

Introduction to Algebraic Topology (Part - I)
Professor Anant R Shastri
Indian Institute of Technology Bombay
Lecture 61
Applications

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The screenshot shows a presentation slide with a blue header and a white body. The header contains a table of contents for the course, with 'Module 61 Applications' highlighted. The body of the slide contains the following text:

We shall begin with a direct application of the last result in the previous section to calculate the fundamental group of any pseudo-graph. This computation itself will then be applied to deduce other results.

Definition 8.10

By a graph we mean a 1-dimensional simplicial complex.

We would like to work with a slightly more general set-up than this, which leads us to the following definition.

At the bottom of the slide, there is a small video inset of Professor Anant R Shastri, a man with white hair and glasses, wearing a white shirt. The slide footer includes the NPTEL logo and the text: 'Anant R Shastri, Retired Emeritus Fellow, Department of Mathem., NPTEL Course on Algebraic Topology, Part I'.

Last time we proved Van Kampen's Theorem and then derived that if you take the fundamental group of the wedge of circles then it is a free group, the rank was equal to number of circles involved there. We shall use that and try to do now the computation of a fundamental group of any one dimensional simplicial complex.

Indeed, we would like to include the case like wedge of circles also. So, we would like to extend the notion of these one-dimensional simplicial complex slightly a little more general. So, such things are called pseudo graphs. By a graph I will mean a one dimensional simplicial complex.

So, a more general thing, a pseudo graph some people may call it just graph itself, but for me a pseudo graph means that you can have a single vertex and then a loop around that or two vertices with many edges between them. These things are not allowed in a simplicial complex, one-dimension simplicial complex remember that. So, this instead of defining like this in an ad hoc fashion, I will do it systematically in a slightly different way which can be useful for us in doing more rigorous mathematics.

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Function Spaces and Quotient Spaces
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Simplicial Complexes II
Covering Spaces and Fundamental Group
G-Coverings and Fundamental Group

Module 56: Proof of Classification
Module 57: Pathways and Free Products
Module 58: Existence of Free Products
Module 59: Free Products and Free Groups
Module 60: Seifert – Van Kampen Theorem
Module 61: Applications
Module 62: Applications, Continued

Anant Shastri

Attaching cells

Definition 8.11
Let $k \geq 0$ be any integer. Let Y be a topological space and $\{f_\alpha\}_{\alpha \in \Lambda}$ be a family of maps $f_\alpha : S^k \rightarrow Y$. Put

$$A := \sqcup_{\alpha \in \Lambda} \mathbb{D}_\alpha^{k+1}$$
i.e., a disjoint union of copies of closed disc $\mathbb{D}_\alpha^{k+1} := \mathbb{D}^{k+1}$ indexed by the set Λ . The space X obtained by attaching $(k+1)$ -cells to Y via the family $\{f_\alpha\}$ is defined to be the quotient of the disjoint union and $A \sqcup Y$ via the identification:

$$x \sim f_\alpha(x) \text{ for all } x \in \partial \mathbb{D}_\alpha^{k+1}.$$

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Introduction
Module 32: G-Coverings
Module 34: Free Products and Path-Loops

So, here is the concept of attaching cells, back to the first chapter wherein and we have defined $A \sqcup Y$ adjunction spaces. So, this is a special case of adjunction spaces now. So, fix an integer k greater than or equal to 0, 0 is also allowed, let Y be a topological space and $\{f_\alpha\}_{\alpha \in \Lambda}$ be an index family of continuous functions from $S^k \rightarrow Y$. So, S^k is the unit sphere of dimension k in \mathbb{R}^{k+1} .

So, I have an index family of functions, k is fixed by the way, α is varying. There are a number of them, it may be 1, 2, 3 and any number of them. Take $A = \sqcup_{\alpha} \mathbb{D}_\alpha^{k+1}$, to be the disjoint union of all these the discs of one dimension higher, indexed again over the same set. The space X obtained by attaching $(k+1)$ -cells to Y via the maps $\{f_\alpha\}$, this is what I am going to define, this entire phrase.

So, this is, by definition, the quotient space of $A \sqcup Y$ by the identifications on the boundary of each of these disjoint discs, namely, I have a map here with: $x \sim f_\alpha(x)$, x is an element of the boundary of the disc $\mathbb{D}^{k+1} = S^k$ and $f_\alpha(x)$ is an element of Y , so identify them. This you do for all $x \in \partial \mathbb{D}^{k+1}$ and for all for all alpha.

