Introduction to Algebraic Topology (Part - I)
Professor Anant R Shastri
Indian Institute of Technology Bombay
Lecture 61
Applications

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Last time we proved Van Kampen's Theorem and then derived that if you take the fundamental group of the wedge of circles then it is a free group, the rank was equal to number of circles involved there. We shall use that and try to do now the computation of a fundamental group of any one dimensional simplicial complex.

Indeed, we would like to include the case like wedge of circles also. So, we would like to extend the notion of these one-dimensional simplicial complex slightly a little more general. So, such things are called pseudo graphs. By a graph I will mean a one dimensional simplicial complex.

So, a more general thing, a pseudo graph some people may call it just graph itself, but for me a pseudo graph means that you can have a single vertex and then a loop around that or two vertices with many edges between them. These things are not allowed in a simplicial complex, one-dimension simplicial complex remember that. So, this instead of defining like this in an ad hoc fashion, I will do it systematically in a slightly different way which can be useful for us in doing more rigorous mathematics.

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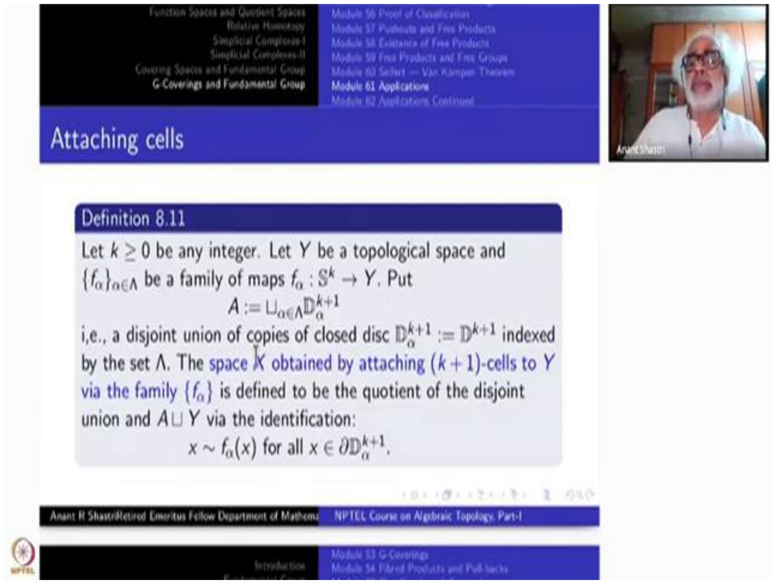

So, here is the concept of attaching cells, back to the first chapter wherein and we have defined $A \sqcup Y$adjunction spaces. So, this is a special case of adjunction spaces now. So, fix an integer k greater than or equal to 0, 0 is also allowed, let Y be a topological space and $\{f_\alpha\}_{\alpha\in\Lambda}$ be an index family of continuous functions from $\mathbb{S}^k \to Y$. So, $\mathbb{S}^k$ is the unit sphere of dimension k in $\mathbb{R}^{k+1}$.

So, I have an index family of functions, k is fixed by the way, α is varying. There are a number of them, it may be 1, 2, 3 and any number of them. Take A=$\sqcup_\alpha \mathbb{D}^{k+1}_\alpha$, to be the disjoint union of all these the discs of one dimension higher, indexed again over the same set. `The space X obtained by attaching $(k+1)$-cells to Y via the maps $\{f_\alpha\}$' , this is what I am going to define, this entire phrase.

So, this is, by definition, the quotient space of $A \sqcup Y$ by the identifications on the boundary of each of these disjoint discs, namely, I have a map here with: $x \sim f_\alpha(x)$, x is an element of the boundary of the disc $\mathbb{D}^{k+1} = \mathbb{S}^k$ and $f_\alpha(x)$ is an element of Y, so identify them. This you do for all $x \in \partial\mathbb{D}^{k+1}$ and for all for all alpha.

If you have just map here, this is what is you would have called adjunction space. But then we can easily convert the situation to that familiar one by taking B to be the subspace $B := \sqcup_\alpha \partial(\mathbb{D}^{k+1}) \subset A$ and $f := \sqcup_\alpha f_\alpha : B \to Y$. So it is an adjunction space nothing else. But this is a very special adjunction space wherein all the discs of the same dimension are coming extra from Y. So, to Y, we have attached this discs, X is the resulting space.

