Introduction to Algebraic Topology (Part I)
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Lecture 7
Computation of π_1 of a Circle

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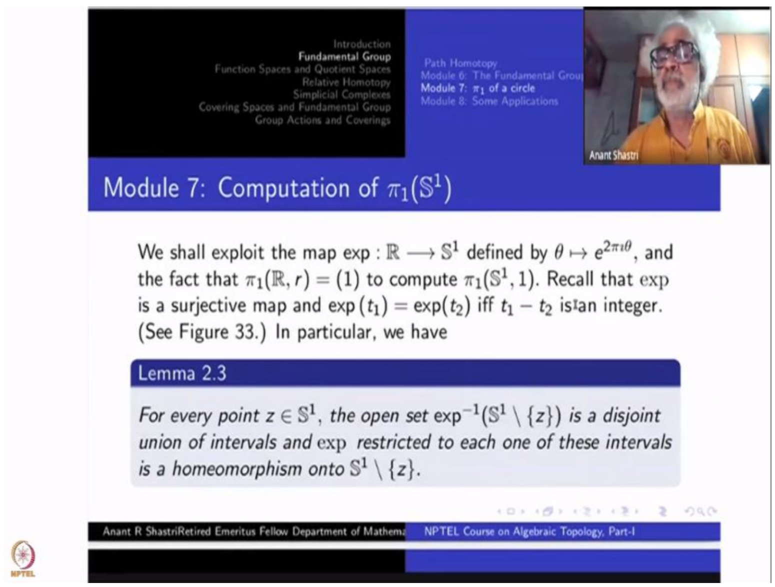

Now, we would like to compute the fundamental group of the circle. The basic tool here is as I have told you the exponential map $Exp = E : \mathbb{R} \to \mathbb{S}^1$, namely, $t \mapsto e^{2\pi\imath t}$ t . The fact that $\mathbb{R}$ is contractible---- therefore, if you take any base point little r, then $\pi_1(R, r)$ is going to be a single point. That is easy to see. Any loop based at a single point we have seen is null homotopic, inside any interval, we have seen that.

So, in particular it is so in $\mathbb{R}$. So, this fact will come to us, very much useful, but when we go down to $\mathbb{S}^1$ under e to the power two pi i t, something strange happens, but not too much strange things. So, it gives you complete control over what is happening in π_1 of $\mathbb{S}^1$ namely the exponential map that gives you the control. So, let us first concentrate on what is the big feature of this exponential map--- it is a surjective function and exponential of t1 plus t2 is what is exponential at t1 into exponential of t2. So, the addition inside $\mathbb{R}$ goes to multiplication inside $\mathbb{S}^1$.

So, it is the group homomorphism. What is the kernel? Kernel is determined by integral multiples of 2pi. Exponential of t1 is equal to the exponential of t2 if t1 minus t2 is an integer because now E(t) is equal to e power 2 pi i t. So, E(t) is equal to 1 if t is just an integer. All the integers go to the same point including the 0, wherever 0 goes to namely 1, so they are all going to the same point. In fact, if the difference is an integer, the image will be the same so this is the meaning of exponential of t1 equal to exponential of t2 if and only if t1 minus t2 is an integer.

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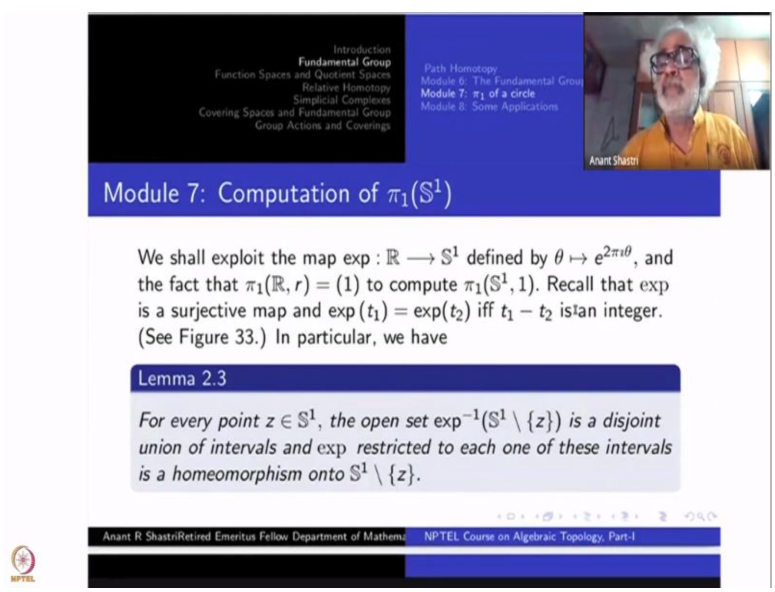

