathbb{S}^1$ is the boundary. This is what we want to prove. There is no map, no continuous function from the entire disk to the boundary such that it is identity on the boundary. Suppose there is such a thing, namely r from $\mathbb{D}^2$ to $\mathbb{S}^1$, a retraction. Let us denote the inclusion map of the boundary $\mathbb{S}^1$ into $\mathbb{D}^2$ by i. Retraction means, r composite i is the identity of $\mathbb{S}^1$. So you can take the base point as 1 comma 0, the unit complex number, on both sides.

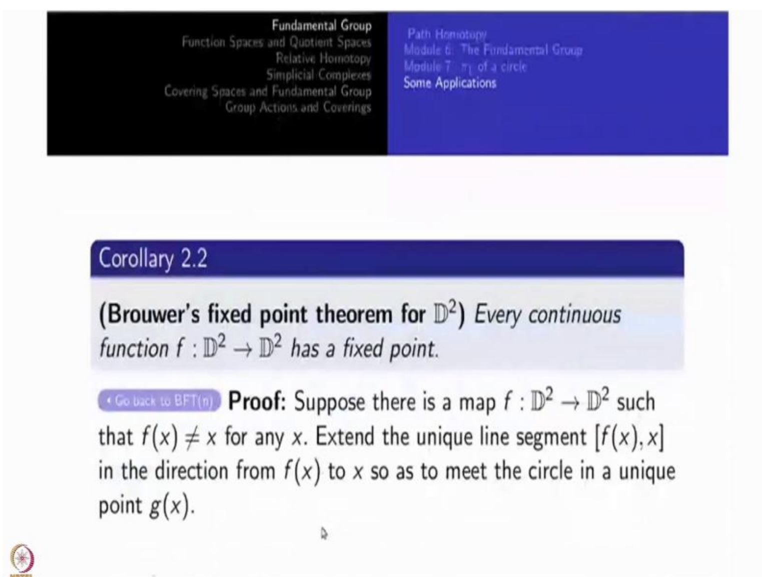
Then what we get? Identity homomorphism of the integers is r composite i check. See identity map induces identity homomorphism on the fundamental groups. But identity map is r composite i. So when you take check of that, it is r check composite i check. i check goes from π_1 of $\mathbb{S}^1$ to π_1 of $\mathbb{D}^2$. Then r check goes from π_1 of $\mathbb{D}^2$ to π_1 of $\mathbb{S}^1$ back. And this is identity because r composite i is identity. Now we will see that there is a contradiction to the algebraic fact that we have proved, that the two end groups here, this one and this one, they are infinite cyclic groups. The middle one is what?

If you take a disk, it is a convex set. So π_1 of this is the trivial group. If you write (0), or multiplicatively, you may write (1), you can write, for a trivial group. From an infinite cyclic group any homomorphism to a trivial group is trivial. Then whatever you compose back here is a trivial map. So composite will be trivial because it is going through trivial group. But we have the identity of Z. Identity, remember, this is infinite cyclic group π_1 of $\mathbb{S}^1$. Identity map is trivial, that is a contradiction.

So look at how a topological fact here, topological hypothesis here converted into some algebraic facts and then used those algebraic facts to conclude that whatever we started here is wrong. If you see, this does not fit here. So you see these are the illustrations of how algebraic topology works. We will have many such things. So π_1 of a disk is trivial group, this is also used. But if we did not know what this group is, we will not have been able to tell anything.

We should know, to have this conclusion, we should know that π_1 of $\mathbb{S}^1$ is a non-zero group, non-trivial group that much you should have known. We did not know that, so you have to compute it. Now, actually now it is infinite cyclic. All that we used is that it is non-trivial. Let us see how to get a very good theorem out of this one, namely Brouwer fixed point theorem. So this corollary was a step towards that one.

(Refer Slide Time: 16:15)



