Introduction to Algebraic Topology (Part I)
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Lecture 8
Computation Continued

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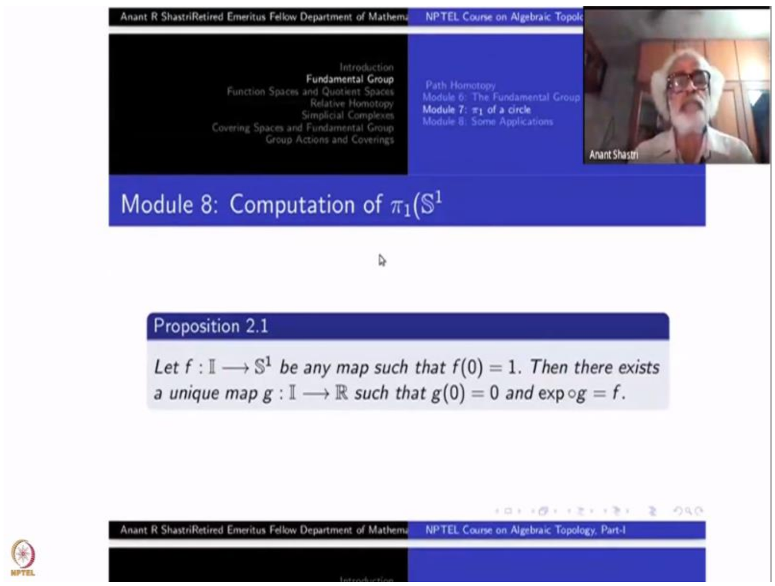

Welcome to lecture number 8 module 8. Last time we started the computation of the fundamental group of a circle, we have introduced exponential function E: $\mathbb{R} \to \mathbb{S}^1$ namely $\theta\theta$ going to $e^{2\pi\imath\theta}$ and verified a few basic properties of this map which is going to be used very heavily in this computation.

We also saw that any two lifts of a function from a connected space to $\mathbb{S}^1$ through the exponential function will differ by an additive constant. In other words, if you fix one point--- the value of one point, then the lift is unique. There cannot be two lifts with the same value at a single point. This much we have seen, which will be used in the proof of the existence theorem.

Now, we want to show that given any continuous map f: $\mathbb{I} \to \mathbb{S}^1$ such that, (let us assume this-- it is just a easy way of carrying on this one) f(0) is 1. (If it is not 1 we can also do that, that is not an essential part). Then there exists a unique map g : $\mathbb{I} \to \mathbb{R}$ such that g(0)

= 0 and E ○ g = f namely every map can be lifted uniquely after specifying the starting point. The unique part we have already seen. We also introduce what is our plan of proof.

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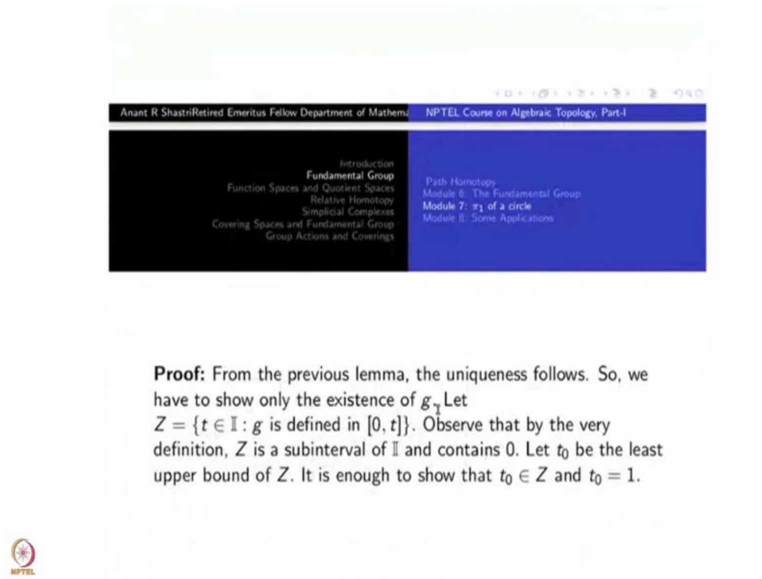

Look at the set Z, the set of all points t inside the closed interval $\mathbb{I}$ such that g is defined in the closed interval [0, t]. So, this is a subset and it is non empty because we can take t = 0 then of course, we know that g(0)= 0 can be defined so that exponential of g(0) which is exponential of 0 is 1 which is f(0). So, that verifies that Z is non-empty. Our idea was to prove that Z is both open and closed. $\mathbb{I}$ being connected, a non-empty open and closed subset must be the whole space.

