Introduction to Algebraic Topology (Part - I)
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Lecture 60
Seifert-Van Kampen Theorem

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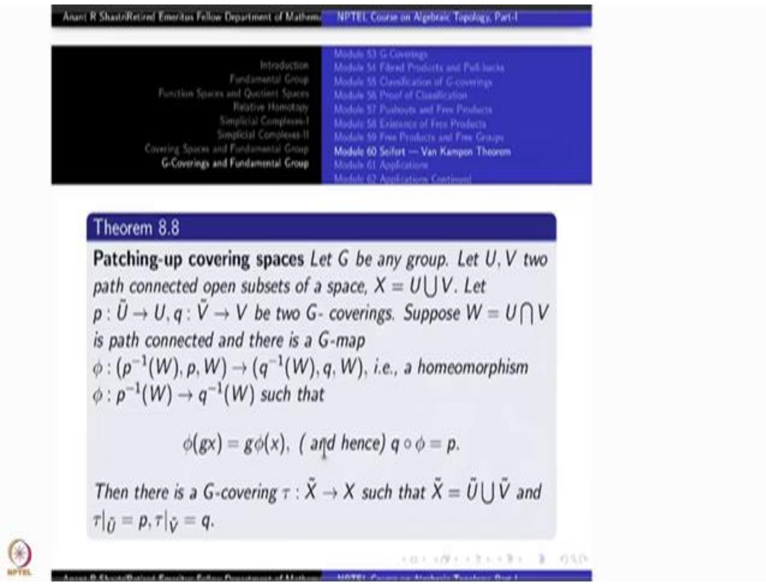

So, at last we are with Seifert-Van Kampen theorem. So, let us begin with another small preparation here-- Patching up of Covering Spaces. Start with a group G and a space X. Let U and V be two path connected open subspaces of X, where $X = U \cup V$. $p : \tilde{U}\ U, q : \tilde{V} \to V$ be G-coverings over U and V respectively.

Suppose $W = U \cap V$ is path connected and there is a G-map between the two coverings restricted over W, namely, $\phi : p^{-1}(W) \to q^{-1}(W)$. That means what? ϕ is a homeomorphism of the top spaces and commutes with the projection maps, $q \circ \phi = p$. Indeed recall that phi of g x must be $\phi(gx) = g\phi(x), x \in p^{-1}(W), g \in G$. That is the definition of G -map.

Then there is a G-covering $\tau : \tilde{X} \to X$ on the whole of X, such that tau restricted to U tilde will be p and tau restricted to V tilde will be q and the total space is exactly the union of U tilde and V tilde, this is the statement. So, that is the meaning of these two coverings on these two open subspaces have been patched up to give a covering on whole space X. Let us see how it is done.

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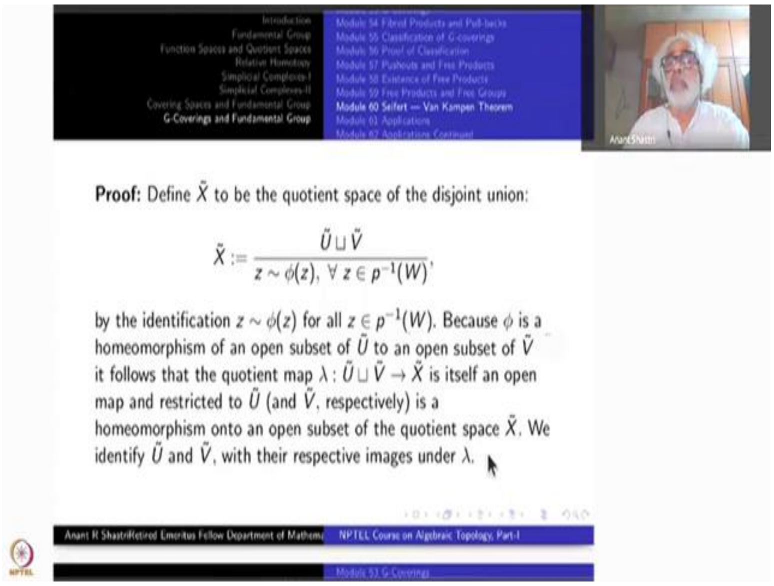