If you have just map here, this is what is you would have called adjunction space. But then we can easily convert the situation to that familiar one by taking B to be the subspace $B := \sqcup_{\alpha} \partial(\mathbb{D}^{k+1}) \subset A$ and $f := \sqcup_{\alpha} f_\alpha : B \rightarrow Y$. So it is an adjunction space nothing else. But this is a very special adjunction space wherein all the discs of the same dimension are coming extra from Y . So, to Y , we have attached this discs, X is the resulting space.

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Module 61 Applications
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Definition 8.12

By a pseudo-graph we mean a space X as in the above definition, where Y is a discrete space and $k = 0$, i.e., X is obtained by attaching 1-cells to a discrete space Y . In this case, we also denote Y by the notation $X^{(0)}$ and call it the 0-skeleton of X or the set of vertices of X . Let $q : A \sqcup X^{(0)} \rightarrow X$ denote the corresponding quotient map.

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Module 59 Free Products and Free Groups
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Attaching cells

Definition 8.11

Let $k \geq 0$ be any integer. Let Y be a topological space and $\{f_\alpha\}_{\alpha \in \Lambda}$ be a family of maps $f_\alpha : S^k \rightarrow Y$. Put $A := \sqcup_{\alpha \in \Lambda} D_\alpha^{k+1}$ i.e., a disjoint union of copies of closed disc $D_\alpha^{k+1} := D^{k+1}$ indexed by the set Λ . The space X obtained by attaching $(k+1)$ -cells to Y via the family $\{f_\alpha\}$ is defined to be the quotient of the disjoint union and $A \sqcup Y$ via the identification:
$$x \sim f_\alpha(x) \text{ for all } x \in \partial D_\alpha^{k+1}.$$

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Module 53 G-Coverings
Module 54 Fibred Products and Pull-backs

Now, by a pseudo graph I mean a space X as in the above definition where I start with Y a discrete space, just a collection of points with discrete topology. And the I take here is 0 that means what I have is this $S^0 = \{-1, 1\}$. Y discrete a space, $\{-1, 1\}$ is also discrete, so any function from a two element set to Y is continuous automatically, either two elements here will go to the same element or they may go to different elements, that is the only two cases.

Then I am attaching 1-cells, $k + 1 = 0 + 1 = 1$. 1-cells what are they? $D_\alpha^1 = [-1, 1]$ the closed interval, the minus 1 goes to some point of Y , the plus 1 goes some other point of Y maybe same point of Y , does not matter, take the quotient space, that will be called as a pseudo graph.

So, we will have this terminology-- instead of writing Y , I will denote Y by $X^{(0)}$, that means the zeroth skeleton just like in the case of simplicial complex. It is just the set of points that is Y to begin with we call it the zeroth skeleton or those points will be called vertices also, this is similar to what we have done in the case of 1- dimensional simplicial complex.

Let then $q : A \sqcup X^{(0)} \rightarrow X$ denote the quotient map. That means what? q restricted to each 1-cell, either endpoints are identified or the endpoints are going to different points, so, depending upon that, either it is actually a homeomorphism or in any case in the interior it is a homeomorphism, endpoints maybe the same in which case the image will be a circle otherwise it will be an arc, it will be edge ordinary edge.

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<ul style="list-style-type: none"> Relative Homotopy Simplicial Complexes I Simplicial Complexes II Covering Spaces and Fundamental Group G-Coverings and Fundamental Group 	<ul style="list-style-type: none"> Module 57 Pathways and Free Products Module 58 Existence of Free Products Module 59 Free Products and Free Groups Module 60 Sylow and Von Klemperer Theorem Module 61 Applications Module 62 Applications: Continued
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Note that $X^{(0)}$ can be identified with a subspace of X , under the quotient map q . Points of $X^{(0)}$ are called the **0-cells** of X or the **vertices**. Also note that for each α , the quotient map q restricted to the interior of \mathbb{D}_α^1 is a homeomorphism f_α into an open subset of X . The images of f_α (and their closures) are called **1-cells** of X or the **edges** of the **pseudo-graph** X .