These things I have sketched here in this background, so at all the integers I have put a bullet -- kind of slightly larger points 0, 1, 2, here minus 1, minus 2, here and so on. All of them are going to this bullet here in the circle. I have taken intervals, say one fourth to three fourth and 1 plus one fourth to 1 plus three fourth and so on. What will they mean, it will be from this if you have to trace all the way up to minus i, this is angle pi by 2, this is 3 pi by 2.

The point one fourth will go to pi by 2 under this map, that is the meaning of this. because I am multiplying by 2pi i, the real number. So, this is the exponential function, it is injective restricted to any open interval of length less than 1. In particular, in 0 to 1 open interval it is injective, 1 and 0 go to the same point. You take any interval which is of length less than 1, then the exponential map is injective because the difference of any two members there is not going to be an integer. That is all. So, this is the property of this function and it would be exploited to the brim.

So, this lemma says the following: for every point z in $\mathbb{S}^1$ the open set exponential inverse of $\mathbb{S}^1$ minus z, throw away one point. What does the inverse look like? It is a disjoint union of intervals and if you take the exponential function restricted to each of these open intervals it is a homeomorphism onto this $\mathbb{S}^1$ minus a single point --- no matter what point is being thrown out.

So, what happens?-- the inverse image of this one single point which way you thrown out, it will be all various points-- look at one single points say r naught, then the next point will be r naught plus 1 (sorry not 2pi i because I have divided by 2 pi--I keep saying 2pi) and previous point will be r naught minus 1 and r naught minus 2, r naught minus 3 and so, on. Difference will be always an integer, where e power 2 pi r naught is your z.

In between these two, interval from r naught to r naught plus 1 it is an injective mapping on to S1 minus z. What is its inverse? Inverse is precisely what you call a logarithm function chosen inside $\mathbb{S}^1$ minus z, log function is not defined on the whole of $\mathbb{S}^1$, you throw just one point it is defined. The complex logarithm of any unit vector has the real part 0, -- it's purely imaginary and you divide by 2pi i, then what you get is the inverse of this one. So, this is all a little bit of complex analysis. That is all that I am recalling here.

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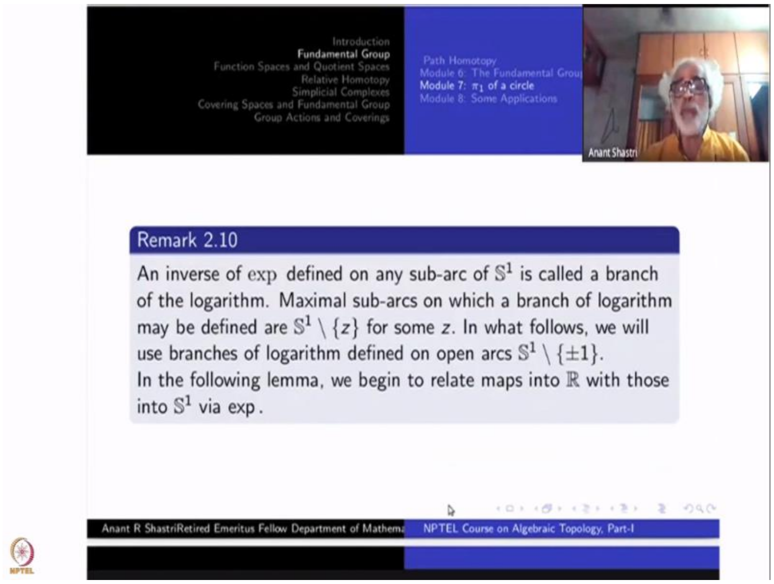

But this property is going to be very, very fundamental for us and inverse of exponential defined on any sub arc of $\mathbb{S}^1$ (any sub arc means at least one point should be missing then you take any subset of that which is connected, that is a sub arc) is called a branch of the logarithm, if you want to use this terminology. Maximal sub arcs on which a branch of logarithm is defined are of the form $\mathbb{S}^1$ minus z. As soon as you include a full circle, it is not defined, you throw a one point it is defined.

