Introduction to Algebraic Topology
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Lecture No. 62
Applications continued

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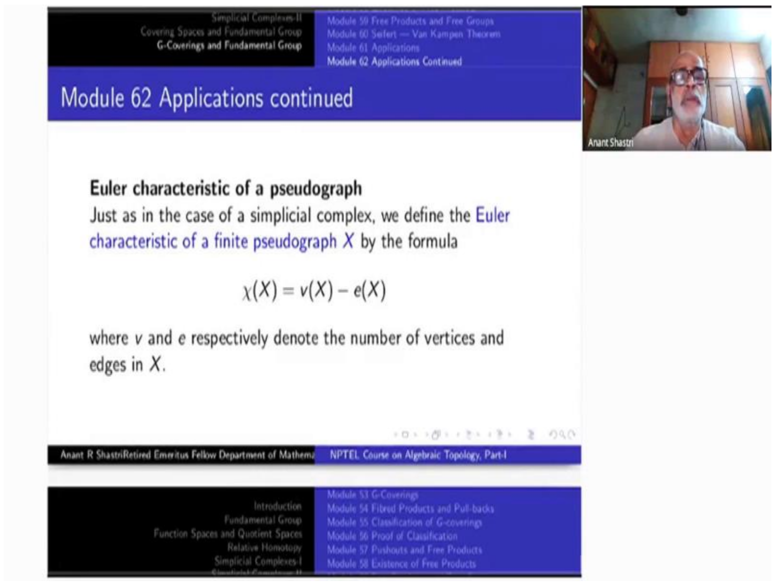

Continuing with the application of various theoretical results that we have proved, today let me first begin with recalling the definition of Euler characteristics which we have defined for a simplicial complex X. Namely, if you denote the number of vertices by $v(X)$, number of edges by $e(X)$ and so on. In the simplicial complex we can denote these numbers more conveniently by $f_0, f_1, \cdots, f_k$ so on, i.e., the number of k-simplexes of X by $f_k = f_k(X)$. Then the alternating sum $\chi(X) := \sum_i (-1)^i f_i = f_0 - f_1 + - \cdots$ is called the Euler characteristic of X.

Exactly same way we can define this for a pseudograph also. This time the number of vertices denoted by $v(X)$ and number of edges denoted by $e(X)$, there is no 2-cells and so on in a pseudograph. So, Euler characteristic of a pseudograph is defined just as $v(X) - e(X)$. It will coincide with the definition of the ordinary pseudograph if this pseudograph happens to be a simplicial complex that is one dimensional simplicial complex. Now, what this has to do with the whatever we have done.

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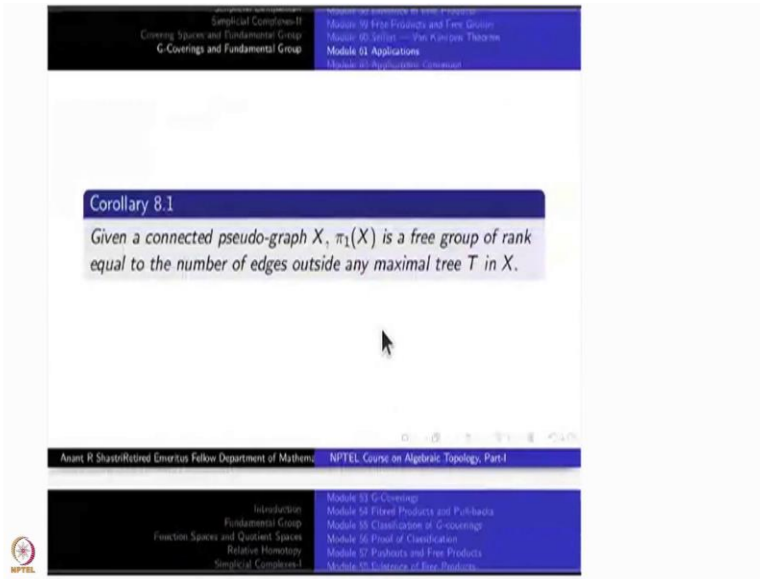

Last time we did one important result about the fundamental group of any pseudograph. Namely, if you start with a connected pseudograph, the fundamental group is a free group and the rank of free group is equal to the number of edges outside any maximal tree T inside X. So, this is what we are going to apply right now and get a nice formula in terms of Euler characteristic for the rank of the fundamental group.

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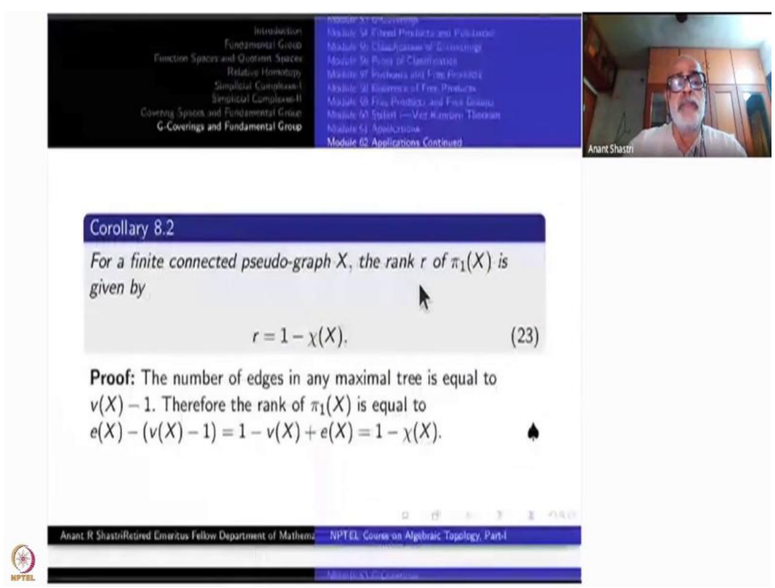

This is the corollary. It is a one line corollary but it is very important and easy to remember. For a finite connected pseudograph the rank r of the fundamental group is given by 1 minus $\chi(X)$. So, how do you do this? Very easy. The number of edges in any maximal tree is equal to $v(X) - 1$. What is $v(X)$? $v(X)$ is total number of vertices. If there is only one vertex there are no edges at all in a maximal tree. If there are two vertices they are connected by 1 edge you cannot connect it by 2 edge, then it would not be a tree at all. Like this you see if there are n vertices you exactly need n minus 1 edges to connect them.

So, if there are $v(X)$ vertices, $v(X)$-1 is number of edges. Since we know that the rank of $\pi_1(X)$ is equal to the total number of edges minus the number of edges inside a maximal tree . So, it is $e(X) - (v(X) - 1)$ which is the same thing as $1 - v(X) + e(x) = 1 - \chi(X)$. That is all. So, the formula is proved.

So, now we will give application of this corollary but this itself was a corollary to this corollary.

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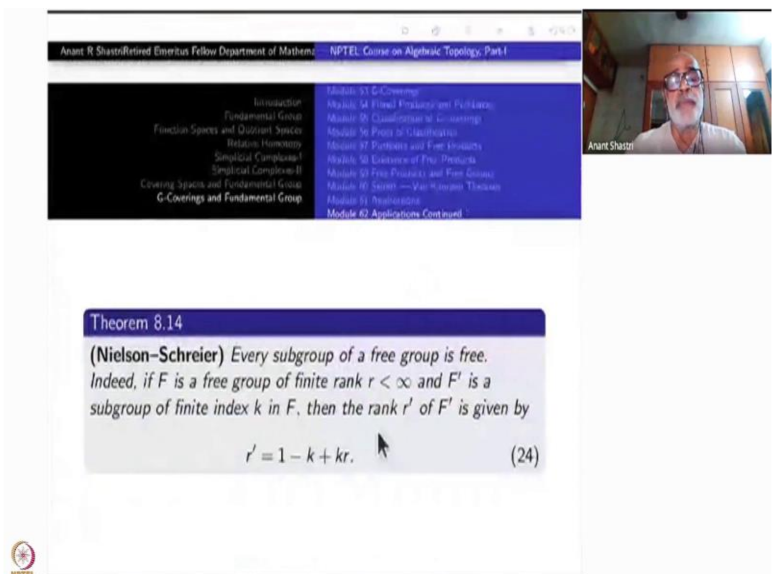

So, now we will give an application to one of the very very important result in group theory. It is called Nielson-Schreier theorem. Both of them have actually done a lot of group theory but their work was all motivated by topology, they have done a lot of topology also. Every subgroup of a free group is free. That is the result we want to see.