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So, $X^{(0)}$ is actually a subspace of X under this quotient map. So, again points of X naught are 0-cells or you can call them as vertices or 0 simplexes and so on. The word `simplex` will not be used here, because this is not a simplicial complex. Indeed, a simplicial complex of one dimension can also be described by this process, only thing is then you have to put some additional conditions. A pseudo graph is in this sense a slight generalization of a 1- dimensional simplicial complex.

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Remark 8.11

(1) A pseudograph is locally path connected. It is connected, it follows that $q(A) = X$ and $q|_A$ itself is a quotient map.

(2) Every 1-dimensional simplicial complex (i.e., a graph) is a pseudo-graph. Indeed in a graph, between two vertices there can be at most one edge whereas in a pseudo-graph several edges are allowed. Also, one can have edges with both their end points the same vertex, which is not allowed in a graph.

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Attaching cells

Definition 8.11

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$$A := \sqcup_{\alpha \in \Lambda} \mathbb{D}_\alpha^{k+1}$$

i.e., a disjoint union of copies of closed disc $\mathbb{D}_\alpha^{k+1} := \mathbb{D}^{k+1}$ indexed by the set Λ . The space X obtained by attaching $(k+1)$ -cells to Y via the family $\{f_\alpha\}$ is defined to be the quotient of the disjoint union and $A \sqcup Y$ via the identification:

$$x \sim f_\alpha(x) \text{ for all } x \in \partial \mathbb{D}_\alpha^{k+1}.$$

Clearly it is locally path connected. It is connected would imply, in particular that $f = \sqcup_{\alpha} f_{\alpha} : B \rightarrow Y$ is surjective and hence the converse is not true. We Shall later see some condition under which it is connected.

If f is not surjective, there may be some extra vertices hanging, if it is connected then you can define this whole thing this quotient space just on A itself, i.e., $q|_A : A \rightarrow X$ itself is a quotient map. How? Whenever two points are mapped onto the same point by f , identify them otherwise you do not identify anything. Where it goes to, alpha alpha goes something Y, some other f beta may also come to that one you identify them, that is the way it has to be done. So, this Y helps to

describe those relations, otherwise it is more complicated within A , but it is done inside A , if you use Y then it is easy to see what is happening.

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Figure 50: Subtree of a Pseudograph

So, here is a picture of a pseudo graph how it could be different from a simplicial complex. So, look at all these heavy lines and heavy line that is simplicial complex but I have attached a loop here, i.e., a one cell here of which the end points have gone to the same point, here also the same thing, here also same thing.

Here they are going to different points, but there is already another edge here. So, these are violating these, these, these things are violating the simplicial complex structure. So, this is the general picture of a pseudo graph, immediately you can see that if I put two more vertices here on this loop, this part becomes a simplicial complex on this part.

Here what should I do? Here also I should put two more vertices, even if I put one it is enough because this point and this point will be now single edge, this is another edge, so this will be like a triangle, so here I can do with just one vertex, putting two more vertices is no problem. What is the meaning of that?

You are as if you are subdividing this pseudo complex, we have not defined anything like this, sub divide the bad loops, a bad edge here and so on to get a simplicial complex, in a hidden way, the one dimensional simplicial complex theory can be applied to pseudo graphs also, I will use this remark again.

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The slide is divided into two main sections. The top section is a table of contents with a dark blue background and white text. The left column lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes I, Simplicial Complexes II, Covering Spaces and Fundamental Group, and G-Coverings and Fundamental Group. The right column lists: Module 53 G-Coverings, Module 54 Fibrations and Path-lifts, Module 55 Classification of G-coverings, Module 56 Proof of Classification, Module 57 Path-lifts and Free Products, Module 58 Existence of Free Products, Module 59 Free Products and Free Groups, Module 60 Seifert-Van Kampen Theorem, Module 61 Applications, and Module 62 Applications - Continual. A small video inset in the top right shows a man with glasses and a white beard, identified as Anant Shastri.

The bottom section has a white background with a blue header that reads "Definition 8.13". The text below the header states: "A connected pseudo-graph is called a **tree** if it is contractible. By a subtree of G , we mean a subcomplex T of G which is a tree." At the bottom of the slide, there is a footer with the NPTEL logo, the name "Anant R Shastri Retired Emeritus Fellow Department of Mathematics", and the course title "NPTEL Course on Algebraic Topology, Part-I".