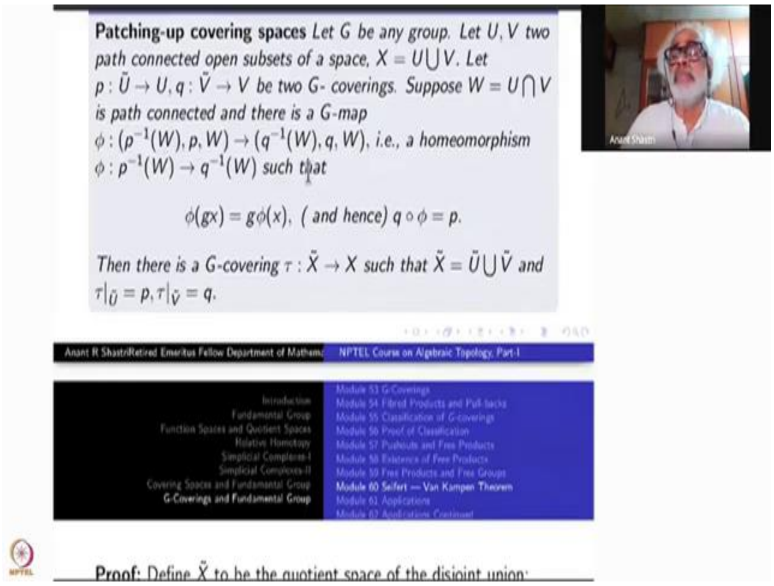

So, $\tilde{X}$ is going to be the quotient of the disjoint union $\tilde{U} \sqcup \tilde{V}$. It is not just the disjoint union because below U and V have intersection. So, on those intersections you must have some identification. Namely, $z \sim \phi(z),\ \forall\ z \in p^{-1}(W)$. Remember this phi is a homeomorphism from p inverse of W to q inverse of W. For all points in p inverse of W, identify with its image under phi, so that $\lambda : \tilde{U} \sqcup \tilde{V} \to \tilde{X}$ is the quotient map, that is the quotient space.

So, by definition X tilde is the quotient of U tilde disjoint V tilde, but see that within U tilde there is no identification. Similarly, within V tilde there is no identification. Therefore, under the quotient map U tilde and V tilde sit inside $\tilde{X}$ and we identify these subspaces of X tilde to be U tilde and V tilde. In that sense X tilde will be a union of U tilde, and V tilde.

So, because phi is a homeomorphism of an open subset of U tilde to an open subset of V tilde. Remember these things are W is an open subset, because W is U intersection V, U and V are themselves open subsets in X. So their inverse images are also open subsets inside U tilde and V tilde. When you identify along an open subset by a homeomorphism, it is very easily check that the quotient map itself is an open mapping. So λ is an open surjective so open quotient map.

Moreover, there is no identification within U tilde and V tilde, so lambda U tilde can be identified with U tilde. And that is what we do, we identify U tilde, V tilde with their respective image under lambda. So, X tilde is defined properly.

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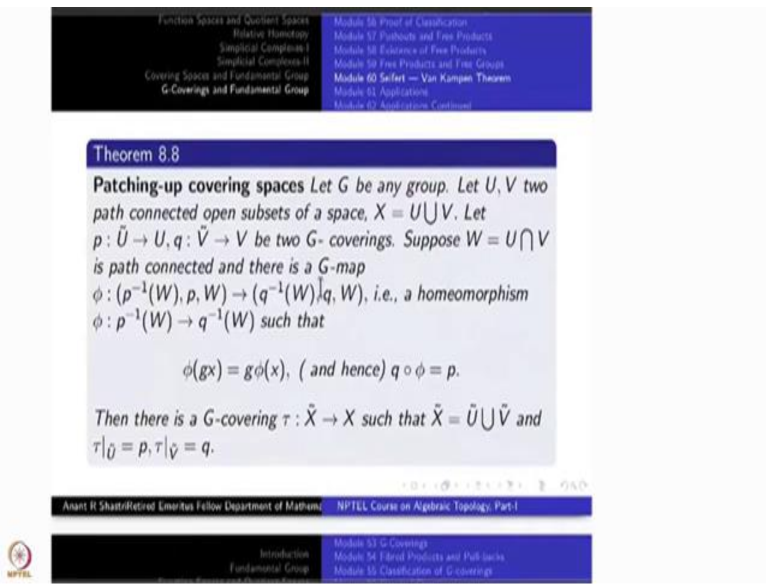