In what follows we will use branches of logarithm defined on open arcs -- two of them -- either throwing 1 or throwing minus 1. By throwing 1, I get 1 arc -- it is a very big arc except 1 point its whole circle and then I take another arc like this throwing away minus 1, so these two things are important, first we will use them.

So the branches why I am saying branches, if you choose say minus 1 to 0 that is one branch, open branch, the same log function will not work when you take 0 to 1 --- that is a different branch. But once you define, once you choose the interval of maximum length, there, it is a 1-1 mapping, so it has an inverse, and that inverse is a branch of the logarithm.

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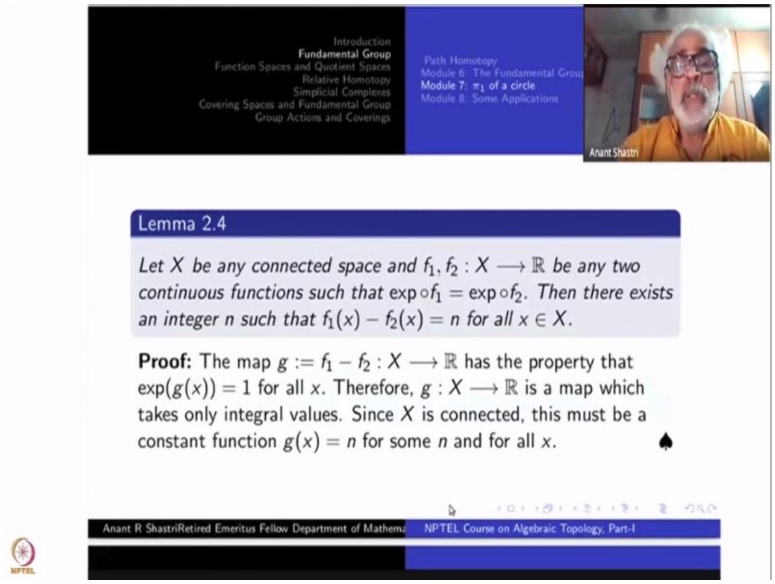

So, let us do something, the first thing is start with any connected space X. Actually we would like to have path connected space but connectedness is enough here. Take two functions X to $\mathbb{R}$, such that when you compose then with exponential, they are the same. Then, there exists an integer n such that f1 of x minus f2 of x is equal to this integral for all x.

The difference is given by one single integer! Look at this one. The exponential function from $\mathbb{R}$ to $\mathbb{S}^1$. This just looks like our fundamental problem in the lifting problem, corresponding to the function p from E to B. E is $\mathbb{R}$ and B is $\mathbb{S}^1$, X is an arbitrary space, we are studying the lifting problems here.

You see the first case of what this theorem says, what this lemma says is, up to an additive constant all the lifts are the same. Take any two lifts f1 and f2 they differ by one single integer; additive integer difference. f1 minus f2 is a constant function n. Is the statement clear? Once the statement is clear, the proof will be as easy as it is.

How do you prove? Look at this difference function g, f1 minus f2, it makes sense because f_i are taking real values. So, the difference makes sense, a difference is also continuous f and g are continuous so the difference is continuous. Now use the property of taking exponential of g. Remember exponential is a homomorphism from additive group to multiplicative group, therefore, exponential of g is exponential of f1 divided by exponential of f2. But they are the same.

So, it’s equal to 1 and this is true for all x. Therefore, the exponential of gx is one, so gx is contained in the set of integers because the exponential of anything is equal to 1 means it must be an integer. This is what we are seeing. Therefore, g from X to $\mathbb{R}$ is a map which takes only integral values but X is connected. If you have a connected space, the image of a connected space under a continuous map must be connected.

So, what is the connected subset of $\mathbb{Z}$ - integers? It is a single point and that point is n, some n. So, for all x, gx must be one single n. So what we have proved now is that lifts of a function taking values in $\mathbb{S}^1$ (they are taking values in $\mathbb{R}$) via the exponential function, are unique up to additive constant.