So, there is a plan using the order in $\mathbb{I}$. We can even simplify this idea even further by the following idea namely, let us look at the supremum, the least upper bound of the set to Z. The set Z is bounded therefore; it has a least upper bound. That upper bound may not be inside Z it will be inside the interval [0 1]. We will claim that this upper bound t_0 is actually inside Z. That corresponds to almost proving that Z is closed, a closed interval.

So, the old idea is slightly changed into proving that the supremum belongs to Z and the supremum is nothing but 1. So, that will prove that Z itself is the whole interval [0, 1] which is the same thing as saying g is defined on the whole of interval [0, 1]. So we want to prove that the supremum of Z is inside Z and that supremum cannot be smaller than 1. It has to be 1 of course. It is bigger than 0 because already 0 is there.

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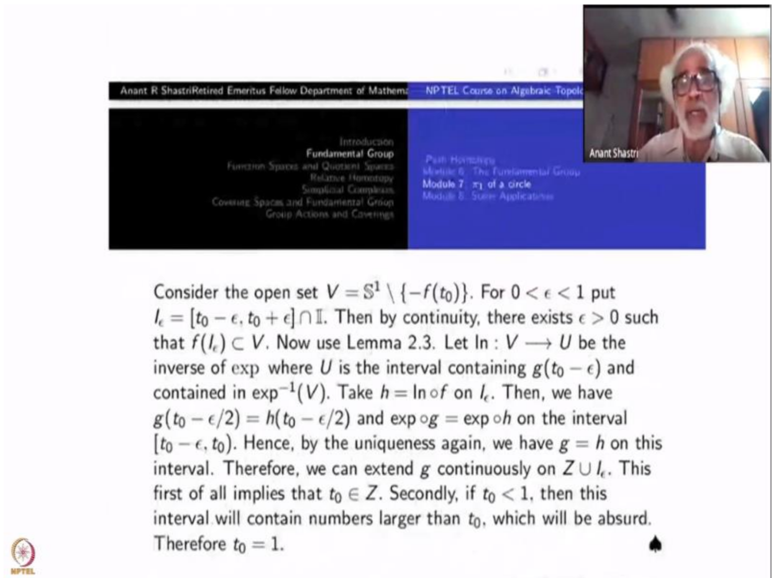

Now, consider the open set V which is $\mathbb{S}^1 \setminus f(t_0)$, where t_0 is some point in the interval which is the supremum of the set Z. $f(t_0)$ is defined. Look at the point, - f(t_0),--- throw away that point. Then you get a big arc. On that arc we have a log function log function is from the arc to $\mathbb{R}$,-- back to $\mathbb{R}$. So, there are many log functions. Which branch you would like to choose? So, let us look at this one.

So, let 0 < epsilon < 1. Let us have this notation namely $I_{\epsilon\epsilon}= (t_0 - \epsilon, t_0 + \epsilon) \cap \mathbb{I}$. I would just like to have this much but it may happen that my choice of epsilon is somewhat big and it will go out of the interval $\mathbb{I}$. So, I will intersect this with $\mathbb{I}$. If your epsilon is sufficiently small this will be completely inside $\mathbb{I}$ there is no need to intersect with $\mathbb{I}$. That is all.

Look at this V. It is a neighborhood of the point t_0. Therefore, by continuity, there exists some $\epsilon\epsilon$ positive such that the entire of $f(I_\epsilon)$ will be contained inside this open set V. So, this is not because of continuity-- it is actually an open set. I am taking this t_0 which goes into that open set.

So, some $f(I_\epsilon)$ is contained inside V--- that is by continuity. V is open, f (t_0) belongs to this set,--- I have thrown away - f (t_0). So, f (t_0) belongs to V. So, $f(I_\epsilon)$ will be contained inside V for some $\epsilon>0\epsilon$. Now, you may use lemma 2.3 . Look at the log functions-- there are several of them-- inverse of the exponential function. There will be one copy from V to U which is the inverse of exp where U is the interval containing g(t_0 -$\epsilon/2$).

g(t_0 - $\epsilon/2\epsilon$) makes sense because t naught is the supremum of the set Z. So, everything smaller than t_0 will be inside Z. Therefore, g is defined there. So, g(t_0 -$\epsilon/2$) makes sense, this will be in one of those various inverse images, so, I am choosing U to be one of those intervals. And they are logarithms now, from V to U will be an inverse of exponential. So it will be a 1 -1 mapping.