Now, on the disjoint union you can take $p \sqcup q : \tilde{U} \sqcup \tilde{V} \to X$=$U \cup V$. But now because p and q are respected by this phi, that $q \circ \phi = p$, therefore, the disjoint union map factors down to give you a map let us say well I have, here I have defined tau, this should be tau, $\tau : \tilde{X} \to X$, $\tau \circ \lambda = p \sqcup q$. This τ itself will be a covering projection. Because, you can just see that if U and, if p and q are covered evenly say that is p is covered by evenly by open subset, by an open subset the same thing will happen for lambda over that open subset, because everything will be inside U tilde.

Same thing will be happening for under q also, if some open subset is evenly covered by q and X is covered by U union V, X is U union V, so, this follows that the union p, q in q defines a covering projection, here in the statement I have put tau, here I put lambda, and you can change this lambda to tau, so that tau is an extension of p as well as q. Same map collapsing to I mean factoring down to the quotient map. So, this was a fairly easy picture, the hypothesis is very important and it does not come freely.

What you have to assume is that intersection is path connected and suppose there is a homeomorphism of the respective things here, of the restricted coverings, restricted coverings are G-coverings they must be isomorphic that means, there is a G map between them, so that you can patch it up along with that.

The beauty of this one is that I have never actually used that there are only two open sets, you could have had X as a union of a family of open sets and assumed this condition for each U_i intersection U_j and perform this quotient space construction. It will work exactly same way for arbitrary things also, this hypothesis should be for each pair (i, j), instead of U and V for each $U_i \cap U_j, i \neq j$ they must be path connected and then there must be homeomorphism like this and under these homeomorphisms you identify them, then you get patched up for covering projection, same proof will work.

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So, this is what the remark here says that if you can replace the two open subsets U, V by a family of open subsets and W with the family UI intersection UJ, then the proof is identical it will work the same way. But the problem is in getting those phi IJ's which are isomorphisms on intersections that problem is not at all obvious, that is a tricky hypothesis there. So that extra hypothesis, if you have some extra hypothesis then only that will come. So, let us see in the usage, what is that extra hypothesis we need?

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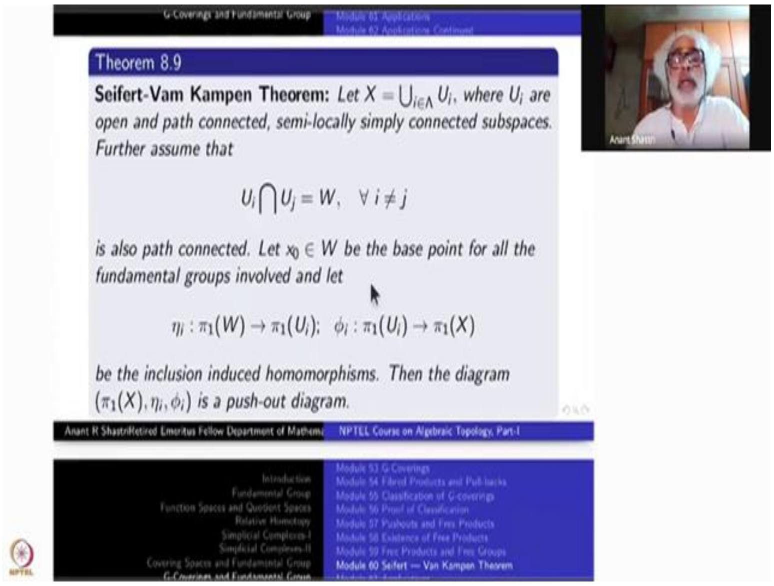

So, here is what we want to apply this patching up. Now, let X be a union of open subsets U_i, index over lambda, where each U_i is path connected, semi locally, simply connected. So, that we will have the simply connected coverings for each of them. Further, assume that intersection

$U_i \cap U_j$ each of them is the same W, this is the extra hypothesis, for every i not equal to j, one single W which is path connected. I allow Ui to be not just one, two member but any members, but all of them should be intersection the same set, same open set, pairwise intersection must be same to me.