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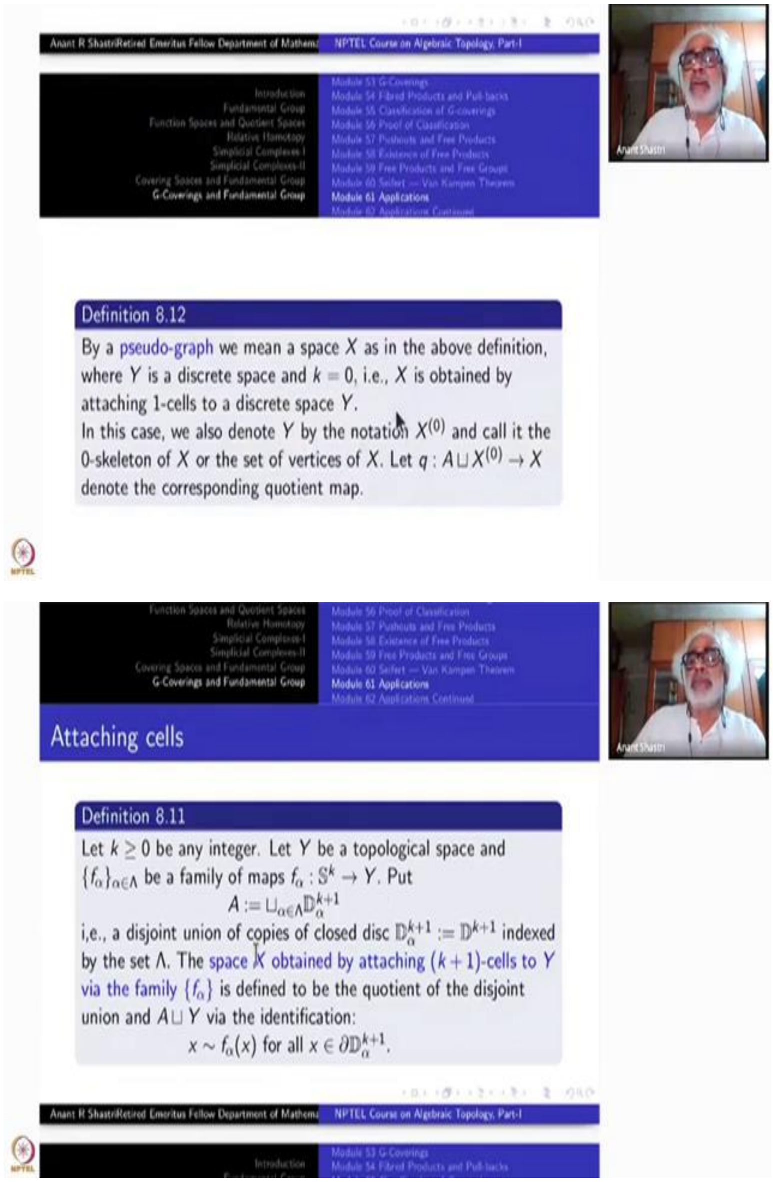

Now, by a pseudo graph I mean a space X as in the above definition where I start with Y a discrete space, just a collection of points with discrete topology. And the I take here is 0 that means what I have is this $\mathbb{S}^0 = \{-1, 1\}$. Y discrete a space, $\{-1, 1\}$ is also discrete, so any function from a two element set to Y is continuous automatically, either two elements here will go to the same element or they may go to different elements, that is the only two cases.

Then I am attaching 1-cells, $k + 1 = 0 + 1 = 1$. 1-cells what are they? $\mathbb{D}_\alpha^1 = [-1, 1]$ the closed interval, the minus 1 goes to some point of Y, the plus 1 goes some other point of Y maybe same point of Y, does not matter, take the quotient space, that will be called as a pseudo graph.

So, we will have this terminology-- instead of writing Y, I will denote Y by $X^{(0)}$, that means the zeroth skeleton just like in the case of simplicial complex. It is just the set of points that is Y to begin with we call it the zeroth skeleton or those points will be called vertices also, this is similar to what we have done in the case of 1- dimensional simplicial complex.

Let then $q : A \sqcup X^{(0)} \to X$ denote the quotient map. That means what? q restricted to each 1-cell, either endpoints are identified or the endpoints are going to different points, so, depending upon that, either it is actually a homeomorphism or in any case in the interior it isa homeomorphism, endpoints maybe the same in which case the image will be a circle otherwise it will be an arc, it will be edge ordinary edge.

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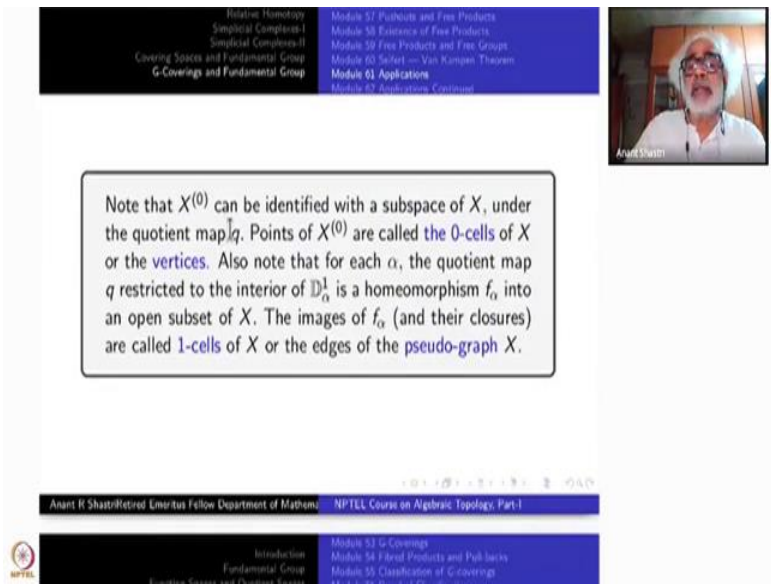

So, X^{(0)} is actually a subspace of X under this quotient map. So, again points of X naught are 0 -cells or you can call them as vertices or 0 simplexes and so on. The word `simplex' will not be used here, because this is not a simplicial complex. Indeed, a simplicial complex of one dimension can also be described by this process, only thing is then you have to put some additional conditions. A pseudo graph is in this sense a slight generalization of a 1- dimensional simplicial complex.