So, g(t_0 -$\epsilon/2$) is such that this interval U is contained inside exponential inverse of V. Exponential inverse of V has all these disjoint intervals. All that I do is now put h equal to logarithm composed with f namely, h = ln ○ f. f is defined on this subset $f(I_\epsilon)$ will go inside V. We will take $f(I_\epsilon)$ going here. Then we have g(t_0 - $\epsilon/2$) will be h(t_0 - $\epsilon/2$) and exponential of g equal to exponential of h on this interval.

Therefore, by uniqueness at one point if they agreed, they must be agreeing everywhere. Hence by uniqueness again we have g equal to h on this interval. There are two functions now, at one point they agree, so they must be agreeing on the interval. Therefore, we can extend the function g on Z, (it is already defined) over Z union $\mathbb{I}_\epsilon$ this interval. So, this is the trick.--- pick up a point that is the end point of the definition, you can extend it slightly that is the whole idea.

Now the end point. I do not know whether g is not defined at the end point. I do not know. So, supremum may not be inside Z, but any smaller value should be ---if you move to the left that will be inside Z. So from that I can extend g on $\mathbb{I}_\epsilon$. That means I epsilon is contained in Z. …..

So, this first of all implies that t_0 is not the end point, if $\mathbb{I}_\epsilon$ contains t_0 in the interior. Then t_0 will not be supremum? Therefore, t_0 must be the endpoint, the endpoint of this interval. The moment t_0 is smaller than 1, $\mathbb{I}_\epsilon$ will be larger. So, it will go beyond t_0 therefore, t_0 is less than 1 will give you a contradiction because there will be a larger number inside Z than t_0. So, t_0 must be 1.

So, in one single go we are getting both--- essentially showing that the set Z is both open and closed. This was easier than that part. So, we are using this one. So, essentially we are using the local behavior of the logarithm function. So, if the lift is defined there, we can extend it to the full interval. Inside that interval, the chosen logarithm function is a homeomorphism. It is a unique way of extending. There is no ambiguity about that one. So, these things are all used very rigorously here. So that proves the existence of lifts. Up to an additive constant they are unique also all. All paths can be lifted up to additive constants. So, now we will make a derivation.

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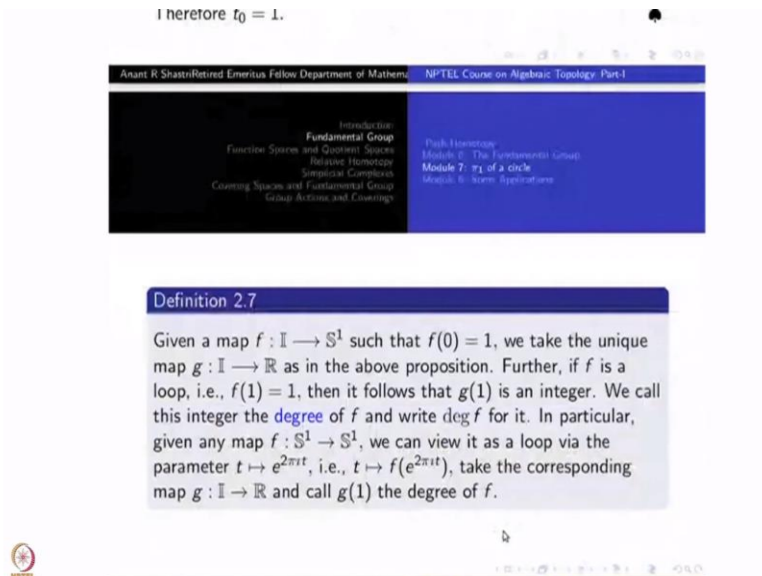

I am prepared this much: namely a passage from loops inside $\mathbb{S}^1$, now we can go to some maps inside $\mathbb{R}$. But what happens if we start with a function from $\mathbb{I}$ to $\mathbb{S}^1$ such that f(0) is equal to f(1). We take the unique map g from $\mathbb{I}$ to $\mathbb{R}$ such that $E \circ g = f$ and $g(0) = 0$. Further, if f is a loop, suppose f(0)=1, you started with a loop means f (1) is also equal to 1, then g (1) will be also an integer because must be equal to 1.

We call this integer the degree of f. Now, this is a strange thing, we took some g but that g is unique and that is why we can define the value of g(1) as the degree of f. In particular, suppose you start with a map from $\mathbb{S}^1$ to $\mathbb{S}^1$, that is a loop. We will remember that one. We have a map from $\mathbb{I}$ to $\mathbb{S}^1$ with both 0 and 1 going to same point, it can be thought of as a map from $\mathbb{S}^1$ to $\mathbb{S}^1$. We can use it as a loop via parameter t going to .