Remember, a free group has basis the subgroup may not have basis coming from this base it need not be a subset of basis. Only thing what we say is every subgroup of a free group is free just like every subgroup of free group which we are dealing with is which we are knowing already.

So, there is a further clause here. Indeed, if F is a free group of finite rank r and F' is a subgroup of finite index, then the ranks can be related by the following formula. Namely, if r' is the rank of the subgroup and k is its index, then r' is equal to $1-k+kr$.

So, we can now derive this by using Euler characteristic and something more that we have done. Everything in the course itself. Namely, we have given a simplicial structure to a covering of a simplicial complex. The simplicial structure coming from a simplicial complex of the given simplicial complex in a very particular way. So, that we are going to use here now.

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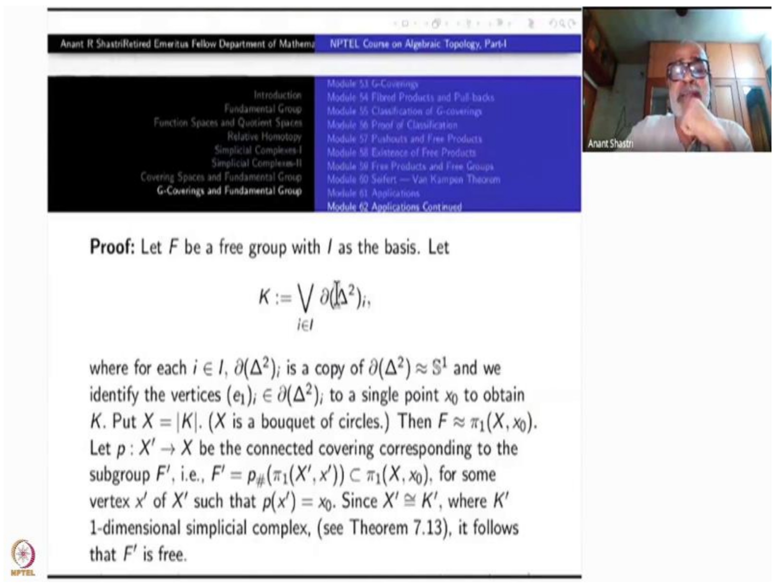

So, let us start with the following. A free group has a basis. Let me denote that base by I, which is just a set. Then I take as many copies as number of elements in I, the boundary of the 2 - simplex, the standard 2 simplex Δ_2. Boundary of the standard 2-simplex, remember, is homeomorphic to he circle $\mathbb{S}^1$. So, it has fundamental group infinite cyclic.

So, what I do is I take as many copies as the number of elements in I, then I performed this bouquet, the one point union of them. Namely, identify one selected vertex say, vertex $(e_1)_i$ form each copy to a single point. Select one point one vertex from each of these copies of the dell bounded delta 2 and identify them to a single point. So, that is the meaning of this wedge of this family $\{\partial(\Delta_2)_i\}_{i\in I}$

A wedge of topological space, in general, is defined by this method. And it is given the usual quotient topology here. So, in this case, each of them is a copy of boundary of delta 2 which is homeomorphic the circle. Then we know that as a topological space |K| is homeomorphic to the bouquet of circles. We have computed the fundamental group of this space X which is nothng but the geometric realization |K| of the 1-dimensional simplical complex.

So, starting with an abstract free group F_I, we have realized it as the fundamental group of a space |K| which is constructed in a very particular manner, it is a 1-dimensional simplicial complex. Now, given any subgroup of the fundamental group we can construct a covering of |K| corresponding to this. This is our general covering space theory. Let $p : X' \to X = |K|$ be the connected covering corresponding to the subgroup $F' \subset F_I = \pi_1(X, x_0)$. What is the meaning of this? Let us take $x' \in X'$ as a base point sitting over the base point $x_0 \in X$, $p(x') = x_0$.

Then, $p_\# : \pi_1(X', x') \to F'$ is an isomorphism. Covering map p induces a homomorphism $p_\# : \pi_1(X', x') \to \pi(X, x_0)$ which is injective and the image is precisely F'. So, this is what the general theory of covering spaces says. That I am going to use it here now.

But now, X' itself has a simplicial complex structure, a 1-dimensional simplicial complex structure, because, X=|K| is a 1-dimensional simplicial complex. Also, how this structure is built? that also we know exactly. Already the fact that $X' = |K'|$ where K' is a 1- dimensional simplicial complex, implies that the fundamental group of X' is free. So, the first part of the theorem is over.

Now, for the more elaborate part, namely, suppose F is a free group of finite rank r (i.e., $\#(I) = r$), then this bouquet will also have exactly r copies of boundary of delta 2, r empty triangle will be there. How many r of them? All of them having one single point as a common vertex.

So, therefore you can count the number of edges and number of vertices here. Then for each vertex in K, K' there will be how many vertices? That depends upon the number of sheets of p. So, how does the number of sheet determined? The index of this group subgroup F' that we have taken, that is equal to the number of sheets of p.

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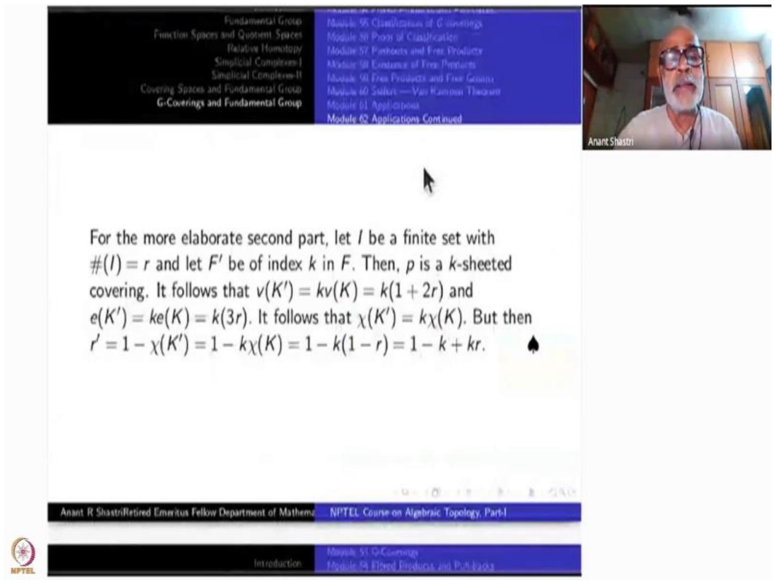

So, there will k many vertices for each vertex below. Similarly there will be k many edges in K' for each edge in K. So, we can now start counting. Let me repeat, for the more elaborate second part, let I be a finite set with cardinality of I equal to r. Let F prime be a subgroup of index k in F. Then p is a k-sheeted covering projection. It follows that the number of vertices in K' is k times the number of vertices in K. But, number of vertices in K is precisely equals to 1 plus 2r. In each triangle there are 3 vertices but one of them is common so that should not be counted repeatedly. So, that is only once and then for other 2 for each of them, there are r triangles, so 2r of them in all. Therefore the total number of vertices in K' is equal to k(2r+1).

Similarly, number of edges inside K' is precisely to k times 3r. So, k times e K what is e K? e K is 3 times each triangle has as many 3 of them as edges. So, they will be 3 times r. So, it is k times 3 times r. So, Euler characteristic $\chi(K') = k\chi(K)$. You do not need all these elaborate thing but I have just written down.

But, then by the earlier formula $r' = 1 - \chi(K')$ $=1 - k\chi(K) = 1 - k(1-r) = 1 - k + kr$. That is the statement in the second part. This completes the proof.