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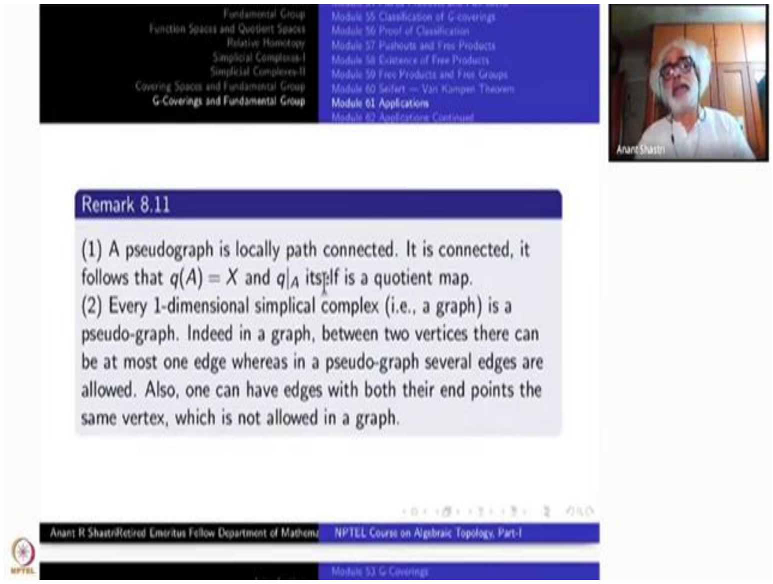

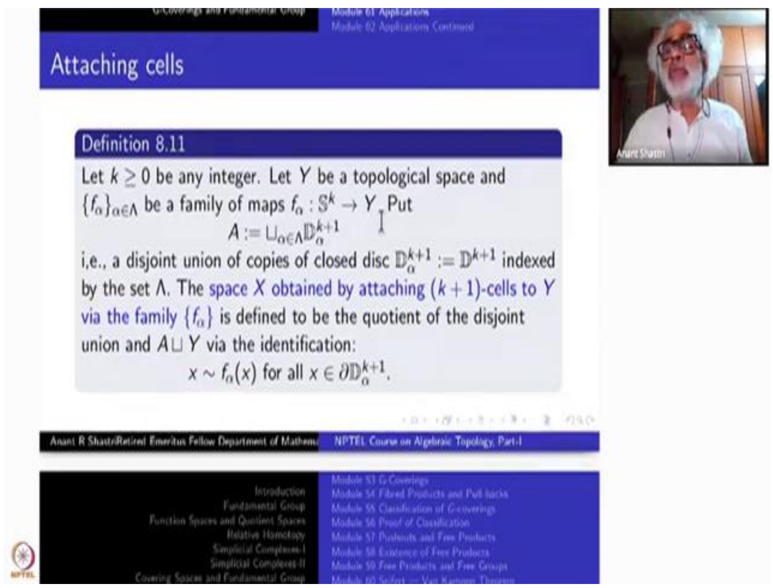

Clearly it is locally path connected. It is connected would imply, in particular that $f = \sqcup_\alpha f_\alpha : B \to Y$ is surjective and hence the converse is not true. We Shall later see some condition under which it is connected.

If f is not surjecive, there may be some extra vertices hanging, if it is connected then you can define this whole thing this quotient space just on A itself, ie., $q|_A : A \to X$ itself is a quotient map. How? Whenever two points are mapped onto the same point by f, identify them otherwise you do not identify anything. Where it goes to, alpha alpha goes something Y, some other f beta may also come to that one you identify them, that is the way it has to be done. So, this Y helps to describe those relations, otherwise it is more complicated within A, but it is done inside A, if you use Y then it is easy to see what is happening.

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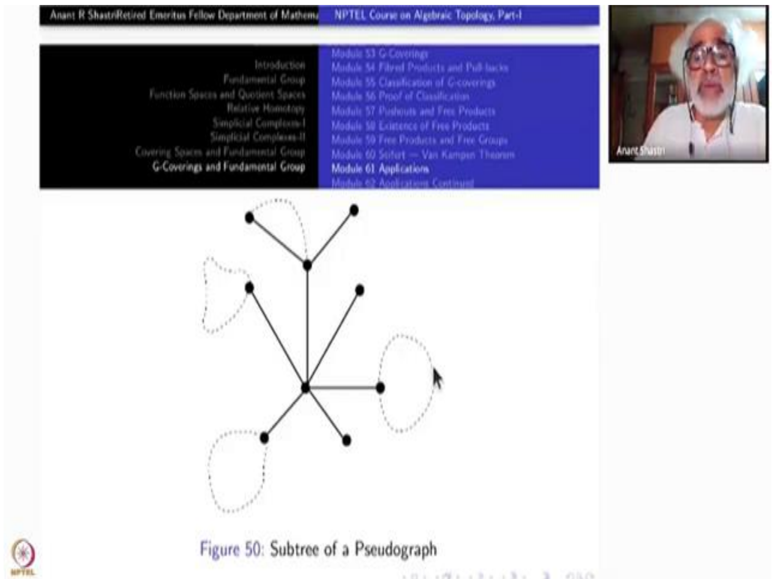

Figure 50: Subtree of a Pseudograph

So, here is a picture of a pseudo graph how it could be different from a simplicial complex. So, look at all these heavy lines and heavy line that is simplicial complex but I have attached a loop here, i.e., a one cell here of which the end points have gone to the same point, here also the same thing, here also same thing.

Here they are going to different points, but there is already another edge here. So, these are violating these, these, these things are violating the simplicial complex structure. So, this is the general picture of a pseudo graph, immediately you can see that if I put two more vertices here on this loop, this part becomes a simplicial complex on this part.

Here what should I do? Here also I should put two more vertices, even if I put one it is enough because this point and this point will be now single edge, this is another edge, so this will be like a triangle, so here I can do with just one vertex, putting two more vertices is no problem. What is the meaning of that?

You are as if you are subdividing this pseudo complex, we have not defined anything like this, sub divide the bad loops, a bad edge here and so on to get a simplicial complex, in a hidden way, the one dimensional simplicial complex theory can be applied to pseudo graphs also, I will use this remark again.