Now, I make one more definition here, namely connected pseudo graph is called a tree, this is the definition, if it is contractible. As a topological space it must be contractible, then automatically it is connected. Of course, such a thing will be called a tree. By a sub tree of G , we mean a sub complex T of G such that it is a tree, it must be a pseudo graph on its own, but it must be sub.

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The slide features a diagram of a pseudo-graph. It consists of a central vertex connected to several other vertices. Some of these connections are solid lines, while others are dashed lines. There are also some loops represented by dashed lines. A mouse cursor is pointing at one of the vertices on the right side of the diagram. The slide includes the same table of contents and video inset as the previous slide. The footer at the bottom of the slide is identical to the previous slide.

For example, in this picture, you can remove this one and look at the rest of the picture that is a sub tree. So that is a sub graph, pseudo sub graph, if I delete this one then also it is true, I can just delete this edge and keep this vertex, but I should not delete this vertex and then I cannot keep this

edge. So, edge has to have complete vertex, where it is attached that is the question. So both endpoints must be somewhere.

So, here it is okay, both endpoints have gone here. So, you can remove this edge, but you cannot just remove a vertex, you can remove a vertex only if it is isolated vertex, if you remove a vertex here, all the edges which are emanating from there has to be removed that is the meaning of sub graph, sub graph, sub pseudo graph.

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The screenshot shows a presentation slide with a table of contents on the left and a video inset of Anant Shrivastava on the right. The table of contents lists modules from 1 to 11. The main content of the slide is:

Lemma 8.2
A connected pseudo-graph is a tree iff it is simply connected.

Proof: We need to prove the 'if' part only. So, let X be simply connected pseudo-graph. Given a vertex $v_0 \in X$, we shall define $H : X \times \mathbb{I} \rightarrow X$ such that $H(x, 0) = v_0$, and $H(x, 1) = x$, $\forall x \in X$. For this, we notice, because of the connectivity of X , that the quotient map restricted to $A = \coprod_{\alpha} J_{\alpha}$ is a surjective closed map and hence, is itself a quotient map onto X , where each $J_{\alpha} = [-1, 1]$ and $f_{\alpha} = q|_{J_{\alpha}} : J_{\alpha} \rightarrow X$ are the characteristic maps.

A connected pseudo graph is a tree if and only if it is simply connected. So, once it is simply connected, it will be actually contractible is what I have to say. If it is contractible simply connected is obvious.

So, I have to prove only the 'if' part. So, start with X a simply connected pseudo graph. Given a vertex v_0 belonging to X , we shall define a homotopy $h : X \times \mathbb{I} \rightarrow X$ such that $h(X \times \{0\}) = \{v_0\}$ and $h|_{X \times 1} = Id_X$ is identity map. For this we first notice that because of the connectivity assumption on X , the quotient map q restricted to A itself is a quotient map. This I have already marked earlier.

Therefore, constructing a map from $X \times \mathbb{I} \rightarrow X$ is the same as constructing it on $A \times \mathbb{I} \rightarrow A$, disjoint union of all these \mathbb{D}_{α}^1 's or J_{α} 's, whatever, copies of the the interval $[-1, 1]$ product with

$[0, 1]$, in a compatible way. For constructing map on a quotient space, always you can go back to original space and then do that. So, I am denoting copies of $[-1, 1]$ by J_α you can denote it by \mathbb{D}_α^1 , it does not matter.

And, let $\hat{f}_\alpha = q|_{J_\alpha} : J_\alpha \rightarrow X$. These are just the extensions of f_α 's the attaching maps. We call characteristic maps, the characteristic maps of the corresponding 1-cell.

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Given any vertex v in X , since X is path connected, there is a path ω_v starting at v_0 and ending at v . We take $H(v, t) = \omega_v(t)$. Now let $f = f_\alpha : \mathbb{D}^1 \rightarrow X$ be the characteristic map of a 1-cell in X . Consider the map $g : \partial(\mathbb{D}^1 \times \mathbb{I}) \rightarrow X$ defined by $g(s, 0) = v_0; g(s, 1) = f(s); g(-1, t) = \omega_{f(-1)}(t)$ and $g(1, t) = \omega_{f(1)}(t)$. Obviously g is continuous.