For example, if your group is a free group of rank 2 and you have subgroup of index 2 then what will be the the rank of this subgroup? Only 1 minus 2 plus 4 equals 3.

So, certainly the subgroup has subgroup has has rank bigger than the rank of the group.

So, this result is used several brancheds of mathematics, differential geometry, algebraic geometry and so on. One of its consequences is called as Riemann -Hurewitz formula.

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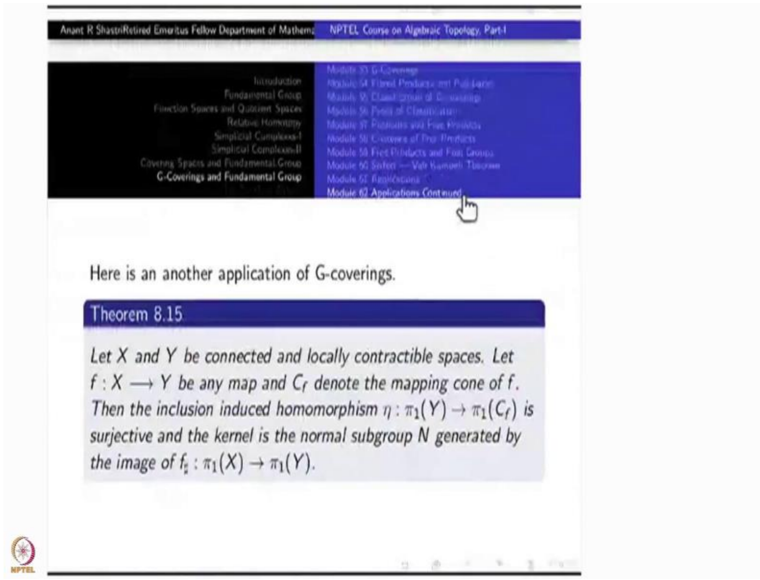

So, now I will give you another important application of this G coverings that we have done. So, here let X and Y be connected locally contractible spaces or we may assume slightly weaker conditions like locally semi locally 1 connected semi locally simply connected. Some such thing you can assume to ensure that simply coverings exists. So, locally contractibility. Let $f : X \to Y$ be any map and Cf denote mapping cone of f. Then the inclusion induced homomorphism $\eta : \pi_1(Y) \to \pi_1(Cf)$ is surjective and the kernel is the normal subgroup N generated by the image of $f_\# : \pi_(X) \to \pi_1(Y)$. Here f is any continuous function.

So, if you want to be very careful here, you have to select a base point here say $x_0 \in X$ and $y_0 = f(x_0) \in Y$ and then you have to write down these points while writing down the fundamental group. Those things are obvious and also always have to be remembered. Here what is more essential and simple I have stated this. Instead of writing all the base points and so on. So, those things will be done carefully in the proof.

Moreover, the entire statement is independent of what base point are taken. If you change the base point in X to say x_1 then you change it in Y also to $f(y_1)$. The statement will be still be true. Therefore, what you remember is this statement. But, a more elaborate a statement has to be also remember always whenever fundamental group is involved you if not be very careful finally you have to put the base points. So, that is the summary of this one.

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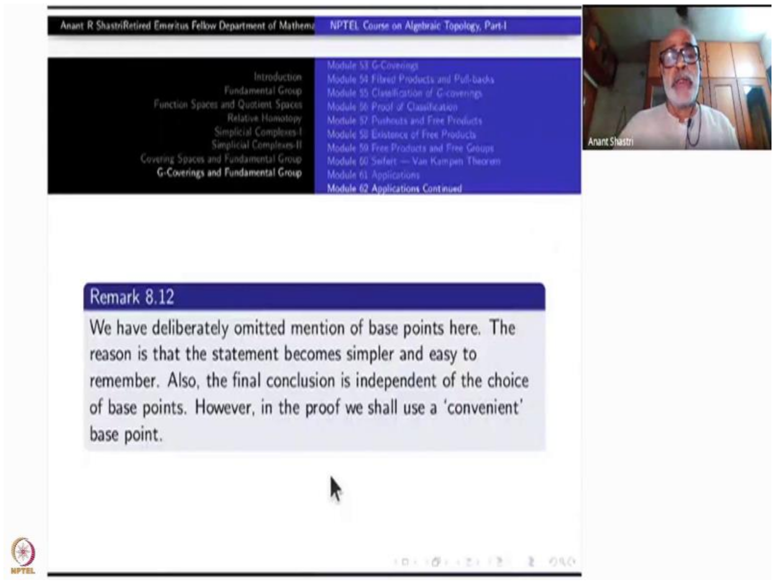

So, I have deliberately omitted mentioning this point so that it will be easy to remember not that you would forget it. You make it a practice not to mention it, yet remembering it.

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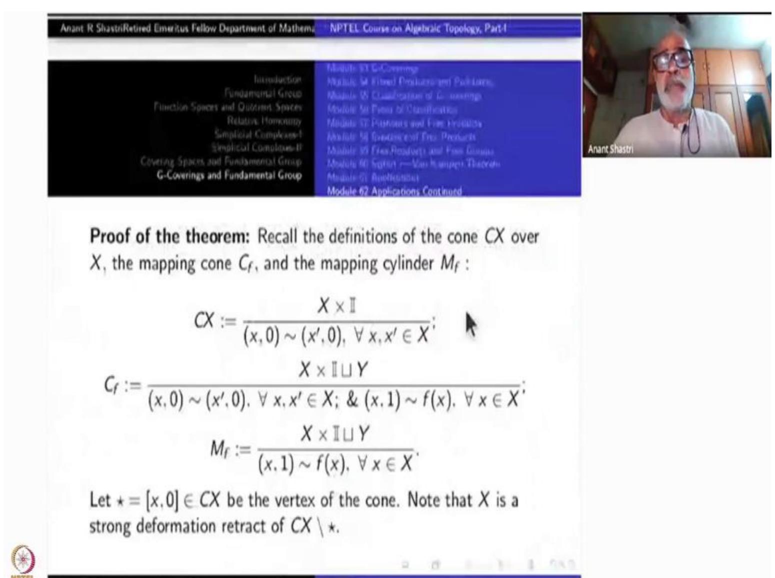

So, let us recall something because it is a long time we have done these things. Let us recall what is the mapping cone, mapping cylinder and so on so that we write down a proof carefully. So, given any topological space X, CX is defined as the quotient of $X \times \mathbb{I}$ wherein we identify all the points $(x, 0)$, to a single point. The mapping cone is defined similarly but little more elaborately. You take $X \times \mathbb{I} \sqcup Y$ and then you identify $X \times \{0\}$ to single point, also you identify $(x, 1) \sim f(x),$ for every $x \in X$. You see the map f is invloved in this definition. Then the mapping cylinder is defined in a slightly different way, namel, the first part of the identification is not taking place at all. Namely, on $X \times \mathbb{I} \sqcup Y$, it is X cross the unit interval perform the identification $(x, 1) \sim f(x)$ for every x in X. I am just recalling this I cannot run the whole theory again. There is a standard notation that all the points x comma 0 are identified to single point that point is denoted by $\star$ and that star is called the apex of the cone. The whole CX is star shaped at this star. Remember that.

Therefore, CX is always contractible. Moreover, if you throw away that point $\star$ then the remaining portion is nothing but $X \times (0, 1]$. Therefore, it is a strong deformation retracts onto X. You can push the entire thing to $X \times \{1\}$ which is homeomorphic to X. X can be thought of as a subspace of CX, via $x \mapsto (x, t)$ where $t \neq 0$. At 0, there is not the whole copy of X but only single point is there.