So, that is t going to f( ). Take the corresponding map $g : \mathbb{I} \to \mathbb{R}$, call $g(1)$, the degree of f. So, here we would like to convert a map from $\mathbb{S}^1$ to $\mathbb{S}^1$ into a map from $\mathbb{I}$ to $\mathbb{S}^1$ and then lift it. From $\mathbb{S}^1$ to $\mathbb{S}^1$ we do not want to lift it. Maps from $\mathbb{S}^1$ to $\mathbb{S}^1$, we cannot lift them because the endpoints may go to a different points. That is the whole idea, of one the endpoints will go to a different point when you lift it. So, to allow that you have to think of this loop as a path from $\mathbb{I}$, a function from $\mathbb{I}$ to $\mathbb{S}^1$ rather than $\mathbb{S}^1$ to $\mathbb{S}^1$. The role is revised here.

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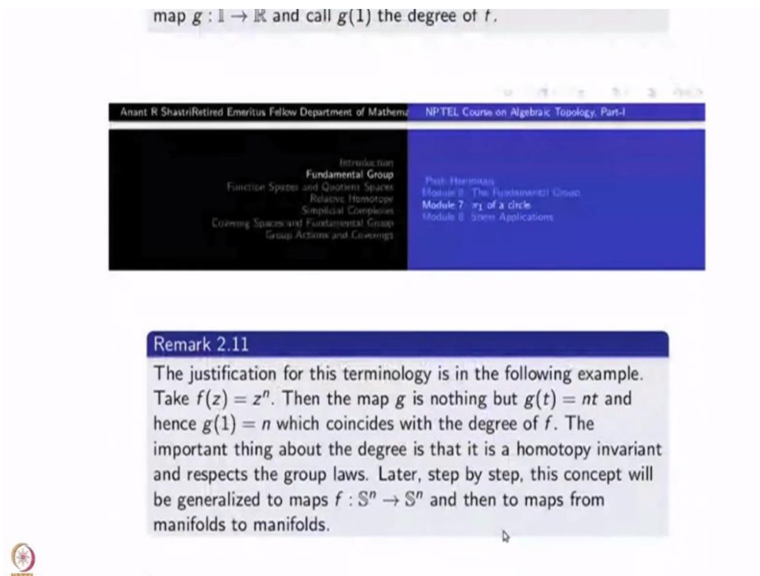

Justification for this terminology, why the term `degree' that needs some explanation? So, here is an example, let us take $f(z) = z^n$ from $\mathbb{S}^1$ to $\mathbb{S}^1$ then the map g is nothing but $g(t) = n \cdot t$ and hence g(1) will be such that is 1. What is g(1)? that is all we have to know.

So, the lift of the map z going to z^n from $\mathbb{S}^1$ to $\mathbb{S}^1$ converted into an map from $\mathbb{I}$ to $\mathbb{S}^1$ is nothing but $n \cdot t$ such that n is the degree, the degree of this polynomial z^n . So, this is it, now, this much justification is good enough. Slowly you will understand that this degree is really a good name. Later, step by step this concept will be generalized to maps from $\mathbb{S}^n$ to $\mathbb{S}^n$ also and then to maps from manifolds to manifolds. This degree concept is very, very important.

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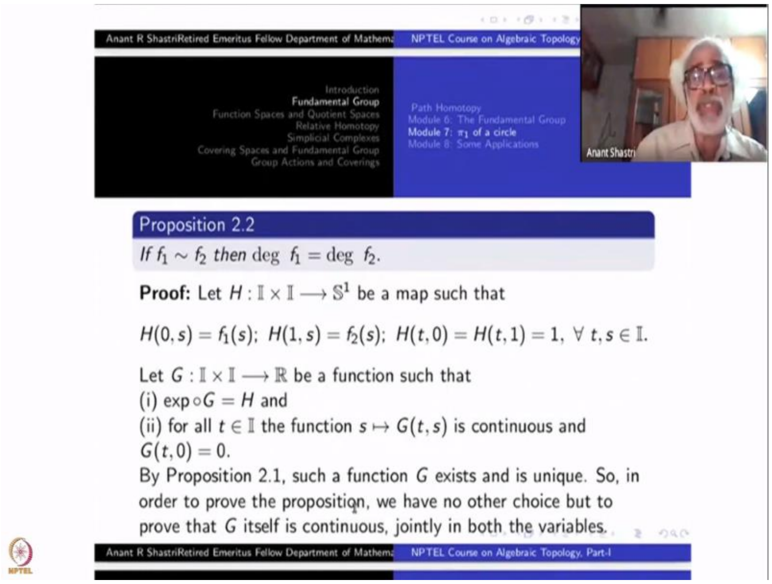

The first thing is we have taken a function from $\mathbb{S}^1$ to $\mathbb{S}^1$, map from $\mathbb{S}^1$ to $\mathbb{S}^1$ and then assigned a degree to it. This degree, this association, the number is a homotopy invariant, path-homotopy invariant. So, this is what we want. So, this is the proposition 2.2 . If f_1 is path Homotopic to f_2 then the degree of f_1 is equal to degree of f_2. Let us go ahead and we will take care of this aspect later on.