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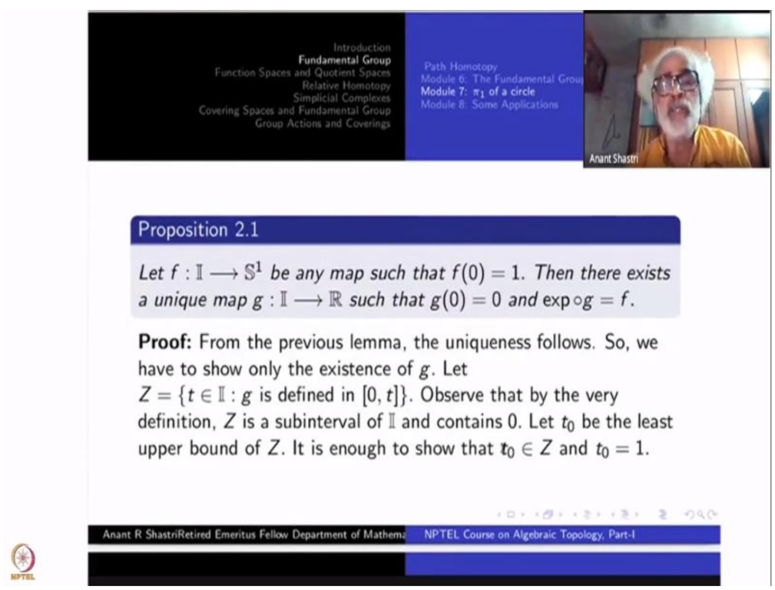

Now, let us look at this proposition one by one you have to add. Let f be a function from $\mathbb{I}$ to $\mathbb{S}^1$. (This is the proof now.) I could have denoted it by omega, but I would like to have function theoretic notation here. Take any map f from $\mathbb{I}$ to $\mathbb{S}^1$, let us take say f0 is 1 this is just to standardize. This is not a very essential thing. So, it is starting at 1. Then there exists a unique map g from $\mathbb{I}$ t to $\mathbb{R}$ such that the starting point is at 0, that is g of 0 is 0 sitting over 1, this 0 is inside $\mathbb{R}$, this 1 is in $\mathbb{S}^1$.

So, the first one, this 1 is a value in $\mathbb{S}^1$ is a unit complex number. 0 is 0 of $\mathbb{R}$ and exponential of g is f. So, this says that every function from $\mathbb{I}$ to $\mathbb{S}^1$ can be lifted. Not only that, it can be lifted, you can choose the starting point to be any integer you want, I have taken it as 0, for f0 is equal to 1. Suppose I have got a lift like this. Then what happens to other lifts. I know by the previous lemma, I have to add an integer and I get it. So whatever the integer I add, g0 will be equal to the corresponding integer. Suppose I subtract n, then f0 sorry g0, g0 may be made into minus n or plus n or any other number.

Therefore, along with this preposition it says that any function, any smooth function from $\mathbb{I}$ to $\mathbb{S}^1$ can be lifted and there were so many lifts, namely, infinitely many lifts, one at each point, one at each integer. So, this will be the meaning of this proposition. So, we have to do it only for one namely g 0 equal to 0, then we are done. Is that clear?

Let us begin. How we are going to do this, but complete proof will be done next time. So, how are we going to do this, so from the previous lemma uniqueness follows. This is what I just told you, there is a unique map. Suppose there is one then another one will be differing by this one by an integer here, but I have fix it g0 equal to 0, so that additive integer n must be 0 that means g1 minus g2 is 0 this means g1 is equal to g2 --- that is the uniqueness.

So, we have to show only the existence. So, what is the idea for proving existence, what we do is we will use the connectivity of $\mathbb{I}$, essentially, but we will do it in a more economic way, look at the set Z of all points t inside $\mathbb{I}$, such that there is a g on interval 0, t with the property namely g0 is 0 and exponential of g is f. You know that there is only one such g, if at all. So, suppose g is defined up to t, 0 to t then you took that t inside Z, Z is a subspace of the interval.

Obviously 0 itself is in Z because I can take g0 equal to 0 and that is all, e power 0, exponential of 0 is 1 we know e power 2pi i times 0 is just 1. So, this set Z is non-empty. In the usual parlor, what we would like to do is that we will show that Z is open and closed, suppose we do that, then because I is connected and Z is non empty, Z must be the whole of I. If Z is the whole of I what happens? 0 to t that t must be taken, can be taken as 1.

So, g is defined on the entire of 0 1. So, that is the solution so this in the scheme of proof this connectivity is used to prove a lot of existence theorems like this in topology, more generally, not for just I but any connected space. That is why we take in the first lemma just connected spaces. But the lifting cannot be done all the time, for that I have to use some special property of I, with that we will come to next time. Thank you.