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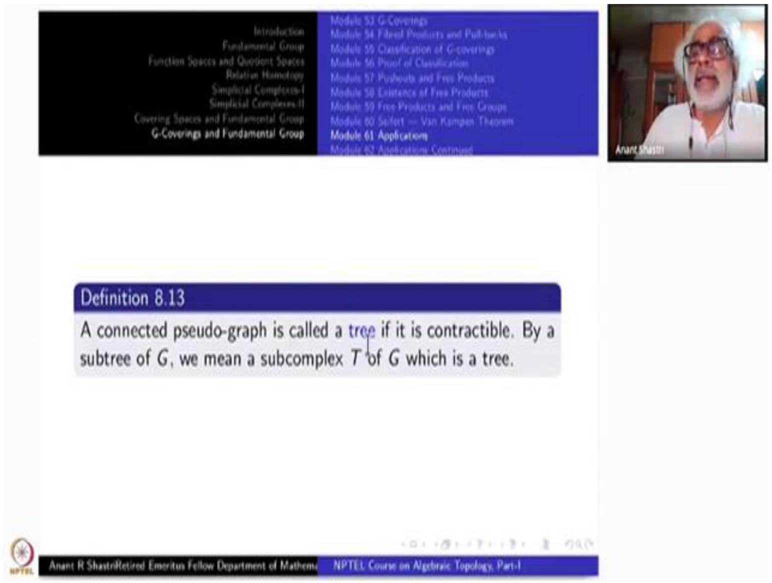

Now, I make one more definition here, namely connected pseudo graph is called a tree, this is the definition, if it is contractible. As a topological space it must be contractible, then automatically it is connected. Of course, such a thing will be called a tree. By a sub tree of G, we mean a sub complex T of G such that it is a tree, it must be a pseudo graph on its own, but it must be sub.

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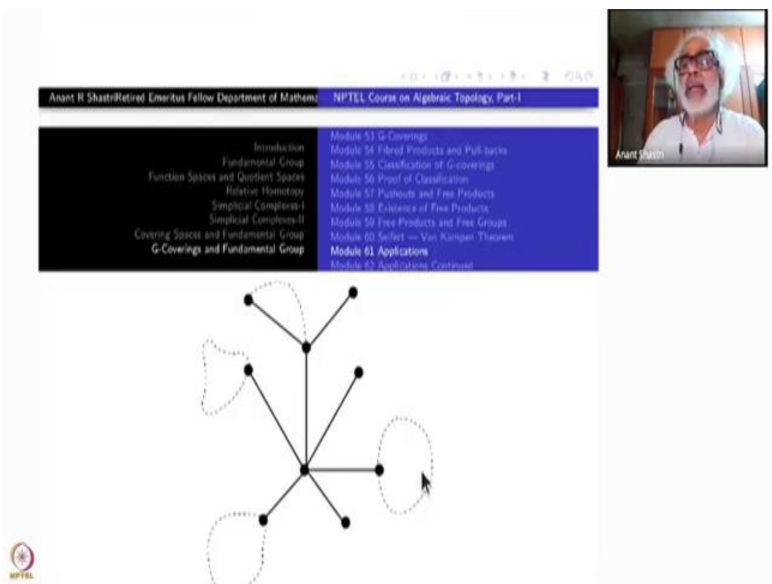

For example, in this picture, you can remove this one and look at the rest of the picture that is a sub tree. So that is a sub graph, pseudo sub graph, if I delete this one then also it is true, I can just delete this edge and keep this vertex, but I should not delete this vertex and then I cannot keep this edge. So, edge has to have complete vertex, where it is attached that is the question. So both endpoints must be somewhere.

So, here it is okay, both endpoints have gone here. So, you can remove this edge, but you cannot just remove a vertex, you can remove a vertex only if it is isolated vertex, if you remove a vertex here, all the edges which are emanating from there has to be removed that is the meaning of sub graph, sub graph, sub pseudo graph.

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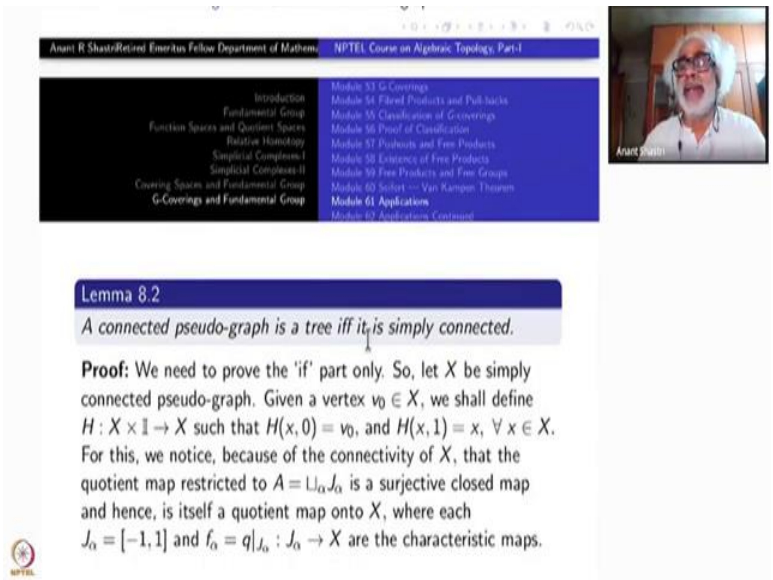

A connected pseudo graph is a tree if and only if it is simply connected. So, once it is simply connected, it will be actually contractible is what I have to say. If it is contractible simply connected is obvious.