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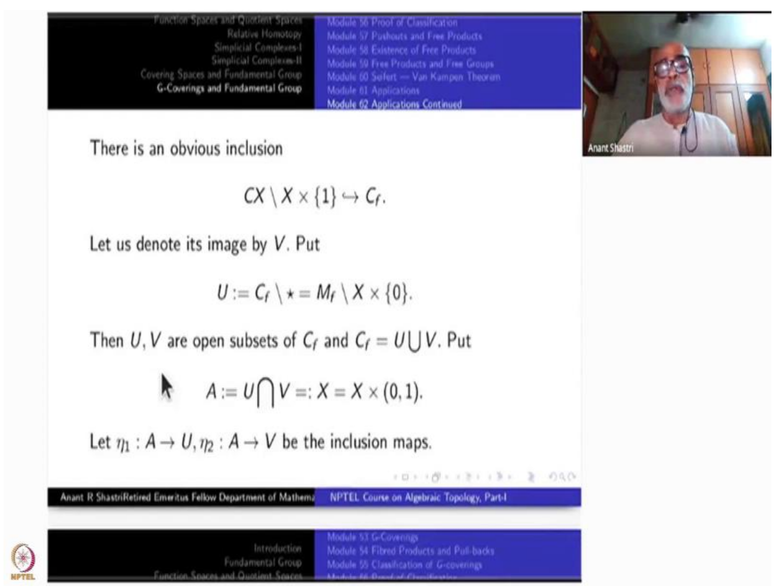

There is also another obvious inclusion map. From CX you throw away the base, that is an open subset anyway. $X \times \{1\}$ is the base. Then the open set $CX \setminus X \times \{1\} \subset Cf$. You can see that CX X cross I disjoin union Y I have taken whatever identification is there those things are also there here. Except that in CX, there is no identification on $X \times \{1\}$, whereas in Cf there is some identification here.

Therefore, if I throw away $X \times \{1\}$, the open part then that is contained inside Cf. So, let us just denote it by $V = CX \setminus X \times \{1\}$, V is temporarily notation that is an open subset. Also look at the open subset obtained by throwing away the star from Cf. If you throw away star, that is same thing as the mapping cylinder U=Mf minus X cross 0. If you identify the whole X cross 0 is single point that is Cf.

So, you are throwing away here you throw away this part also that is Mf. So, U and V are both open subsets of Cf and $Cf = U \cup V$. So, so far what I have done is setting up notations for application of Van-Kampen theorem. Like you had this in the case of spheres, then you removed North Pole South Pole and then you took the intersection and so on. That is the kind of thing we are doing here again.

So, intersection let me denote it by $A = U \cap V$. This is all just convenient temporary notation A is nothing but $X \times (0, 1)$ So, let us have these notations also $\eta_1 : A \to U, \eta_2 : A \to V$, these are the inclusion maps. So, let me show you the picture first.

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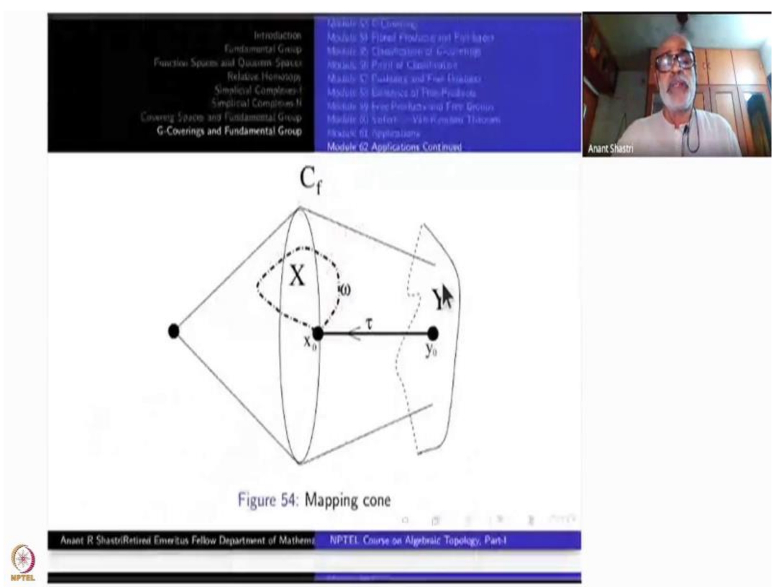

So, picture is here this is the entire mapping cone. So, this is the star so this is X cross let us say X cross half half way. This whole thing is 0 to 1 here. So, at there is at 0 there is 1 identification at X1 also there ia identification here X will be identified with f of X. So, I have taken X naught as a base point for X but this is actually I have put it X cross half half way. I could have put it at any level t not equal to 0 or not equal to 1. Anywhere I can put it is a same thing. So, from here to here this entire thing this is my V and from here to the rest of thing this is my U.

So, union is the whole space see here interaction will be remove this point remove this part Y and this open part which is nothing but X cross open interval 0 1. So, we will come back to this this picture again.

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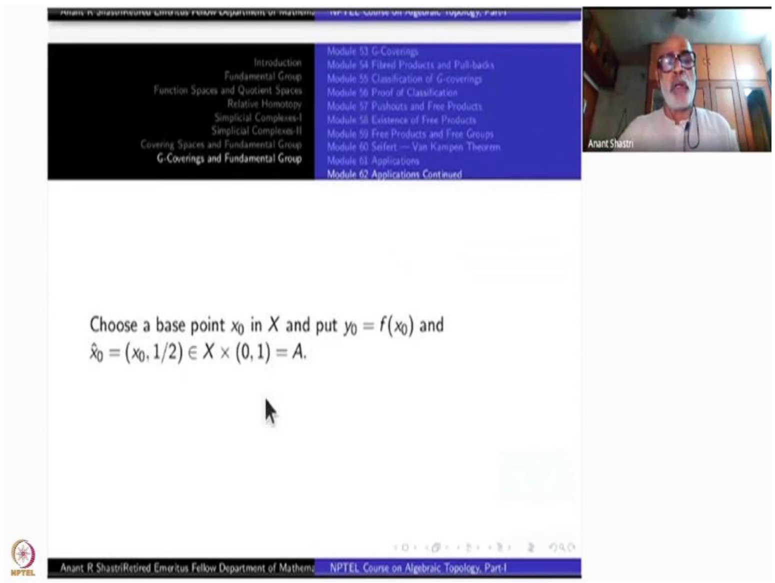

So, right now this is what I have said. Therefore, so, one more thing, namely, I have shown x_0 as a base point, whatever it is does not matter, $y_0 = f(x_0)$. But, $x_0 \in X$ whereas I am interested in $X \times \{1/2\} \subset U \cap V = X \times (0,1)$. Therefore I have take the base point to be $\hat{x}_0 = (x_0, 1/2)$ So, that belongs to A= $A = X \times (0,1)$. Our obvious choise of taking $X \times \{0\}$ to represent a copy of X does not work here. So, we are now identifying X with $X \times \{1/2\}$.

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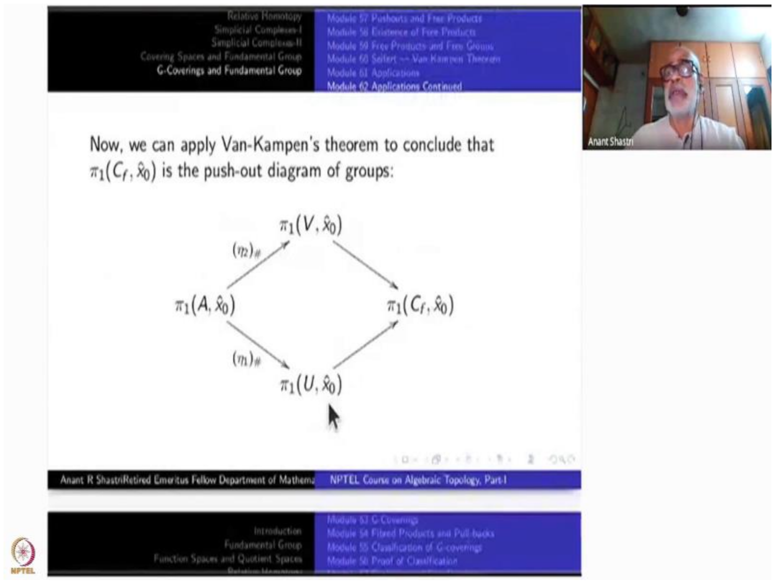

Once you have this, we have this Van Kampen's theorem. The Cf is written as union of U and V it is a common base point $\hat{x}_0$ belonging to all of them. I take pi 1 intersection of these two is A. So, pi 1 of A x naught eta 1 check eta 2 check then these are also inclusion maps so I am not writing them separate notation is not necessary.