Given any vertex v in X , since X is path connected, there is a path $\omega_v : \mathbb{I} \rightarrow X$ starting at v_0 and ending at v . Fix those paths, there may be many paths, I do not care, take some paths like this and fix them once for all. Now, you define $H(v, t) = \omega_v(t)$. For each vertex v in X , I have defined this one, i.e., for each vertex v , the map H is defined on $\{v\} \times \mathbb{I}$.

Now, look at $\hat{f}_\alpha : J_\alpha \rightarrow X$, f_α from \mathbb{D}^1 to X . Again, I am writing \mathbb{D}^1 or J , minus 1 plus 1 is here, be the characteristic maps from one cell of X . Fix an α and let us drop this notation temporarily. Define $g : \partial(-1, 1] \times [0, 1] \rightarrow X$ as follows: $g(s, 0) = v_0;$

$g(s, 1) = \hat{f}_\alpha(s); g(-1, t) = \omega_{\hat{f}_\alpha(-1)}(t); g(1, t) = \omega_{\hat{f}_\alpha(1)}(t)$. Clearly g continuous and you may think of this as a loop at v_0 in X . Now X is simply connected. Therefore, this loop can be, this function can be extended over $\mathbb{D}^1 \times \mathbb{I}$. A function which is defined on the boundary of, boundary of \mathbb{D}^1 cross \mathbb{I} which is \mathbb{D}^2 (19:27) can be extended to inside because it is null homotopic. So, X is simply connected is used here, g is continuous, so you can extend it.

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<ul style="list-style-type: none"> Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes I Simplicial Complexes II Covering Spaces and Fundamental Group G-Coverings and Fundamental Group 	<ul style="list-style-type: none"> Module 53 G-Coverings Module 54 Filtered Products and Push-outs Module 55 Classification of G-coverings Module 56 Proof of Classification Module 57 Pushouts and Free Products Module 58 Existence of Free Products Module 59 Free Products and Free Groups Module 60 Seifert-Van Kampen Theorem Module 61 Application Module 62 Applications Continued
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Since X is simply connected, it follows that we get an extension of g to $\hat{g} = \hat{g}_\alpha : \mathbb{D}_\alpha^1 \times \mathbb{I} \rightarrow X$. Put

$$G : \sqcup_\alpha \mathbb{D}_\alpha^1 \times \mathbb{I} \rightarrow X; \quad G|_{\mathbb{D}_\alpha^1 \times \mathbb{I}} = \hat{g}_\alpha, \forall \alpha.$$

Then G factors down to define a continuous map $H : X \times \mathbb{I} \rightarrow X$ as required.

Module 61 Application Continued

Given any vertex v in X , since X is path connected, there is a path ω_v starting at v_0 and ending at v . We take $H(v, t) = \omega_v(t)$. Now let $f = f_\alpha : \mathbb{D}^1 \rightarrow X$ be the characteristic map of a 1-cell in X . Consider the map $g : \partial(\mathbb{D}^1 \times \mathbb{I}) \rightarrow X$ defined by $g(s, 0) = v_0; g(s, 1) = f(s); g(-1, t) = \omega_{f(-1)}(t)$ and $g(1, t) = \omega_{f(1)}(t)$. Obviously g is continuous.

You do this for each α . Since X is simply connected, you have an extension $\bar{g}_\alpha : \mathbb{D}_\alpha^1 \times \mathbb{I} \rightarrow X$ of $g = g_\alpha$ for each alpha. Now, put $G = \sqcup_\alpha \bar{g}_\alpha : A \times \mathbb{I} \rightarrow X$, you take the disjoint union over this one so that on each restriction it is g alpha. All that you have to observe is that wherever you have identified, it is the same old thing for each of them, so it is compatible. Therefore, this factors down to a continuous map $H : X \times \mathbb{I} \rightarrow X$ such that $H \circ (q \times Id) = G$. By the very choice $H(s, 0) = G(q \times Id(s, 0)) = g(s, 0) = v_0$.

And $H(\hat{f}_\alpha(s), 1) = H(q \times Id(s, 1)) = g_\alpha(s, 1) = \hat{f}_\alpha(s)$. So that proves that X is contractible.