So, I have to prove only the `if' part. So, start with X a simply connected pseudo graph. Given a vertex v_0 belonging to X, we shall define a homotopy $h : X \times \mathbb{I} \to X$ X such that $h(X \times \{0\}) = \{v_0\}$ and $h|_{X \times 1} = Id_X$ is identity map. For this we first notice that because of the connectivity assumption on X, the quotient map q restricted to A itself is a quotient map. This I have already marked earlier.

Therefore, constructing a map from $X \times \mathbb{I} \to X$ is the same as constructing it on $A \times \mathbb{I} \to A$, disjoint union of all these $\mathbb{D}^1_\alpha$'s or J_α's , whatever, copies of the the interval $[-1, 1]$ product with

$[0, 1]$, in a compatible way. For constructig map on a quotient space, always you can go back to original space and then do that. So, I am denoting copies of $[-1, 1]$ by J_α you can denote it by $\mathbb{D}^1_\alpha$, it does not matter.

And, let $\hat{f}_\alpha = q|_{J_\alpha} : J_\alpha \to X$. These are just the extensions of f_α's the attaching maps. We call characteristic maps, the characteristic maps of the corresponding 1-cell.

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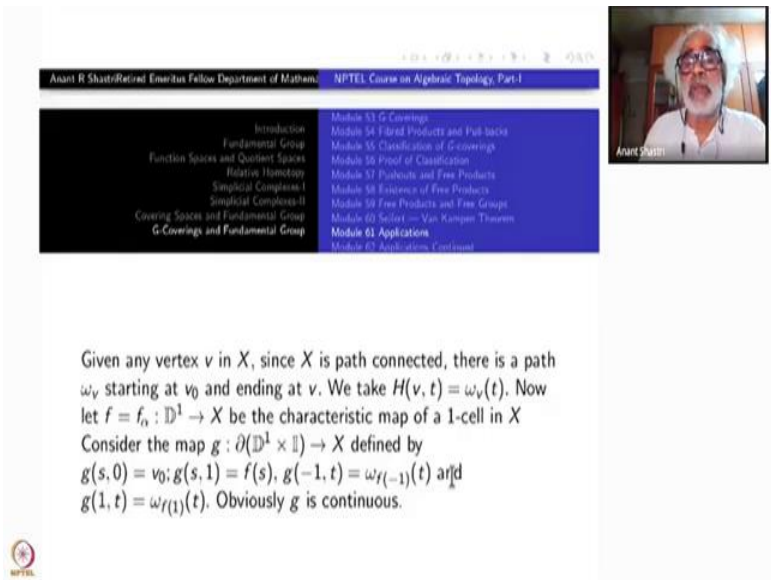

Given any vertex v in X, since X is path connected, there is a path $\omega_v : \mathbb{I} \to X$ starting at v_0 and ending at v. Fix those paths, there may be many paths, I do not care, take some paths like this and fix them once for all. Now, you define $H(v, t) = \omega_v(t)$. For each vertex v in X, I have defined this one, i.e., for each vertex v, the map H is defined on $\{v\} \times \mathbb{I}$.

Now, look at $\hat{f}_\alpha : J_\alpha \to X$, f alpha from D1 to X. Again, I am writing D1 or J, minus 1 plus 1 is here, be the characteristic maps from one cell of X. Fix an α and let su drop this notation temporarily. Define $g : \partial(-1, 1] \times [0, 1]) \to X$ as follows: $g(s, 0) = v_0$; $g(s, 1) = \hat{f}(s); g(-1, t) = \omega_{\hat{f}(-1)}(t); g(1, t) = \omega_{\hat{f}(1)}(t)$. Clearly g continuous and you may think of this as a loop at v_0 in X. Now X is simply connected. Therefore, this loop can be, this function can be extended over $\mathbb{D}^1 \times \mathbb{I}$. A function which is defined on the boundary of, boundary of D1 cross I which is D2 (())(19:27) can be extended to inside because it is null homotopic. So, X is simply connected is used here, g is continuous, so you can extend it.

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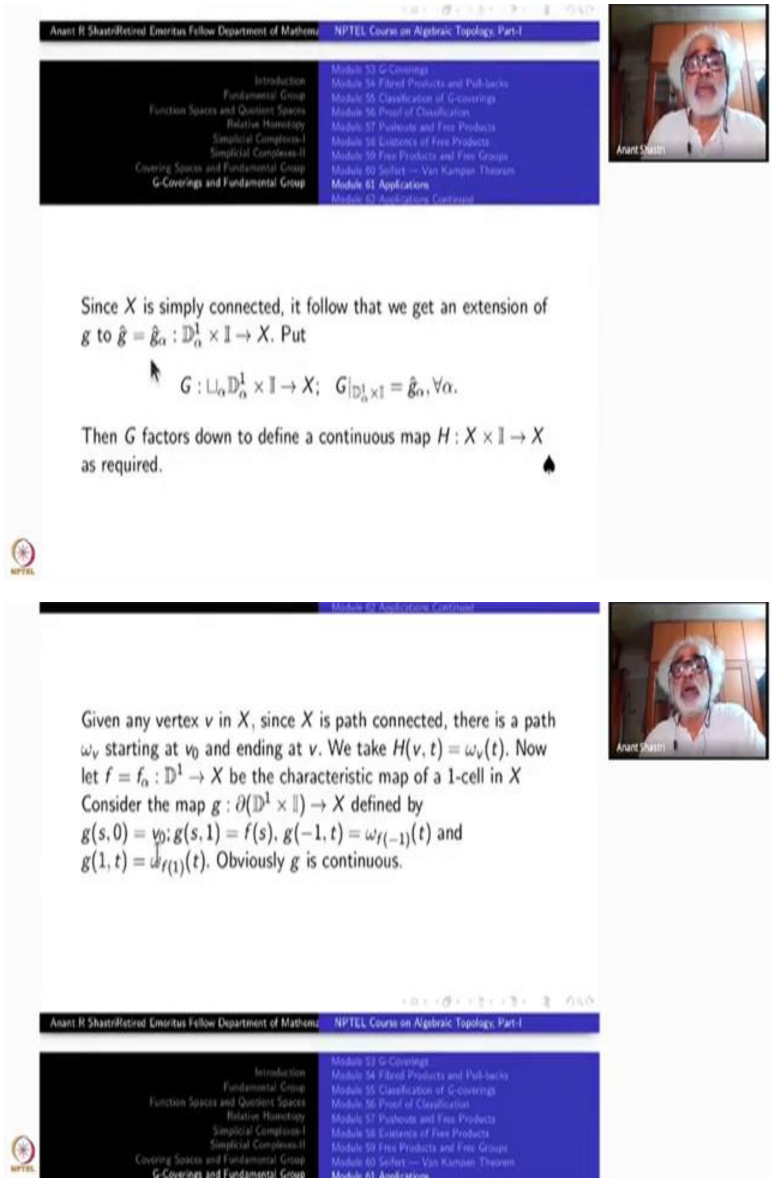