Choose a base point inside W to be x_0 and let this be the base point for all the spaces while cconsidering the fundamental group. So, that will not be written each time because one single same x_0 is there, x_0 is in W, x_0 in Ui, Uj and x_0 is in x. So, everywhere I am taking the base point x_0, no need to writing down that element.

Let $\eta_i : \pi_1(W) \to \pi_1(U_i), \phi_i : \pi_1(U_i) \to \pi_1(X)$ be the homomorphisms induced by the inclusion maps, W is included in the Ui, look at the corresponding homomorphism in pi 1 similarly here, they are inclusion induced maps. Do not confuse them for monomorphisms, they may not be monomorphisms that is the crux of the matter, the inclusion induce homomorphisms.

Then look at this collection $\{\pi_1(X), \eta_i, \phi_i\}$ this becomes a diagram of groups. What we want to show is that this is a push out diagram, this is the statement. In other words, once you know pi 1 of Ui and the pi 1 of W and their homomorphisms etc pi 1 of X can be completely determined, it has to be the push out, this is the Vam Kampen theorem, computing the fundamental group of the union by knowing them on each individual open subsets, so that is vaguely but this is the full rigorous statement here.

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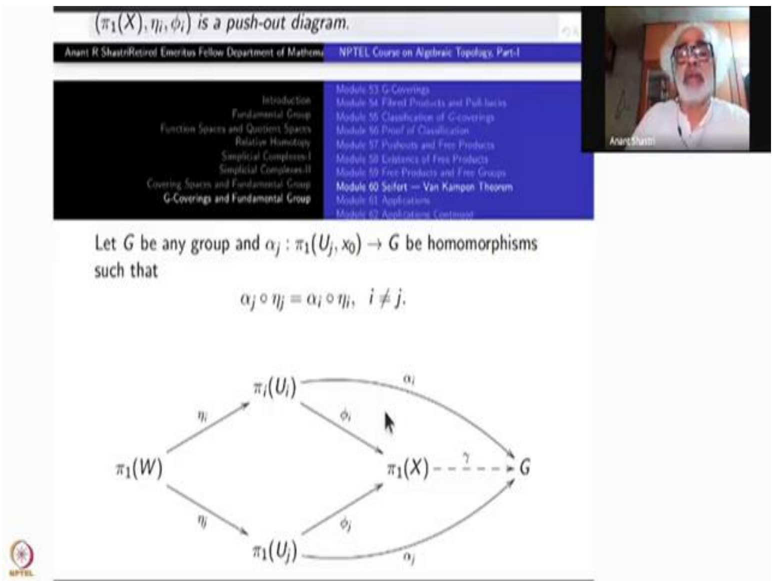

So, let us go to the proof. Take any group G, take al family of homomorphisms $\alpha_j : \pi_i(U_j) \to G$ such that $\alpha_j \circ \eta_j = \alpha_i \circ \eta_i, i \neq j.$ So, this is the data for push out, this is another diagram. If the original diagram is a push out diagram then this is what it have, namely, it should admit a unique homomorphism $\gamma : \pi_1(X) \to G$ here such that these triangles are commutative. This is meaning of pi 1 of X is the pushed out.

So, given such a data, we must produce γ so that is our task now. So, what is our tool? the tool is the classification for G-coverings or G-coverings interpreted in terms of homomorphisms from the fundamental group into whatever G that we have taken. And the correspondence is canonical and this is what we are going to use again and very strongly now.