Let us start with a homotopy: Let $H: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{S}^1$ be a map such that H (0, s) =f_1(s), H (1, s) = f_2 (s), H (t, 0) and H (t, 1) equal to 1 for every t and s. That is the definition of a path homotopy, from f_1 to f_2. Now, let $G : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{R}$ be a function such that exponential of G equal to H. And for all t inside $\mathbb{I}$ the function s going to G (t, s) is continuous and G (t, 0) = 0.

I am taking a function $G : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{R}$, I am not telling that is continuous. But if you fix t, then as a function of s this is continuous. And G(t, 0) is 0 for each, each t. I am defining a lift of the function H(t, s), where s goes to H (t, s), t is fixed, so that is a path, and that path can be lifted. Each fixed path for then, (first coordinate is fixed,) you can lift it by the previous proposition such a G is defined. G(t, 0) , the starting point is 0 that is what I do in proposition 2.1 Such a function G exists.

So, in order to prove a proposition, we have no other choice, but to prove that G itself is continuous. If I show that G itself is continuous, then what I get is a Homotopy of lifts, this is what we wanted to prove. So, if capital G itself is continuous, and exponential of G is H, then capital G will be a lift of H. So, we have no other choice, but to prove that G itself is continuous, jointly in both variables. That is the point. Only when you fix t, s going to G(t, s), is continuous. That much we know.

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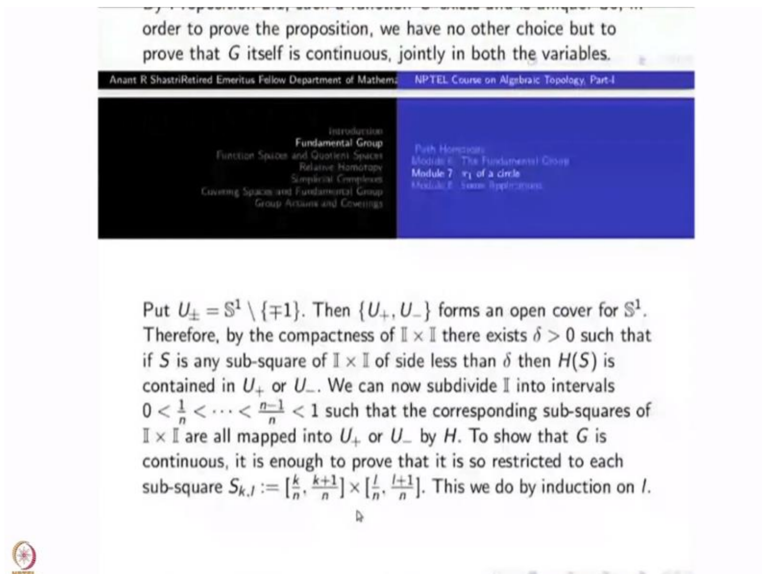

So, this is what we are going to prove now. Put $U\pm$ after throwing away ± 1namely U_+ is $\mathbb{S}^1 \setminus \{-1\}$ and U_- is $\mathbb{S}^1 \setminus \{+1\}$, this minus is throwing away the point (set−theoretic minus). So these two are open arcs U_+ and U_- They will cover the whole of $\mathbb{S}^1$. The inverse image of these things under capital H (started with a homotopy H) that will be an open cover of $\mathbb{I} \times \mathbb{I}$.

So, for compact sets with an open cover you have the Lebesgue number, there exists a number, δ positive, called the Lebesgue number for this cover such that if S any sub square of $\mathbb{I} \times \mathbb{I}$ of side length less than δ then the entire of H (S) is contained inside either U_+ or

U_- . So how do I manage this one? All that I have to do is take the Lebesgue number and take $\delta\delta$ less than the Lebesgue number divided by √2.

Then every sub square of $\mathbb{I} \times \mathbb{I}$ of side length less than $\delta\delta$ has its diameter less than √2 δ δ which will be less than the Lebesgue number. Therefore, they will be contained inside $H^{-1}(U_+)$ or $H^{-1}(U_-)$ -- all these sub squares. This is the same thing as saying that H of this sub square is contained inside either U_+ or U_- . So, let me show you the diagram first then explain this a bit more clearly.