Inclusion induced maps here. The statement of Van Kampen's theorem is that pi 1 of Cf is pushpout i.e., the amalgamated free product, namely, it is free product of this group with this group modulo the normal subgroup generated by the image of this minus this and so on. So, that is what we have written down here.

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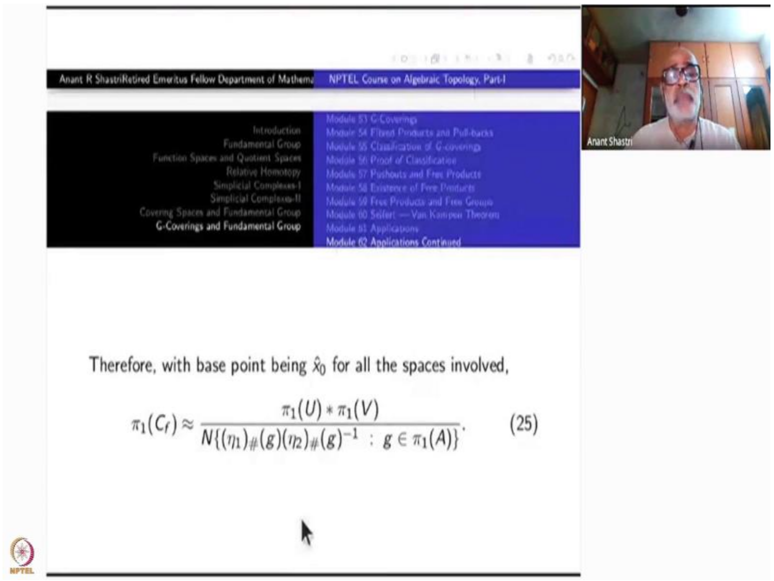

It is pi 1 of U star pi 1 of V modulo the N is normal subgroup generated by eta 1 check g multiply by eta 2 check g inverse that is one single element at g belongs to pi 1 of A. Take all of them then take the normal subgroup generated by them. The quotient of this is pi 1 of Cf. I can put the x naught hat here x naught hat, x naught hat here all this. Base point can be always written down this written down here. There is one (())(26:51) but now we know all these things so we have figured out what these things are.

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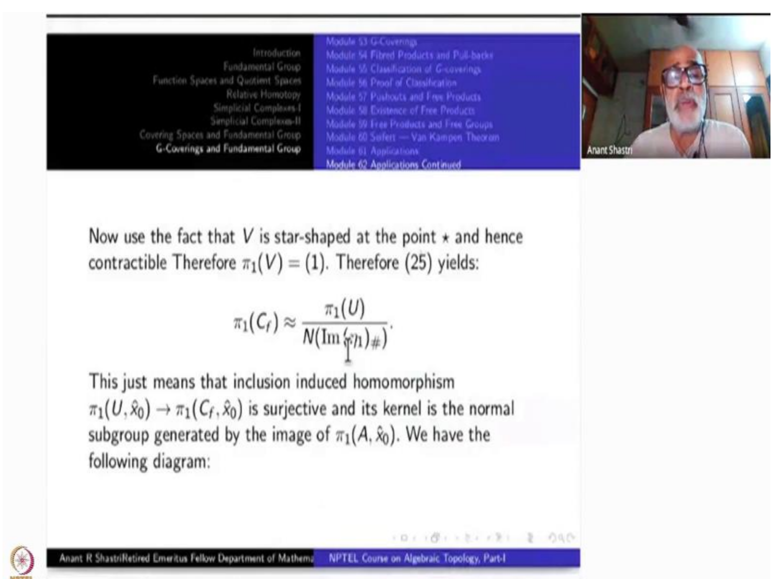

Now use the fact that V is star-shaped at the $\star$. Every star shaped space is contractible. In particular its fundamental group is trivial here. Therefore, this statement here becomes simpler. Whatever we have proved it gives you this pi 1 of V there is no need to write there is a trivial group so it is pi of U and eta 2 of g there is no need to write. They are all trivial elements it is the normal subgroup generated by eta 1 of g as g varies over this, which is the same thing as normal subgroup generated at the image of pi 1 of A under $(\eta_1)_\#$.

So, image of pi 1 of A under eta 1 check. So, this is the neat expression that we have got. Pi 1 of Cf is the quotient of pi 1 of U by normal subgroup generated by eta 1 check. We are not yet through. So what is this U? What is this eta 1 etcetera you have to figure out. Because, these are not in the beginning of our statement in the statement only X and Y and f are there, so everything you have to convert it into X and Y and the map f.

So, look at inclusion induced homomorphism $\pi_1(U) \to \pi_1(C_f)$. Remember this is not given to be monomorphism or anything. It is given by the inclusion map. That homomorphism is surjective and its kernel is the normal subgroup generated by $\pi_1(A)$ So, if we go back to this part you can ignore the upper part here. This part is an exact sequence which is surjective kernel of this map is this one that is the meaning of this one. So, we have to do one more step here to understand in terms of X and Y that is the that is the major part of the thing is over. We have to figure out that the correct thing is arrived.

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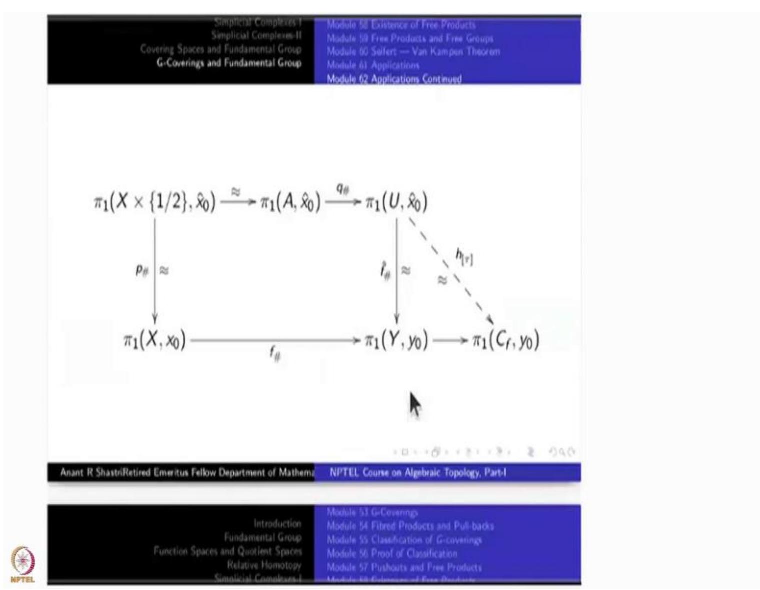

So, this is the picture going back to the the mapping cone. $X \times \{1/2\}$ is sitting in the middle of A with $\hat{x}_0 = (x_0, 1/2)$ as the base point. This is a deformation retract of A. Therefore the inclusion induced homomorphism is an isomorphism here. Then A itself is contained inside U so this is pi 1 of U here and this is that is the identification here. So, there is q check, the quotient. Inclusion followed by the the quotient restricted to the subspace that is an inclusion. So, here it is just the porjection map, forget the t-coordinate half, $(x, 1/2) \mapsto x.$ This is the projection map so the induced homomorphism is again an isomorphism on pi 1.

Now, you see pi 1 of X x naught so this is nothing but this copy of this here. From here I have f check to pi 1 of Y so what is the map here. Remember, the mapping cone whole mapping cone there is a deformation but here I have removed $\star$ just the one open part here that is all. So, mapping cone that is a map any bracket x comma t going to f x this is very different map and that is why I have written here $\hat{f}$ so this is $\hat{f}$ is a strong deformation retraction on to Y. So, $\hat{f}_\#$ is an isomorphism on the fundamental group.