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The slide displays a table of contents for the NPTEL course on Algebraic Topology, Part I, presented in a dark blue box. The table lists various modules and topics, with 'Module 33 G-Coverings' highlighted in white. Below the table of contents, a separate blue box contains 'Proposition 8.1', which states: 'Let T be a subtree of a pseudo-graph G . Then the quotient map $G \rightarrow G/T$ is a homotopy equivalence.'

Anant R Shastri Retired Emeritus Fellow Department of Mathem. NPTEL Course on Algebraic Topology, Part I	
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Proposition 8.1
Let T be a subtree of a pseudo-graph G . Then the quotient map $G \rightarrow G/T$ is a homotopy equivalence.

The next result. Let T be a subtree of a pseudo graph G . Then a quotient map G to G by T is a homotopy equivalence. If it is a tree then it is contractible. We have seen long back, a result when you can collapse a contractible subspace to get the quotient map will be a homotopy equivalence. So, I would like to recall this, namely, when the inclusion map of this contractible subspace into the whole space must be a cofibration. Remember that theorem, so use that theorem.

To conclude that $G \rightarrow G/T$, the quotient map is homotopy equivalence, namely when you collapse a tree, tree means it is contractible thing. Like an edge can be collapsed, union of two edges at a vertex, if they do not form a loop that can be collapsed and so on.

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Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

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Anant Shastri

Proof: If G is a graph, this follows from the fact $T \subset G$ is a cofibration (See Theorem 4.4.)
 In the general case, note that by adding one or two extra vertices on all edges of G which are not in T we can make G into a graph.

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So, I have to use the result on cofibration. If G is a graph then we know that the inclusion map of T into G is a cofibration. By the way, this cofibration result was done only for simplicial complexes. But our situation can be converted easily to the case of a simplicial complex because a pseudo graph can always be subdivided, by putting extra vertices, and made into a simplicial complex.

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The screenshot shows a presentation slide with a blue header and footer. The header contains the following text: "Simplicial Complexes II", "Covering Spaces and Fundamental Group", "G-Coverings and Fundamental Group", "Module 30 Free Products and Free Groups", "Module 30 Subsets - Van Kampen Theorem", "Module 31 Applications", and "Module 32 Advanced Concepts". The main content of the slide is a white box with a blue border containing the text: "Theorem 8.12", "Let X be a connected (non empty) pseudo-graph, and T_0 be a subtree in it. Then there exists a subtree T in X containing T_0 , and such that T contains all vertices of X ." A mouse cursor is pointing at the text. The footer contains the text: "Anant R Shastri Retired Emeritus Fellow Department of Mathematics", "NPTEL Course on Algebraic Topology, Part I", and "NPTEL". The video inset shows a man with a white beard and glasses, identified as Anant Shastri.

Now the next result. Let X be a connected nonempty pseudo graph. Let T_0 be a subtree in it. Then there exists a subtree in X , that is actually T , containing this given T_0 such that this T contains all the vertices of X . This is one way of telling that that this is a maximal tree, you cannot make it into a larger tree by putting extra edges, because when you put all the vertices that are there, as soon as you put one extra edge there will be a loop.

So, let us prove this one, a rigorous proof is required now, using Zorn's lemma or some such thing. No hand waving can be done. But in the finite case you can actually do this, there is no problem, for finite case you do not need Zorn's lemma.

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Proof: Let \mathcal{T} be the collection of all subtrees in X which contain T_0 . Order it by inclusion. We shall apply Zorn's lemma. So, let $\{T_i\}$ be a chain in \mathcal{T} . We claim $T = \cup_i T_i$ is a tree. Given any point $x \in T$ say $x \in T_i$. Then there is a path in T_i from x to some point in T_0 which will be also a path in T and so, T is connected. Now suppose $\omega : I \rightarrow T$ is a loop. Since its image is compact, it follows that the image is contained in the union of finitely many closed 1-cells, which are all contained in some T_i . But then this loop is null homotopic in T_i and hence null homotopic in T as well.

So, let \mathcal{T} be the collection of all subtrees in X which contain T_0 . Partially order it by inclusion. Now, we shall apply Zorn's lemma and then conclude that there is a maximal one. All members of \mathcal{T} contain T_0 so maximal will also contain T_0 .