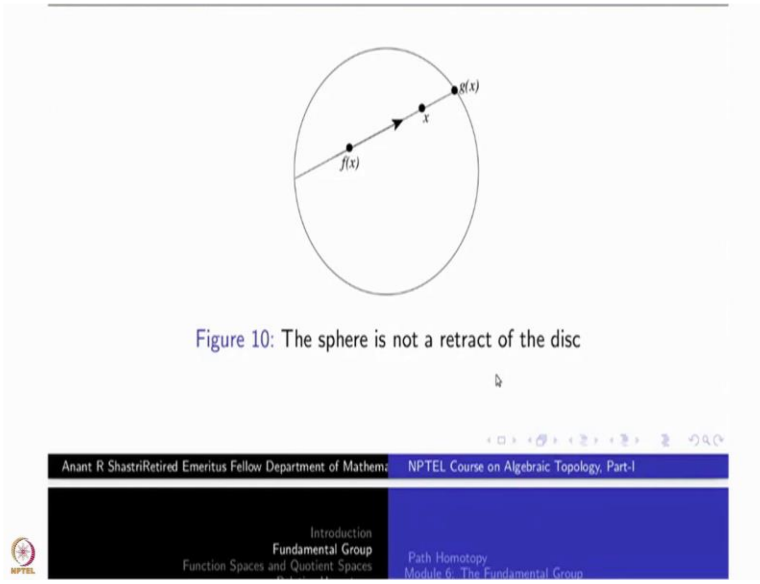
Brouwer fixed point theorem says that for any disk, a continuous function, self-map, has a fixed point. This you have studied for closed interval 0 1, or any closed interval [a ,b] also, in your Real Analysis course by just using Intermediate value theorem. But Intermediate value theorem will not work in the case of of $\mathbb{D}^2$ because there is no order in higher dimensional disks, $\mathbb{D}^2$, $\mathbb{D}^3$, etc,. It will not work. So what is involved here? It will come out if you look at these kinds of proofs?

So how to show that any continuous function from $\mathbb{D}^2$ to $\mathbb{D}^2$ has a fixed point? Its again, by contradiction. Suppose there is a map such that f x is never equal to x for any of x. x and f x are different points of the disk. Any two points inside $\mathbb{R}^2$ will determine a unique line. So look at that unique line. Inside that line you can take only the portion between f x and x. So, that is a segment, line segment f x to x. Now you can give it a direction, namely trace it from f x to x. (You can trace it from x to f x, or f x to x.)

So you trace it from f x to x. Keep extending it, the same line till you hit the circle, the unit circle, the boundary of the disk. The boundary of the disk will be hit by, whether you go forward or backward, in two different points. I want the forward point, so from f x to x. That is, nearer to x than to f x. There are two points on this line segment, the two points on the circle. Intersection of the circle with the entire line. So I want the point which is towards the point x rather than towards the point f x. That is all.

Let us call that point g x. So I have constructed a function, x going to g of x, out of the function f only after assuming that f has no fixed points. The whole thing, you know, geometrically it is easy to see that whatever you have done is a continuous operation, namely g will also be continuous. But you do not have to rely on your geometric intuition here. You can actually prove that g is a continuous function because f is a continuous function. Let us see how.

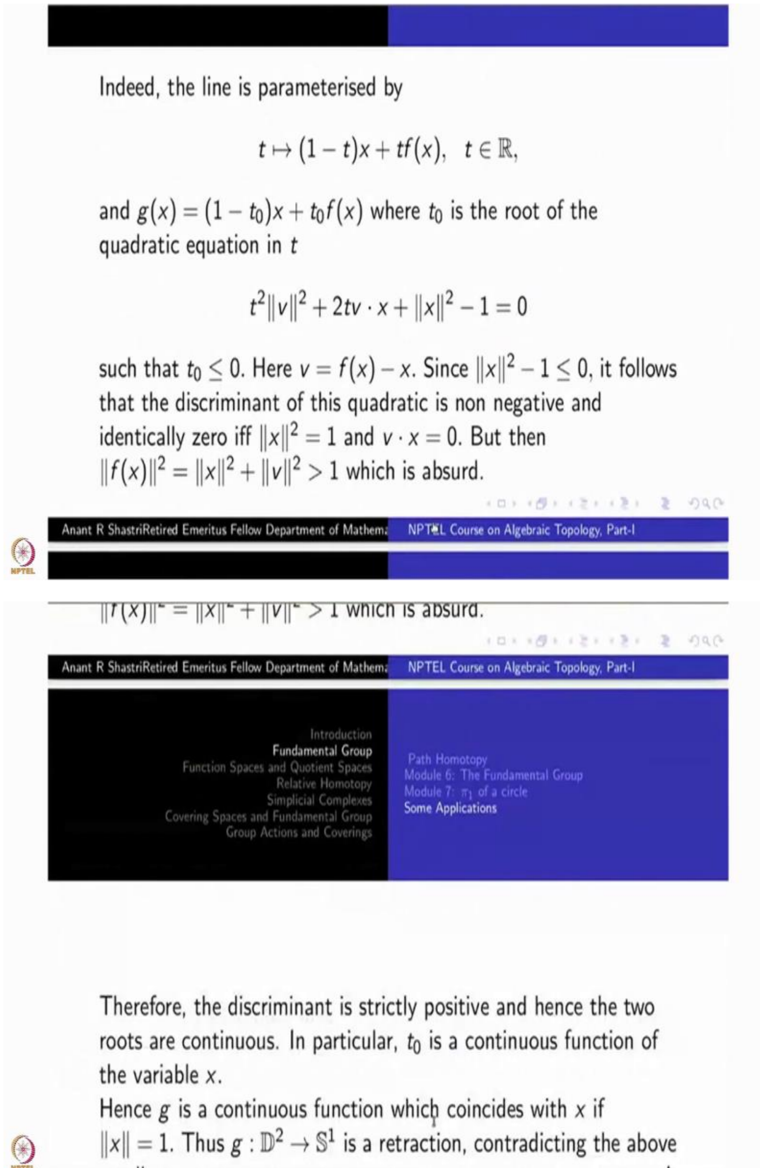
(Refer Slide Time: 20:08)