So, what we know this Cf here is sitting here that is an inclusion map here but here I have taken different map here. So, inclusion map is factored by this way it goes into Y and then it comes here. So, this is a commutative diagram here. This is surjective and its kernel is precisely this one this is what we have seen from pi 1 of A. Pi 1 of A is isomorphism to this is isomorphism to this one.

So, the problem here is you see that this is x naught twiddle here is f check this is y naught so what is its base points where do they go. That required all these detailed explanation. So, you still may wonder if still have some doubt you can use the last part of the diagram namly, the commutaive trianlge. That is strictly not necessary, never mind , you have completed a proof this part is over now.

I have taken trouble to explain exactly what happens to these base points under these maps. Because, this is in some other map here f twiddle of that comes to y naught that is fine. So, you have to understand that this map is a homotopic equivalence so it is an isomorphism. But why it should be the same thing why this diagram is commutative.

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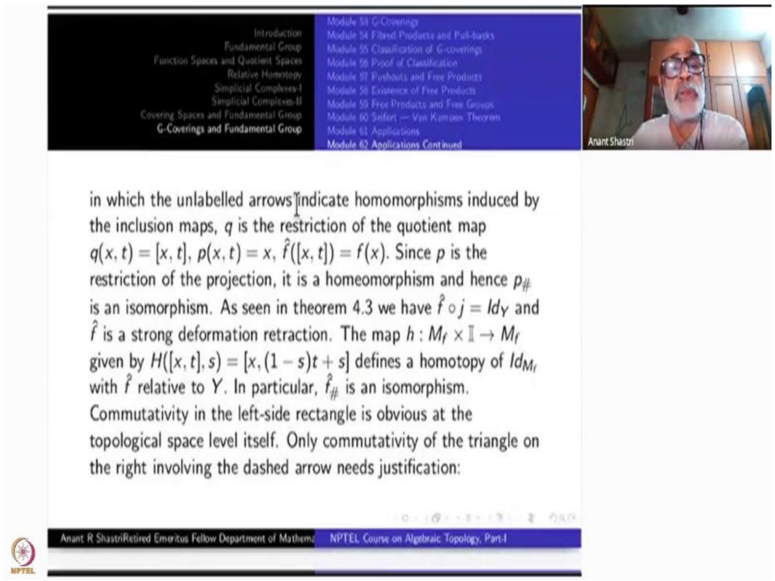

So, here I have elaborated this let us go through this once again. The unlabeled arrows indicate homomorphism induced by the inclusion maps here. What are they? This one there is only one this one here this is something this dot dot dot I will explain it. And q is the restriction of the quotient map $q(x,t) = [x,t]$, x comma t is or notation is x comma t in bracket round bracket changes to square bracket. $p(x,t) = x, \hat{f}(x,t) = f(x)$, all this I have told you.

Since, p is the restriction of the projection to $X \times \{1/2\}$, it is a homeomorphism and hence $p_\#$ c is an isomorphism. As seen in an earlier theorem, namely, about the homotopy properties of the mapping cylinder, if you do not remember this is what it is. Namely, $\hat{f} \circ j$ is identity of Y. And $\hat{f}$ is a strong deformation retraction. The map j not included here in the diagram. This j is the inclusion from Y to mapping cylinder U, up above so j followed by $\hat{f}$ is identity map of Y and this is a strong deformation retraction this is what I am going to use now.

So, but I will be write down what is that strong deformation retraction here. Let us take $H : M_f \times \mathbb{I} \to M_f$ given by $H([x,t],s) = [x,(1-s)t+s)]$. This defines a homotopy of identity of Mf with $\hat{f}$ restricted to Y. What are points here? When t equals to 1 that is the point of Y. So, on Y it is the identity map this is the meaning of this one.

When t equal to 1, it is 1 minus s plus s which is just 1. So, you can extend it by identity on Y. In rest of the time it is homotopy when s equals to 0, RHS is $[x,t]$, so it is identity map of the mapping cylinder itself. When $s = 1, H([x,t],1) = [x,1] = f(x) = \hat{f}([x,1]).$ So, in particular , it follows that $\hat{f}_\# : \pi_1(M_f, \hat{x}_0) \to \pi_1(Y, y_0)$ is an isomorphism. Note that $\hat{f}(\hat{x}_0) = f(x_0) = y_0.$

Commutativity of the left side rectangle is obvious at the topological level itself. look at this X comma half goes to X here X comma t bracket X comma t goes to x. Here, x comma open x comma half open there is no identification goes to x of bracket here. So, this is obvious so this f of that. Once some diagram is commutative at the topological level, the diagram of induced homomorphisms will be commutative at the homotopy groups level.

Only commutativity of the triangle on the right side involving the dashed arrow has to be justified. What is this one? So, this is remember, $h_{[\tau]}$ What is this τ? So, I will explain that one now. This changes the base point $\hat{x}_0 = [x_0, 1/2]$ to $[x_0, 1] \sim y_0$. Remember, U is a subspace of C_f but this is not the inclusion induced homomorphism. Here the base points are changed.

Both $\hat{x}_0 = [x_0, 1/2], y_0 = f(x_0) = [x_0, 1]$ and you have the arc here $[x_0, t]$ from one point to the other. You trace it this way the other way around that is the path τ. Whenever you have a path , $h_{[\tau]}$ defines an isomorphism of the fundamental groups effecting a change of base points. Take a loop here, start at the other base point here now go via tau to the first base point and trace the loop comeback by the tau inverse. So, that is the definition of $h_{[\tau]}$ if you remember that, that is fine, otherwise I have just recalled it to you now.

So, what I am going to use this thing. We get a homotopy here to push this $\hat{x}_0$, by a homotopy to this point y_0. So, that is precisely the role of this homotopy. Namely, as if the base point is slowly moved along this point to this point that is the homotopy I want write it down and then you are commutative this were the same thing.

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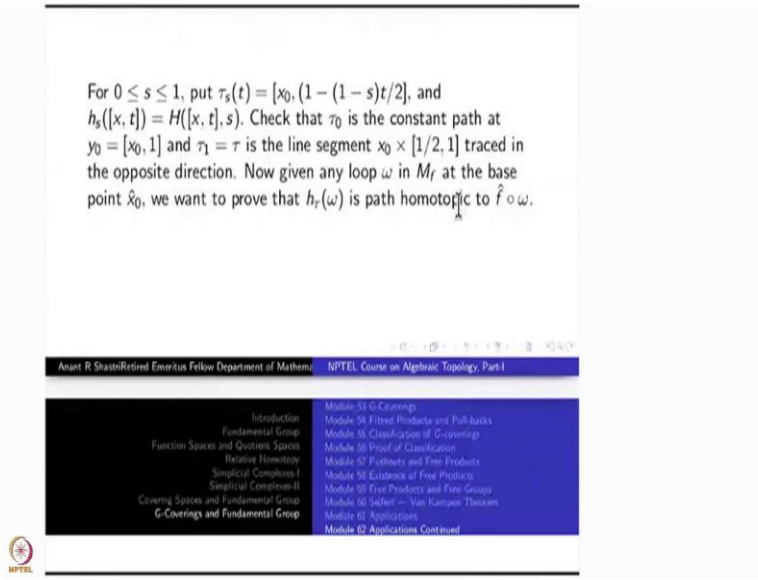

So, for $0 \le s \le 1, \tau_s(t) = [x_0, 1-(1-s)t/2], 0 \le t \le 1$. So, I have taken a family of paths τ_s. Now let us look at s is equal to 0. $\tau_0(t) = [x_0, 1-t/2]$. That is precisely this this arc traced from 1 to ½. So $\tau_0 = \tau$. What is τ_1? When s equal to 1, we have $\tau_1(t) = [x_0, 1] = y_0$ which is the constant path at y_0 So, I begin with the full path τ and slowly I trace it upto $(1+s)/2$ only, up till here and finally this is the constant path at y_0.