You do this for each α. Since X is simply connected, you have an extension $\bar{g}_\alpha : \mathbb{D}^1_\alpha \times \mathbb{I} \to X$ of $g = g_\alpha$ for each alpha. Now, put $G = \sqcup_\alpha \bar{g}_\alpha : A \times \mathbb{I} \to X$, you take the disjoint union over this one so that on each restriction it is g alpha. All that you have to observe is that wherever you have identified, it is the same old thing for each of them, so it is compatible. Therefore, this factors down to a continuous map $H : X \times \mathbb{I} \to X$ such that $H \circ (q \times Id) = G.$ By the very choice $H(s,0) = G(q \times Id(s,0)) = g(s,0) = v_0.$

And $H(\hat{f}_\alpha(s), 1) = H(q \times Id)(s,1)) = g_\alpha(s,1) = \hat{f}_\alpha(s).$ So that proves that X is contractible.

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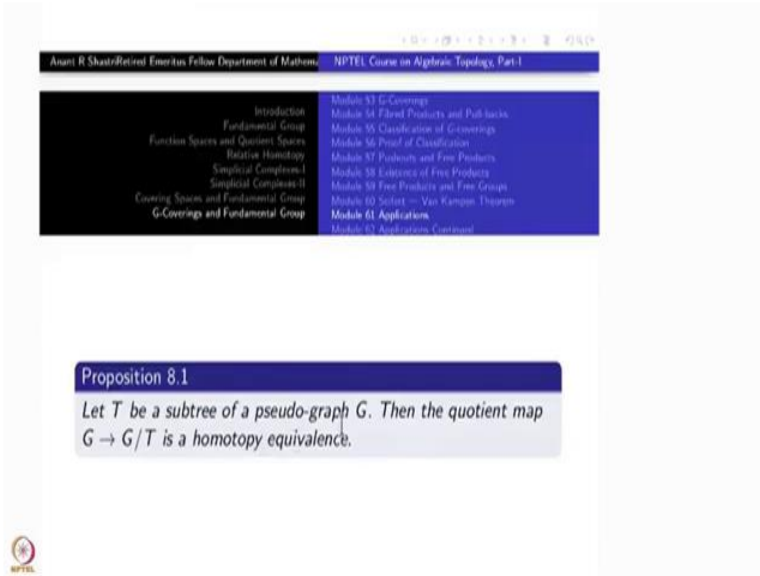

The next result. Let T be a subtree of a pseudo graph G. Then a quotient map G to G by T is a homotopy equivalence. If it is a tree then it is contractible. We have seen long back, a result when you can collapse a contractible subspace to get the quotient map will be a homotopy equivalence. So, I would like to recall this, namely, when the inclusion map of this contractible subspace into the whole space must be a cofibration. Remember that theorem, so use that theorem.

To conclude that $G \to G/T$, the quotient map is homotopy equivalence, namely when you collapse a tree, tree means it is contractible thing. Like an edge can be collapsed, union of two edges at a vertex, if they do not form a loop that can be collapsed and so on.

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So, I have to use the result on cofibration. If G is a graph then we know that the inclusion map of T into G is a cofibration. By the way, this cofibration result was done only for simplicial complexes. But our situation can be converted easily to the case of a simplicial complex because a pseudo graph can always be subdivided, by putting extra vertices, and made into a simplicial complex.

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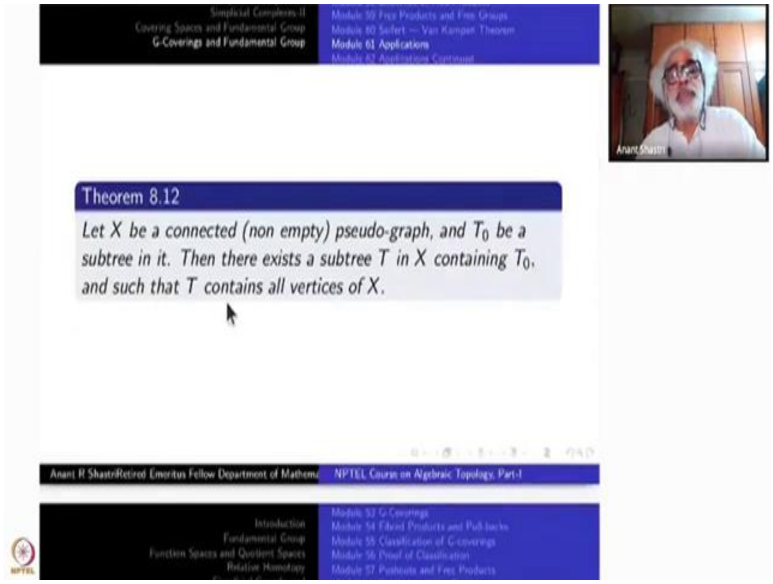

Now the next result. Let X be a connected nonempty pseudo graph. Let T_0 be a subtree in it. Then there exists a subtree in X, that is actually T, containing this given T_0 such that this T contains all the vertices of X. This is one way of telling that that this is a maximal tree, you cannot make it into a larger tree by putting extra edges, because when you put all the vertices that are there, as soon as you put one extra edge there will be a loop.