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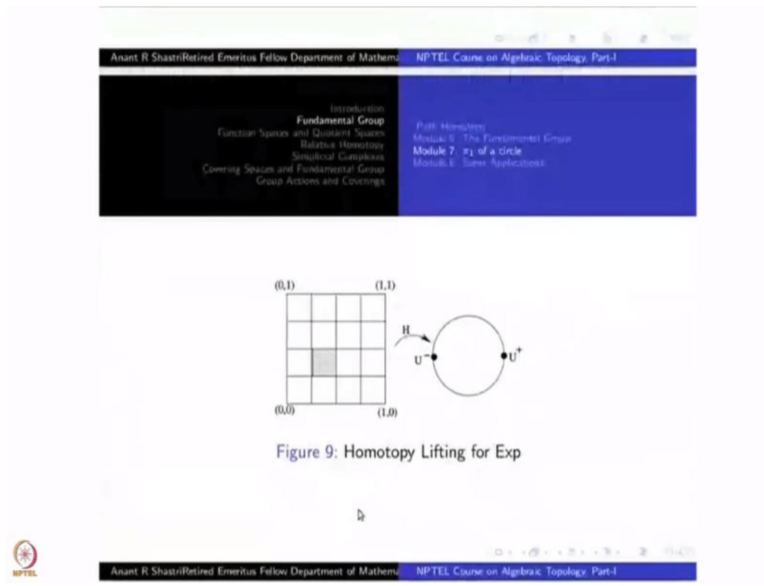

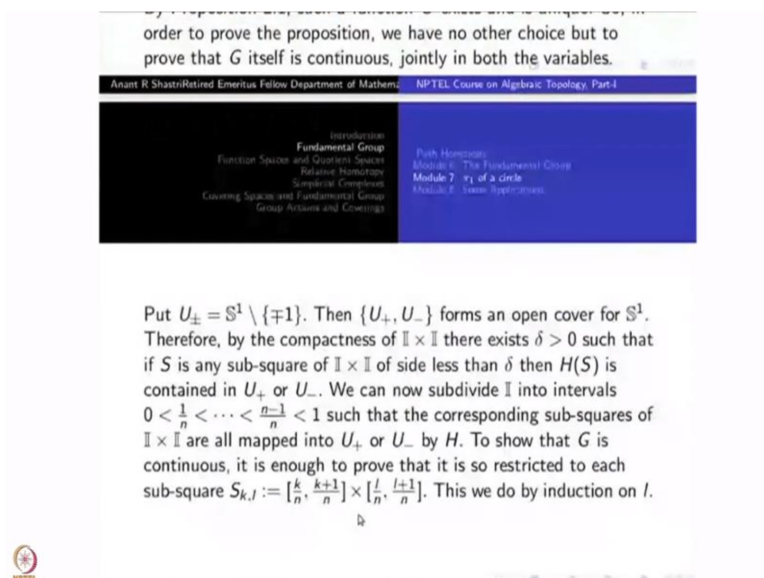

So, what I have done is I have cut down $\mathbb{I} \times \mathbb{I}$ into smaller squares, each small square here, under H, will go either inside this whole U_+ here or the U_-. Remove this point and take all this part either it avoids this -1 or it avoids + 1. It will not be a subset like this starting from here to here, that is the whole idea. It avoids either -1 or avoids the +1, the image of each squared here. Is that understood?

So this is what the Lebesgue number does in several cases in analysis. So, this is part of analysis here. So once I have such a δ what I do is, I will cut down the square. Cut down $\mathbb{I}$ into, you know intervals like $0 < 1/n < \cdots < (n-1)/n < 1$. I have to choose n to be sufficiently large so that 1/n is less than this δ that is all. Then each square will be having side length less than δ, so this hypothesis will be true.

Now, to show that G is continuous: G is defined as a function but we have to show G is continuous. It is enough to prove that restricted to every sub square, G is continuous. How does a subsquare look like? Something like $[k/n, (k+1)/n] \times [l/n, (l+1)/n]$, where l and k are between 0 and n-1. On each square if it is continuous then you are done. There are finitely many squares, they are all closed squares, so, I have to show that H is continuous on each square.

So, how do we start? We start from the bottom here, we first show that the function H is continuous on each of these bottom squares. Using that we will do the same for the second stage, then third stage, fourth stage, and so on. So, proving the continuity of H will be also done inductively on this, the value of s here.