So, that is the family of arcs $\{\tau_s\}$. Check that tau naught is a constant path at y naught equal to x naught comma 1 tau 1 is tau in the segment x naught cross half 1. Trace in the opposite direction. Now go back to whatever deformation you have written earlier, put $h_s = H(-s)$, and we shall use this.

Now given any loop omega in Mf like this picture which is omega here at base point x naught inside the whole of Cf that will be converted into, by pushing the base point here to all the way to y_0, a loop based at y_0. There are two things you have to do here. So, given a loop omega in Mf at the base point x naught hat we want to prove that h tau omega is homotopic to f hat composite omega f hat composite omega is a path is a loop in Y at the base point Y naught.

If you show this one then this diagram commutativity of this path follows this h tau. Whenever you push this loop here it is loop here inclusion map followed by this one that is fine. So, this is what we have to prove.

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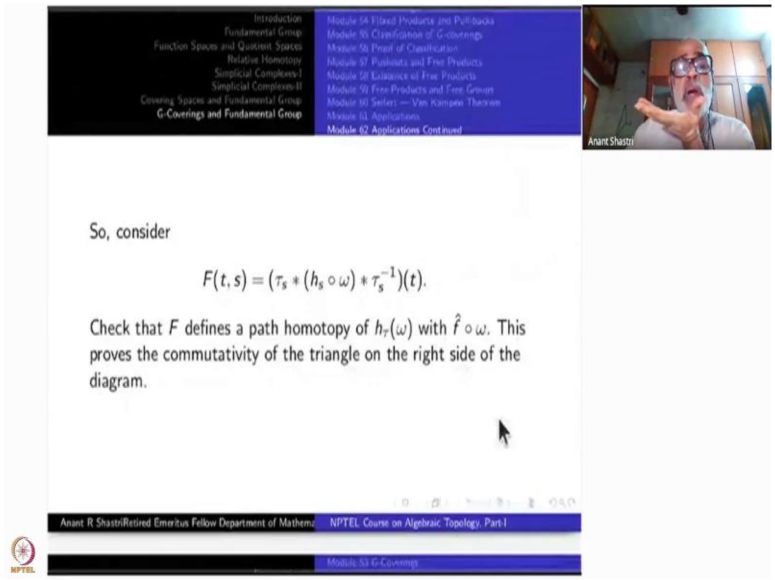

And here is the formula let us check. Consider $F(t,s) = (\tau_s * (h_s \circ \omega) * \tau_s^{-1})(t)$. Remember that composition $*$ of paths make sense only when the end point th first on eis the same as the starting point of the next. So that is what you have to take case: write tau s star hs composite omega star tau s inverse t. Check that F is a homotopy as required for h tau of omega to a f hat of omega.

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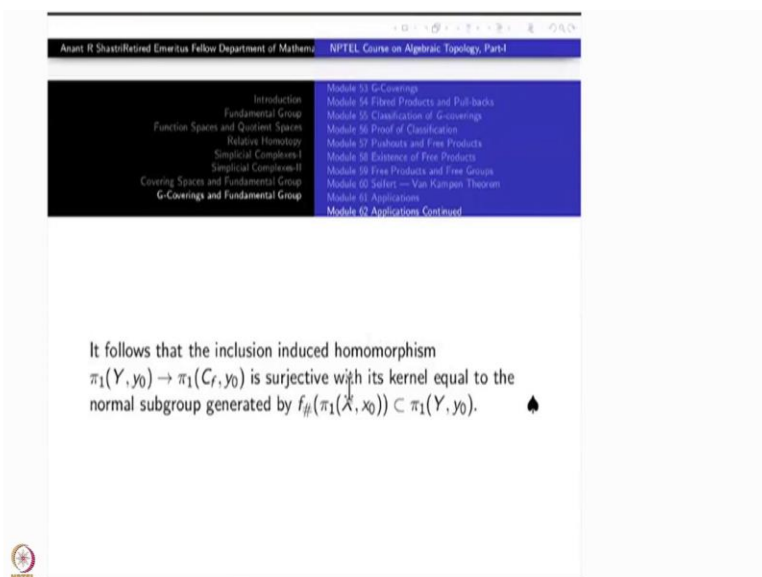

So, it follows that the inclusion induced homomorphism $\pi_1(Y, y_0) \to \pi_1(Cf, y_0)$ is surjective with its kernel equal to the normal subgroup generated by this one.

So, we shall come to the last result of this series now. As a special case of this general theorem take the space X to be a circle and f be any map into Y.

That is a special case. When you take a cone over that it is nothing but the space obtained by attaching a two cell to Y. Because, the cone over $\mathbb{S}^1$ is nothing but the 2-disc. So, that is a two cell being attached via the map f to Y so that is the special case. So, we shall slightly generalize this one not just one at a time but several 2-cells being attached. That is the next result here.

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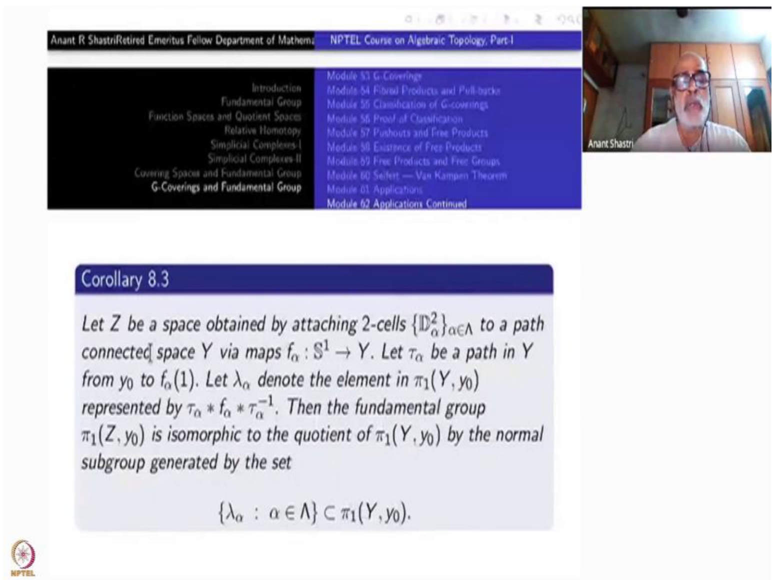

So, let Z be a space obtained by attaching 2-cells $\mathbb{D}^2_\alpha$'s, not just one, but several of them indexed over Λ, to a path connected space Y via the attaching maps $f_\alpha : \mathbb{S}^1 \to Y$. Let τ_α be a path in Y from the base point $y_0 \in Y$ to $f_\alpha(1)$. This 1 is indicating complex number 1$\in \mathbb{S}^1$ which is being taken as the base point for each copy of $\mathbb{S}^1$. Let, $\lambda_\alpha \in \pi_1(Y, y_0)$ denote the element represented by the loop $\tau_\alpha * f_\alpha * \tau_\alpha^{-1}$.

Note that f_α is a loop at $f_\alpha(1) \in Y$. So take start at y_0, take τ_α, follow it by f_α, and then trace τ^{-1} to come back to y_0. That is λ_αThen the fundamental group $\pi_1(Z, y_0)$ is isomorphic to the quotient of $\pi_1(Y, y_0)$ by the normal subgroup generated by all the elements $[\lambda_\alpha]$, as α runs over Λ. So, take the normal subgroup generated by this collection, go modulo that, that is pi 1 of Z.

So, the missing thing here if you look at is that in the previous theorem where X is connected Y is also connected and you are given the base points and the map respected them. But now, here the family $\{\mathbb{D}^2_\alpha\}\}$, by definition, is disjoint, the union will not be connected.

So, except that it is similar to the previous theorem so we have to be careful how this can be done. We cannot directly apply the case when there is only member in Λ. In that case, of course, it is direct application of previous theorem. Then it is image of image of this pi 1 of S 1 pi 1 of S 1 which is infinite cyclic then normal subgroup generated by that one that is precisely to what it is.