So, let us prove this one, a rigorous proof is requiresd now, using Zorn's lemma or some such thing. No hand waving can be done. But in the finite case you can actually do this, there is no problem, for finite case you do not need Zorn’s lemma.

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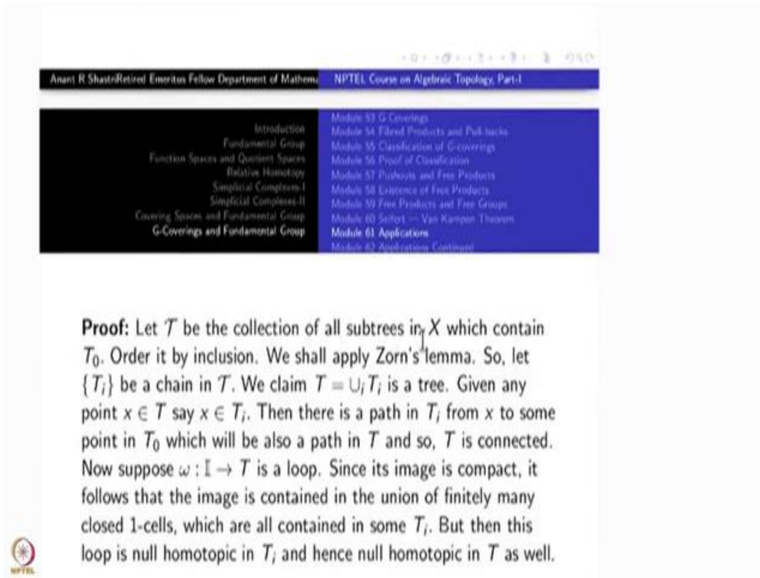

So, let $\mathcal{T}$ be the collection of all subtrees in X which contain T_0. Partiallly order it by inclusion. Now, we shall apply Zorn's lemma and then conclude that there is a maximal one. All members of $\mathcal{T}$ contain T_0 so maximal will also contain T_0.

So, let $\{T_i\}$ be a chain in $\mathcal{T}$, in this collection. Chain means what? Totally order subcollection. We claim that $T = \cup_i T_i$ is a tree. So, how do we know that it is a tree? First of all, given any point in this union, it will be in one of the Ti's, because it is a union. Then there is a path in T_i from X to some point in T_0, because the tree T_i is a connected, which will be also path in T and so T is connected. Ti is a tree, so Ti's are connected but now I have proved that T is connected.

Now, suppose ω is a loop in T. A loop means what? It is continuous function from I to T. Therefore, the image must be compact. We know that any compact subset is contained in a finite sub pseudo graph. It follows that the image is contained in the union of finitely many closed 1-cells. Closed 1 cells means what? I have told you that it is just either edges or full circles which are all contained in some Ti because Ti's are only finitely many of them let us say all the vertices are in T 1, T 2, …. and then take the maximum of them in that Ti they will be all be there. But then this loop is inside Ti, it null homotopic in Ti. But Ti is a sub space of T, so it is null homotopic in T.

Therefore, every chain has an upper bound, so that is sufficient condition for Zorn's lemma, Zorn's lemma will tell you that there is a maximal tree. A maximal element of $\mathcal{T}$ is nothing but a maximal subtree of G which contains T_0.

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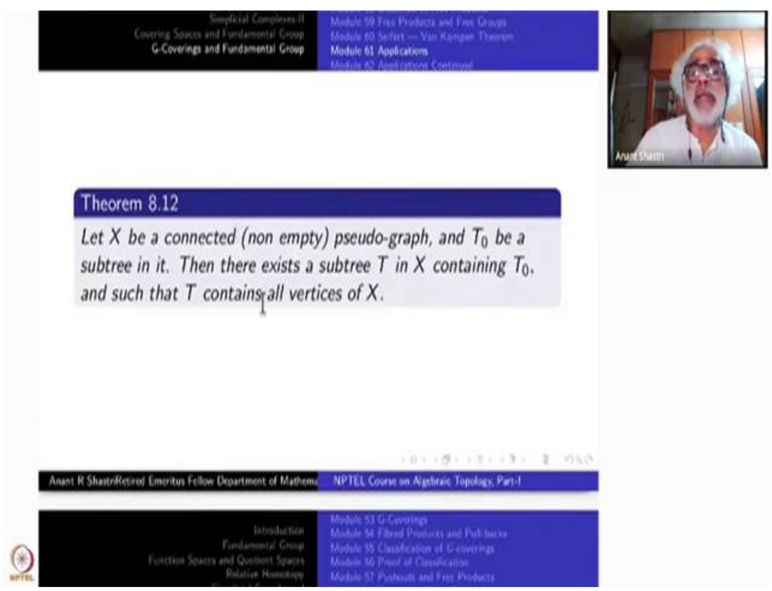

Now, suppose there is another vertex which is not in this maximal T. That would mean what? Because X is connected from that extra vertex you must be having a sequence of edges all the way from the extra vertex and coming to this T. That would mean that you will get a vertex v somewhere along such that v is not in T but the next edge, say, s, will have the other vertex inside

T. So, this is what it means. What we have done so far? We have seen that there is an edge s which will have one endpoint in T and the other end is not in T. From the endpoint which is not in T you can collapse the whole edge inot T, contract. That means, T is a strong deformation retract of $T \cup s$.