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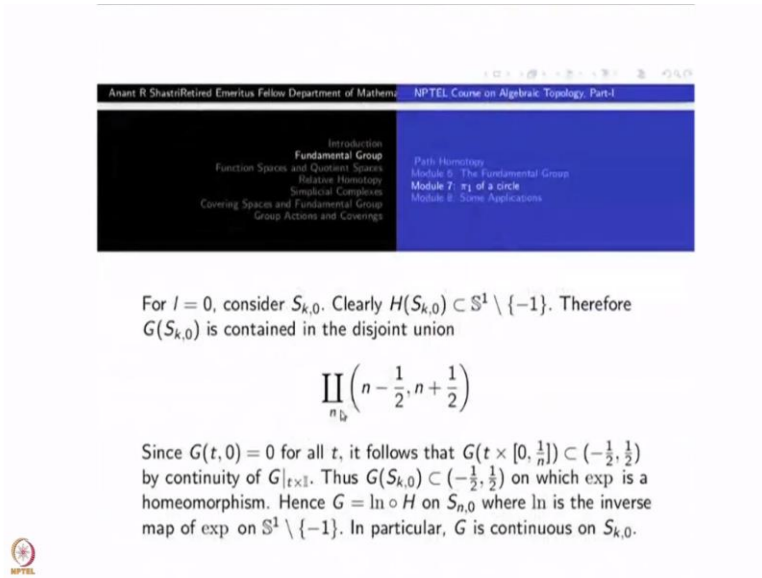

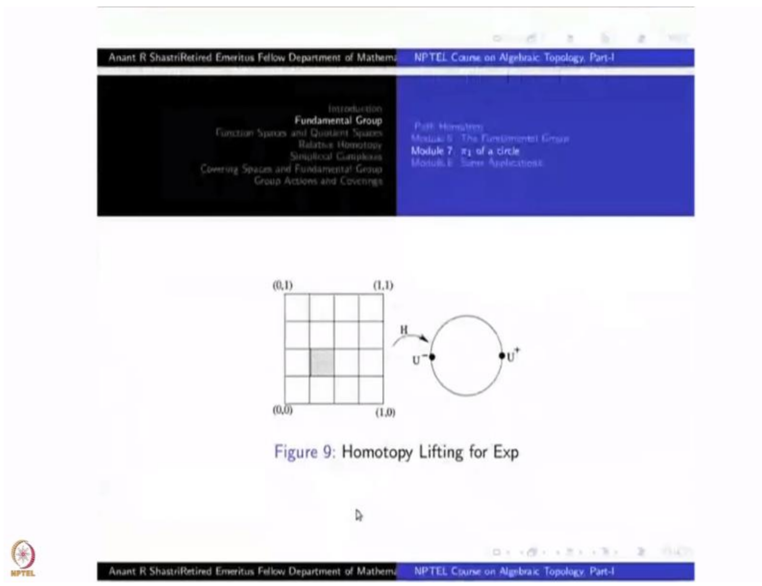

So, what have we done?. Take l equal to 0. Consider $S_{k,0}$. It may be this or this square here or this or this or whatever one of them. $S_{k,0}$, this whole thing (remember all these are initial points, they are here they are mapped to this point), therefore, this entire thing cannot avoid this point 1, which means they have to avoid -1.

Because they cannot avoid +1 because all these points are mapped to this, this point here. Therefore, G($S_{k,0}$) these things are contained in a disjoint union of open sets. The inverse image of U_+, inverse image of U_+ is nothing but all (n-½, n+½) because half integer-points go to -1 under exponential function. So, they are avoided, so all these intervals of length 1, any two of them disjoint, one of them must contain G($S_{k,0}$) because G (t, 0) is g(0) on that square, (all these lines) see. That is the trick here.

G may not be continuous on $S_{k,0}$ and the whole of it, but on each vertical line it is continuous, and the vertical lines are connected. So, if it is contained in the union like this, it must be contained in one of them, only one of them. Which one? Wherever the starting point is, the starting point we know where they are, they have been chosen inside 0, starting point little g has been chosen to be starting point of that.

So, that is what, G(t, 0) is 0 for all t. So, we know that G(t × [0, 1/ n]) is contained in the interval (-½, ½), by the continuity of G restricted to t ×𝕀. This is what is happening. So, G($S_{k,0}$) must be inside of (-½, ½). Once you have one single thing, the exponential function is a homeomorphism with its inverse being log here. Therefore, G itself will look like log ○ H but log function is continuous here.

Therefore, G is continuous on the whole of $S_{k,0}$. ------ Now, what happens? In particular, if you look at this line, this is s = 1/n , So, on this line it is continuous. Now use this fact, and proceed, the proof is the same.