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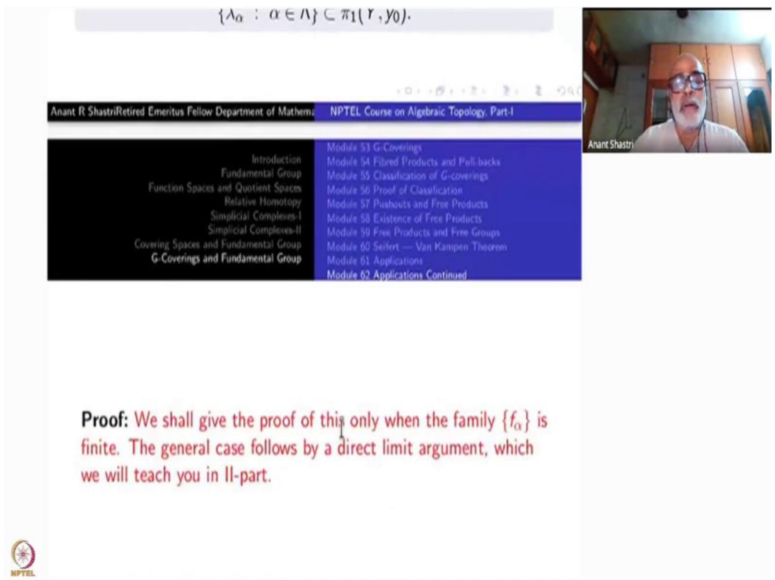

Now, I have to be cautious. We shall give the proof of this statement only when Λ is finite. The general case follows by what is called a direct limit argument which we shall teach you in the second part of this course. Not only that, in the second part, the attaching cell etcetera itself will be formalized into what is called a CW complex.

Not only just attaching one cells and two cells you will do as in K cells in general and then we study those things very elaborately. So, you may check that this last result here is a motivation for the second part. So, let us workout this one when the family $\{f_\alpha\}$ is finite.

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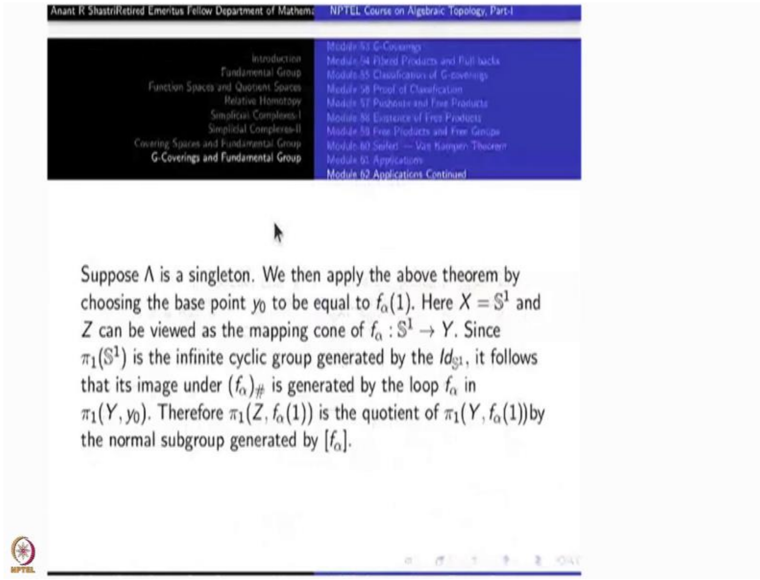

First of all I want to repeat. Suppose Λ is a singleton. Then we can apply the above theorem by choosing base point $y_0 = f_\alpha(1)$. We are free to choose the base point. Here, $X = \mathbb{S}^1$, and Z can be viewed as the mapping cone of f_α.

Since, $\pi_1(\mathbb{S}^1, 1)$ is infinite cyclic group, generated by the inclusion map $\mathbb{S}^1 \to \mathbb{S}^1$. You may denote it by $Id_{\mathbb{S}^1}$ and [$Id_{\mathbb{S}^1}$] will denote a generator for $\pi_1(\mathbb{S}^1, 1)$. It follows that its image under f alpha check is nothing but the class $[f_\alpha]$, as a loop, since $f_\alpha \circ Id = f_\alpha$ itself. Therefore, $\pi_1(Z, f_\alpha(1))$ is the quotient of $\pi_1(Y, f_\alpha(1))$ by the normal subgroup generated by $[f_\alpha]$ You do not have to take all the elements in the image, because all other elements are powers of f alpha. So the case when lambda is a singleton is over.

Now what we are going to do, we do a simple induction. The only problem is each time the base point will have to be changed. and therefore it is important to realize that our earlier theorem was `base point free'. It is the unique feature of this result that any base point will do. So, that is why I have taken so much of trouble in that one. Let us now use it here.

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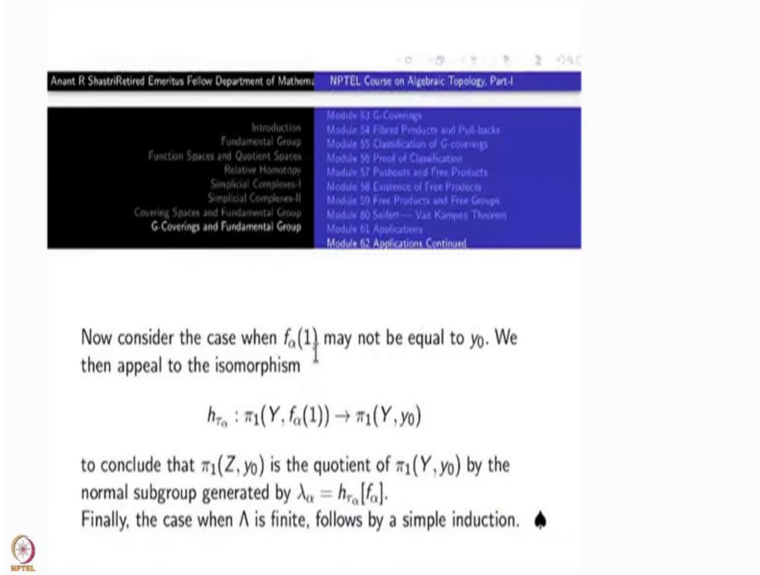

Now consider the case when f alpha 1 may not be equal to y naught. We then appeal to the isomorphisms $h_{[\tau_\alpha]}$'s. τ_α are some fixed paths in Y from y_0 to $f_\alpha(1)$. Then $h_{[\tau_\alpha]} : \pi_1(Y, f_\alpha(1) \to \pi_1(Y, y_0)$ is an isomorphism which we want.

Once this is an isomorphism a normal subgroup here will go to normal subgroup here. And in this case, what is are the normal subgroups? Here it is generated by $[f_\alpha]$ and under $h_{[\tau_\alpha]}$ this element goes to $[\lambda_\alpha]$ and then normal subgroup on the other side generated by $[\lambda_\alpha]$.

So, this will give you now the statement for y naught. Where y naught is a fixed point base point for Y independent of what f alpha. But, now this is applicable for all the other alphas also, when Λ is not a singleton.

So, you can apply one by one. First label the elements of $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$. First attach $\mathbb{D}^2_{\alpha_1}$ via f_{α_1}, next attach $\mathbb{D}^2_{\alpha_2}$ via f_{α_2} and so on. What you get is that the fundamental group of the secod one is the quotient of the fundamental group of the first one by the normal subgroup generated by $[\lambda_{\alpha_2}]$. So, together the the fundamental group fo the second one will be the quotient of the fundamental group of Y by the nromal subgroup genrated by $\{[\lambda_{\alpha_i}] \; : i = 1, 2\}$So, the case is got by a simple induction.

The infinite case does not immediately follow from this. Even in the the countable. So, for that, you need what is called as direct limit arguments. So, that will be a part of the next course the second course to this one. So, you are welcome to attend that. Thank you, I have enjoyed teaching you people. Hope you all learnt something from this course. Thank you.