So first of all, let us have a look at what we have done in this picture, x is not equal to f x but both of them are in the disk. I am not saying that they are in the interior of the disk. They are in the closed disk. Then some of them may be on the boundary also, does not matter. The picture shows both of them are inside, that does not mean that they should be always so.

Extend that line in the direction f x to x and hit the point g of x which is on the boundary. So this function g is expressed in this picture by geometric means. What I want to say is, you can actually write down a formula for g in terms of x and f x. Finally it is a function of x only. Because f x is already a function of x. So let us see how.

(Refer Slide Time: 21:16)



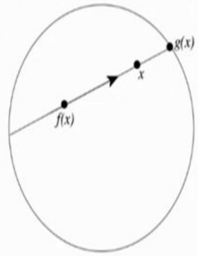


Figure 10: The sphere is not a retract of the disc

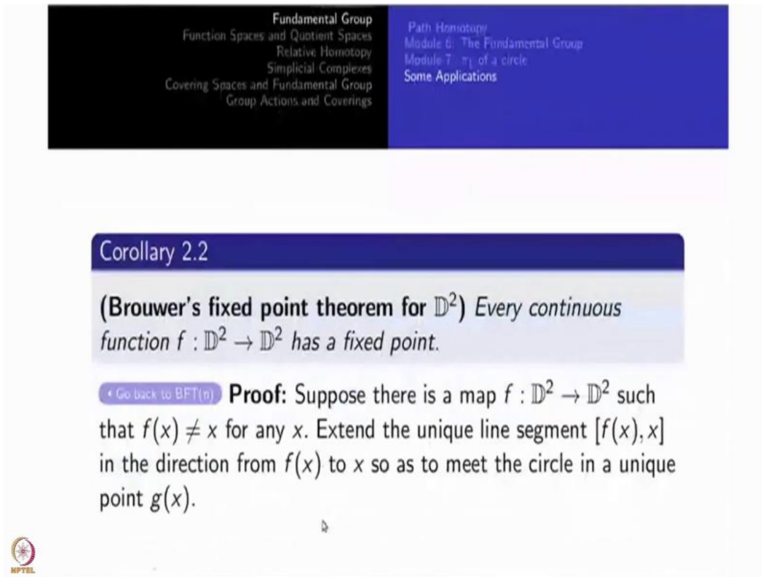
Corollary 2.1

The boundary of the disc $\mathbb{D}^2$ is not a retract of $\mathbb{D}^2$.

Proof: If $r : \mathbb{D}^2 \to \mathbb{S}^1$ is a retraction, and $\iota : \mathbb{S}^1 \to \mathbb{D}^2$ is the inclusion map, then we have $r \circ \iota = Id_{\mathbb{S}^1}$. Therefore, (taking the base point to be $(1,0)$ everywhere) on the fundamental groups, we have

$$Id_{\mathbb{Z}} = r_\# \circ \iota_\# : \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{D}^2) \to \pi_1(\mathbb{S}^1)$$

which is absurd, since $\pi_1(\mathbb{D}^2) = (1)$. ♠



So the parameterization of this entire line is: 1 minus t times x plus t times f x, where t ranges over ℝ. Given any two vectors u and v, t times this plus 1 minus t times that gives you the entire line passing through the endpoints of these vectors u and v. So that is the vector algebra, so 1 minus t times x plus t times f x, this is the line segment. This is the entire line. But I do not want the entire line. I want t naught such that the norm square of this is equal to one. That t naught is the root of a quadratic equation in terms of t.

What is that equation? The norm of gx must be equal to 1 because it is on the circle. The norm of this vector is nothing but ---you see we can rewrite this whole thing as-- (taking v equal to fx minus x) $t^2\|v\|^2 + 2t(v\dot{x}) + \|x\|^2 - 1$=0.

t square into norm of v square plus twice t into v dot x plus norm of x square minus 1 equal to zero. So t naught is a root of this. When t equal to t naught you get this one. So this is the equation. It is a quadratic equation. Norm v is given. What is norm v square? Namely, $\|v\|^2 = \|f(x)\|^2 - f(x) \cdot x + \|x\|^2$.