So, let $\{T_i\}$ be a chain in \mathcal{T} , in this collection. Chain means what? Totally order subcollection. We claim that $T = \cup_i T_i$ is a tree. So, how do we know that it is a tree? First of all, given any point in this union, it will be in one of the T_i 's, because it is a union. Then there is a path in T_i from X to some point in T_0 , because the tree T_i is a connected, which will be also path in T and so T is connected. T_i is a tree, so T_i 's are connected but now I have proved that T is connected.

Now, suppose ω is a loop in T . A loop means what? It is continuous function from I to T . Therefore, the image must be compact. We know that any compact subset is contained in a finite sub pseudo graph. It follows that the image is contained in the union of finitely many closed 1-cells. Closed 1 cells means what? I have told you that it is just either edges or full circles which are all contained in some T_i because T_i 's are only finitely many of them let us say all the vertices are in T_1, T_2, \dots and then take the maximum of them in that T_i they will be all be there. But then this loop is inside T_i , it null homotopic in T_i . But T_i is a sub space of T , so it is null homotopic in T .

Therefore, every chain has an upper bound, so that is sufficient condition for Zorn's lemma, Zorn's lemma will tell you that there is a maximal tree. A maximal element of \mathcal{T} is nothing but a maximal subtree of G which contains T_0 .

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The slide is from an NPTEL course on Algebraic Topology, Part I, presented by Anant R Shastri, a Retired Emeritus Fellow at the Department of Mathematics. The slide content is as follows:

Table of Contents:

Introduction	Module 53 G-Coverings
Fundamental Group	Module 54 Filtered Products and Path-lifts
Function Spaces and Quotient Spaces	Module 55 Classification of G-coverings
Relative Homotopy	Module 56 Proof of Classification
Simplicial Complexes-I	Module 57 Products and Free Products
Simplicial Complexes-II	Module 58 Subgroups of Free Products
Covering Spaces and Fundamental Group	Module 59 Free Products and Free Groups
G-Coverings and Fundamental Group	Module 60 Serret-Van Kampen Theorem
	Module 61 Applications
	Module 62 Applications Continued

Main Text:

By Zorn's lemma there is a maximal element in \mathcal{T} say T . We claim that all vertices of X are in T . If this is not true there will be an edge s which will have one end point in T and another not in T . But then it is easily seen that $T \cup s$ is a deformation retract of $T \cup s$ and hence $T \cup s$ is also a tree contradicting the maximality of T . ♠

Theorem 8.12

Let X be a connected (non empty) pseudo-graph, and T_0 be a subtree in it. Then there exists a subtree T in X containing T_0 , and such that T contains all vertices of X .

The video inset shows Anant R Shastri speaking.

Now, suppose there is another vertex which is not in this maximal T . That would mean what? Because X is connected from that extra vertex you must be having a sequence of edges all the way from the extra vertex and coming to this T . That would mean that you will get a vertex v somewhere along such that v is not in T but the next edge, say, s , will have the other vertex inside

T. So, this is what it means. What we have done so far? We have seen that there is an edge s which will have one endpoint in T and the other end is not in T . From the endpoint which is not in T you can collapse the whole edge into T , contract. That means, T is a strong deformation retract of $T \cup s$.

Therefore, $T \cup s$ is also contractible. Therefore, $T \cup s$ is a larger tree in X . That is a contradiction to the fact that T is maximal. Therefore, there are no more vertices. So, that completes the proof that given any tree there is a largest tree that contains it and that largest tree contains all the vertices of X .

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The screenshot shows a presentation slide from an NPTEL course. At the top, it identifies the speaker as Anant R. Shastri, a Retired Emeritus Fellow from the Department of Mathematics, and the course as 'NPTEL Course on Algebraic Topology, Part-I'. Below this is a table of contents with two columns. The first column lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes-I, Simplicial Complexes-II, Covering Spaces and Fundamental Group, and G-Coverings and Fundamental Group. The second column lists: Module 32 G-Coverings, Module 33 Fibred Products and Pullbacks, Module 34 Classification of G-Coverings, Module 35 Proof of Classification, Module 37 Pushouts and Free Products, Module 38 Existence of Free Products, Module 39 Free Products and Free Groups, Module 40 Seifert - Van Kampen Theorem, Module 61 Applications, and Module 62 Applications Continued. To the right of the table of contents is a small video inset of the speaker, Anant Shastri. Below the table of contents, a blue header box contains 'Theorem 8.13', and a white text box below it states: 'Every connected pseudo-graph is homotopy equivalent to a bouquet of circles.'