Look at one of the squares here, the same argument will tell you that it will be continuous on each of these squares. In particular, it will be continuous on this line. Once you have that it will give you the hypothesis for the next squares and so on. So, induction starts.

It will be continuous on each of these squares, this argument will be used in a much larger context later on, as an argument, this is all we need. But details will be slightly more complicated technicalities. We will do a big generalization of this one later on, when we study covering spaces.

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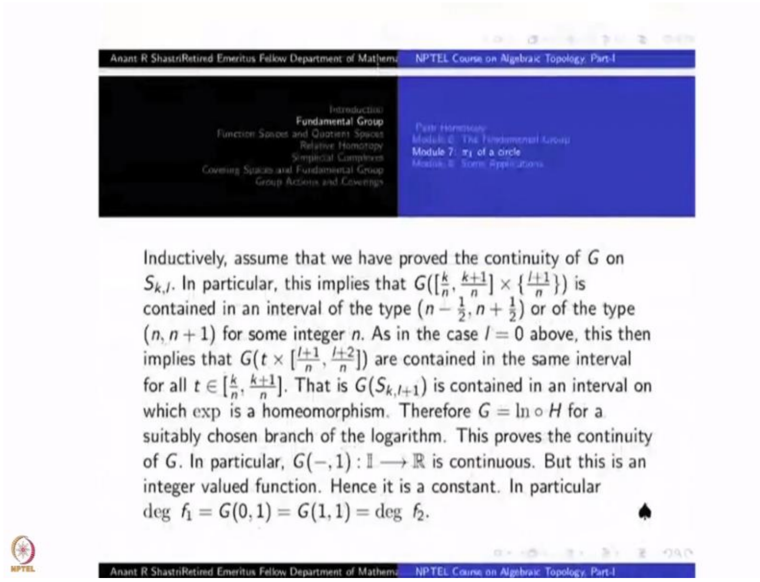

So, what we have got is: G itself is a continuous function on all of $\mathbb{I} \times \mathbb{I}$ to $\mathbb{R}$ exponential of G is the the original H. Now look at the degree of f_1 which is G(0, 1) but that will also be equal to G(1, 1), but that is the degree of f_2. Difference is a constant, that constant must be the same for both of them because they are all in one single G continuous function difference is the constant integer that is what we knew.

So, that integer must be the same. You just look at the end point of G for each t That will give you the various integer degrees of H(t, s) for fixed t. If t= 0 that will be the degree of f_1 and t=1 will be the degree of f_2. This is what we are doing here. G(0, 1) will be the degree of f_1, G(1, 1) will be that of f_2. And then the whole of G (t, 1) is continuous.

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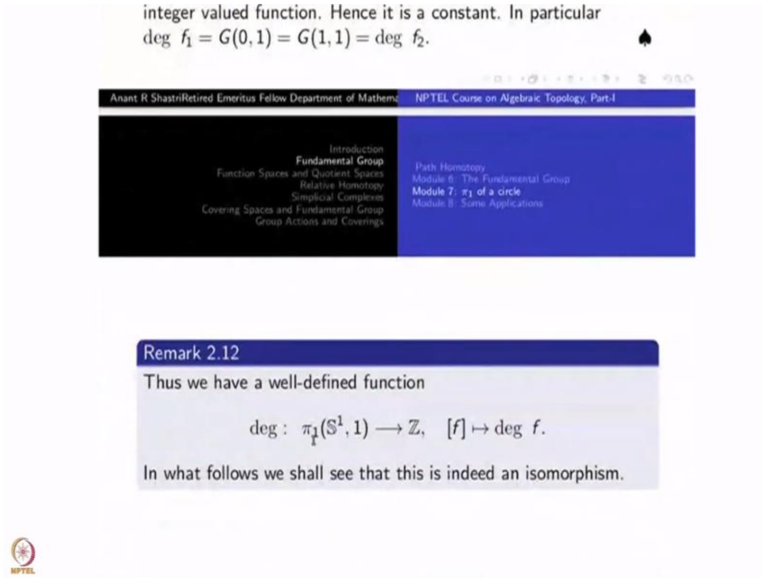

Thus we have a well-defined function from the set of path homotopy classes of maps from $\mathbb{S}^1$ to $\mathbb{S}^1$ to integers namely the class of f goes to the degree of f. Irrespective of what element of the class you choose, because they will be homotopic, the degree will be the same.

So, this is our first attempt in the computation of this group. We have got a function on it to $\mathbb{Z}$. The next step is that we want to show that it is a group homomorphism and it is surjective and injective. So it is an isomorphism, that we will do in the next module. Thank you.