So I have put what is this v? v is the vector f x minus x. You re-check what we have done here. 1 minus t times x plus t times f x can be written as x plus t times f x minus x. That is why I have taken that. You rewrite it with fx minus x as v. Then the computation of this norm becomes easy. So, solutions of a quadratic equation wherein the coefficients are functions of x automatically will be smooth functions, if the coefficients are smooth functions.

If they are continuous, the solutions will also be continuous. Because we can get them by a formula, b plus minus square root of b square minus 4 times a c divided by 2 a. Whatever that is, that is what we have to do. Now x squared minus 1 non-positive, because x is inside the circle, in the disk of the radius 1. It follows that the discriminant of this quadratic equation is non-negative and identically 0 if and only if norm x square is equal to 1 and v dot x is 0. (v dot x square is equal to norm v square times norm x square minus 1 implies both sides are zero. But then what happens is modulus of f x square will be equal to modulus of x square plus modulus of v square. (v is f x minus x and if v is perpendicular to x then norm of f x square will be norm of x square plus norm of v square.) But then norm of x square plus norm of v square will be greater than 1 which is absurd, because this whole thing is less than 1, that will not happen. Therefore, discriminant is strictly positive and hence the two roots are continuous.

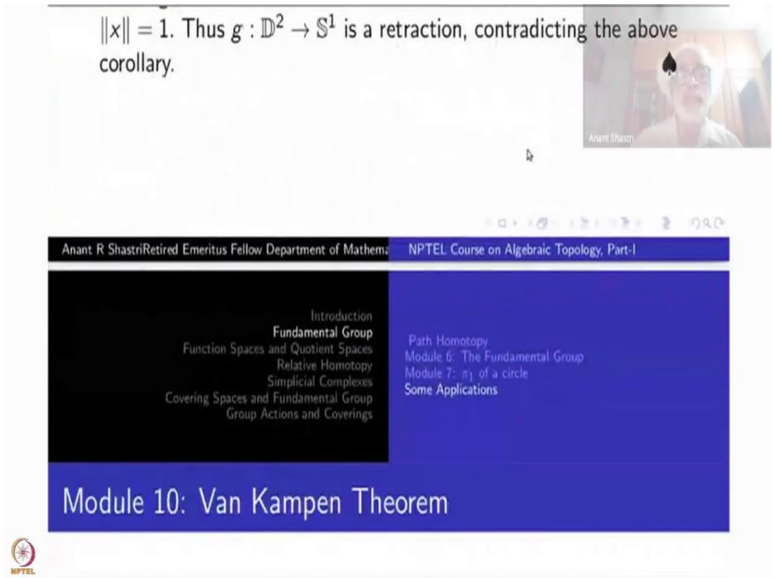
See whenever a root is equal to 0, there will be a problem about continuity,-- functions are continuous or discontinuous. If it is strictly positive, you can take the square root, that will be positive function. It will be a continuous function. So I have given you full justice why the roots of this polynomial, of this quadratic equation are continuous functions of the variable x here. The discriminant is strictly positive and hence the two roots are continuous.

In particular, the t naught which is a solution of the equation is a continuous function of the variable x. It depends upon x. We are writing it as a constant. But as x changes, the point g x will also change. This g is nothing but, you know, g is nothing but 1 minus t naught times x plus t naught times f x, finally. So g is continuous function.

Now you can verify this either by using the quadratic equation or by directly using the picture. If x is already on the circle, f x may be anywhere in the disc. What happens to the line segment? When you extend it, x itself is a point on the circle. Beyond that it will go away from the disc. So this intersection is g x will be equal to x, g x is equal to x if and only if x is on the circle. I have given a function g which is identity on the circle and it is a continuous function from the entire disk to circle, g takes values in the circle.

Its domain is the entire disk. And on the circle it is identity. And that is a contradiction to the first corollary here, there is no such function, there is no such retraction of $\mathbb{D}^2$. So why we get a contradiction? We started with an assumption, that assumption must be wrong: `f x is not equal to x for any x' was the assumption. What is the conclusion? There must be a point x such that f x is equal to x. So that is the conclusion.

(Refer Slide Time: 28:28)



Next time we will give another method to compute the fundamental groups. Both these methods will be, later on, strengthened far beyond what you see in the beginning. So, next time we will prove what is called Van Kampen Theorem, starting with a very simple version of that-- there are many other versions later on. Yeah, thank you.