In particular, there is always such a thing if we start with a single vertex. Single vertex is a tree after all. So, every connected pseudo graph is homotopic to a bouquet of circles. This is the corollary of whatever we have done finally. How does you get it? Start with the pseudo graph, take any vertex you want. There will be a tree containing that. That tree I will take the maximal one, which will have all the vertices in it, all the vertices of G are there.

This is a tree I can collapse it. Then $q : G \rightarrow G/T$ is a homotopy equivalence. What will happen to G/T ? All those edges which are not in T they would become circles, every edge which is in T has become a single point, at that single point I will have some circles.

What are these circles? Only for those edges, I did not say that all the edges are inside T all the vertices are inside T , all those edges which are outside T they will become circles now. So, G

by T is a wedge of circles, namely, one point union of circles, the number of circles will be precisely equal to the number of edges which we have missed from T .

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Anant R Shastri Retired Emeritus Fellow Department of Mathematics, NITEL Course on Algebraic Topology, Part-I	
Introduction	Module 31 G-Coverings
Fundamental Group	Module 32 Free Products and Pull-backs
Function Spaces and Quotient Spaces	Module 33 Classification of G-coverings
Relative Homotopy	Module 34 Proof of Classification
Simplicial Complexes-I	Module 35 Pushouts and Free Products
Simplicial Complexes-II	Module 36 Existence of Free Products
Covering Spaces and Fundamental Group	Module 37 Free Products and Free Groups
G-Coverings and Fundamental Group	Module 38 Seifert — Van Kampen Theorem
	Module 39 Applications
	Module 40 Applications Continued

Proof: Starting from $T_0 = \{v_0\}$, any single vertex, take a tree T in X containing all vertices of X . Then the quotient map $X \rightarrow X/T$ is a homotopy equivalence. All the vertices have been identified to a single vertex $*$. So are all the edges in T and so the edges in $X \setminus T$ are the ones which survive and the two points of these edges are identified with $*$. Therefore X/T is the bouquet of circles with the number of circles equal to the number of edges in $X \setminus T$. 🔥

This is the gist of this thing. We are starting with $T_0 = \{v_0\}$, any single vertex, take T to be a maximal tree containing all the vertices, a tree containing all the vertices. Then $q : X \rightarrow X/T$ is a homotopy equivalence. All the vertices have been identified to a single vertex, because T contains all the vertices.

So, all the edges in T and also have become to single point, so the edges in X minus T are the ones which survive and how they survive? They become circles and 2 endpoints of these edges are identity to single point. Therefore, X by T is a bouquet of circles. Therefore, what is the conclusion? You take a pseudo graph which is connected, take any point in X , $\pi_1(X, v_0)$ is a free group, that is the conclusion. So, let us go to the next theorem.

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The slide is a screenshot from an NPTEL course. At the top, it identifies the speaker as Anant R. Shastri, a Retired Emeritus Fellow from the Department of Mathematics, and the course as 'NPTEL Course on Algebraic Topology, Part-I'. Below this is a table of contents with two columns. The left column lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes-I, Simplicial Complexes-II, Covering Spaces and Fundamental Group, and G-Coverings and Fundamental Group. The right column lists: Module 3: G-Covering, Module 5: Free Products and Pushouts, Module 56: Classification of G-coverings, Module 56: Proof of Classification, Module 57: Pushouts and Free Products, Module 58: Existence of Free Products, Module 59: Free Products and Free Groups, Module 60: Seifert-Van Kampen Theorem, Module 61: Applications, and Module 62: Additional Content. Below the menu is a blue header for 'Corollary 8.1' followed by the text: 'Given a connected pseudo-graph X , $\pi_1(X)$ is a free group of rank equal to the number of edges outside any maximal tree T in X .' The NPTEL logo is visible in the bottom left corner of the slide.

Given a connected pseudo graph π_1 of X is a free group of rank equal to the number of edges outside any maximal tree in X . Therefore, this number is independent of the choice of the maximal tree that we make. By the way the maximal tree may not be unique, you can think about that, but the number of edges outside because of this will be the same that is the beauty.

So, that is something which we have now. Next time we will use this one to prove a big theorem in group theory and some more topology later on. So, that will be the last module, the last lecture for this course. Thank you.