Therefore, $T \cup s$ is also contractible. Therefore, $T \cup s$ is a larger tree in X. That is a contradiction to the fact that T is maximal. Therefore, there are no more vertices. So, that completes the proof that given any tree there is alargest tree that contains it and that largest tree contains all the vertices of X.

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In particular, there is always such a thing if we start with a single vertex. Single vertex is a tree after all. So, every connected pseudo graph is homotopic to a bouquet of circles. This is the corollary of whatever we have done finally. How does you get it? Start with the pseudo graph, take any vertex you want. There will be a tree containing that. That tree I will take the maximal one, which will have all the vertices in it, all the vertices of G are there.

This is a tree I can collapse it. Then $q : G \to G/T$ is a homotopy equivalence. What will happen to G/T? All those edges which are not in T they would become circles, every edge which is in T has become a single point, at that single point I will some circles.

What are the these circles? Only for those edges , I did not say that all the edges are inside T all the vertices are inside T, all those edges which are outside T they will become circles now. So, G by T is a wedge of circles, namely, one point union of circles, the number of circles will be precisely equal to the number of edges which we have missed from T.

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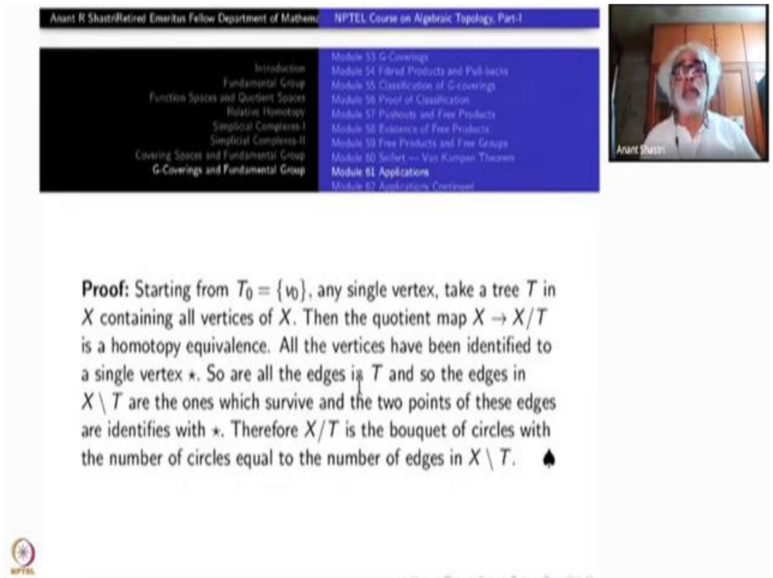

This is the gist of this thing. We are starting with $T_0 = \{v_0\}$, any single vertex, take T to be a maximal tree containing all the vertices, a tree containing all the vertices. Then $q : X \to X/T$ is a homotopy equivalence. All the vertices have been identified to a single vertex, because T contains all the vertices.

So, all the edges in T and also have become to single point, so the edges in X minus T are the ones which survive and how they survive? They become circles and 2 endpoints of these edges are identity to single point. Therefore, X by T is a bouquet of circles. Therefore, what is the conclusion? You take a pseudo graph which is connected, take any point in X, $\pi_1(X, v_0)$ is a free group, that is the conclusion. So, let us go to the next theorem.

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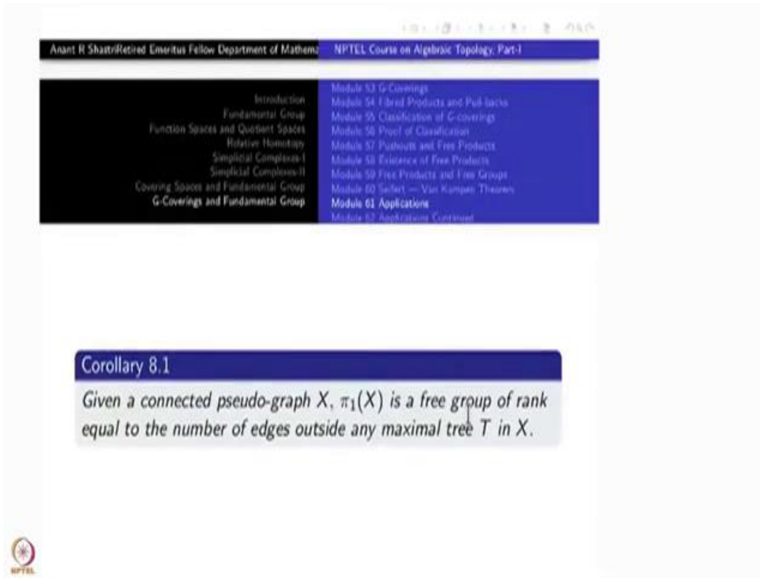

Given a connected pseudo graph pi 1 of X is a free group of rank equal to the number of edges outside any maximal tree in X. Therefore, this number is independent of the choice of the maximal tree that we make. By the way the maximal tree may not be unique, you can think about that, but the number of edges outside because of this will be the same that is the beauty.

So, that is something which we have now. Next time we will use this one to prove a big theorem in group theory and some more topology later on. So, that will be the last module, the last lecture for this course. Thank you.