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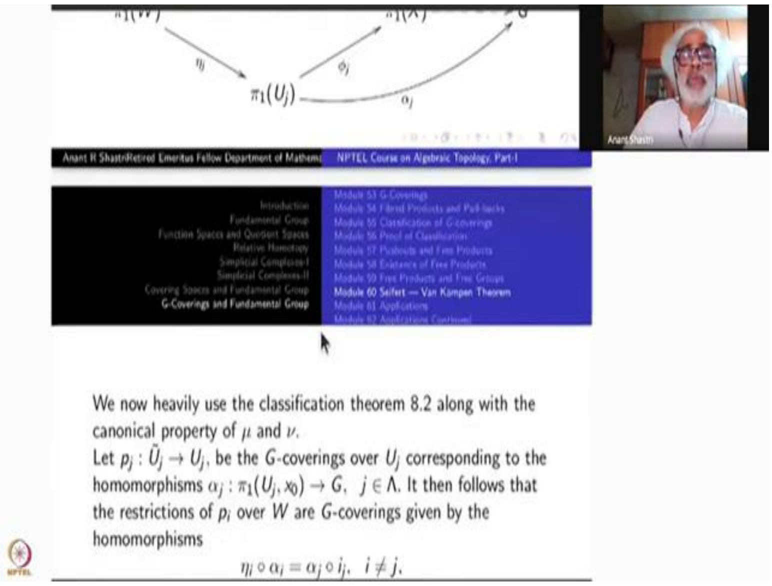

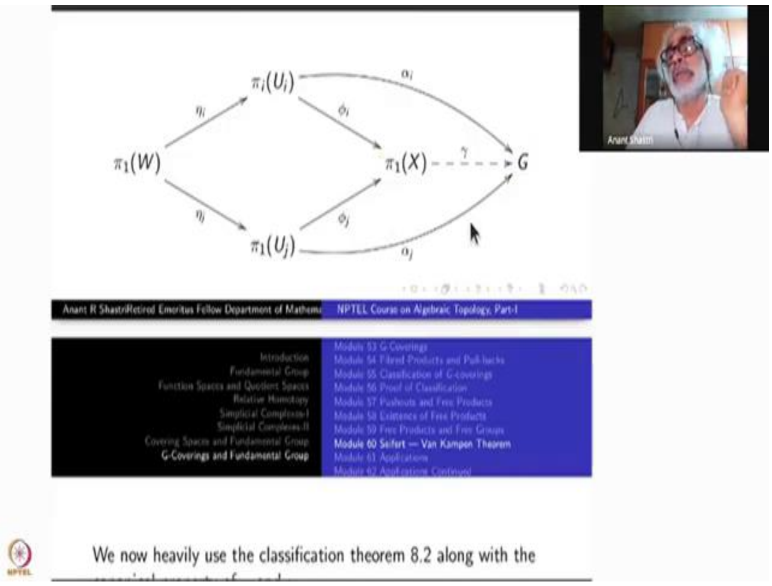

So, to get this gamma. So, we are heavily using the classification theorem 8.2 along with the canonical property of mu and nu. So, what are these coverings in terms of homomorphisms? For on each Uj there is a homomorphism from $\alpha_i : \pi_i(U_i) \to G$. This will give you a unique G-covering on U_i. Unique in the sense isomorphism class, isomorphism class of a G-covering, every homomorphism corresponds to exacty one isomorphism class of G-covering.

Let me denote that G- covering by $p_i : \tilde{U}_i \to U_i$ for each i. Now, take alpha j if you compose it with eta j, what do I have? I have the same thing as alpha i and then eta i composite alpha i. These two homomorphisms are the same means, if I take this covering and pull it back over here on W, if I take this covering and pull it back here over W they are isomorphic because the corresponding homomorphism $\pi(W) \to G$ are the same.

So, this is the hypothesis that we get, this W is connected space here, you have to come here, to come here you need a homomorphism from pi 1, to do that you need a base point here, common base point. So, we have done not only just a common base point, but a small neighborhood and at neighborhood, W, is the same for all Uj that was the crucial fact here.

So, now, I have got a clooection of $\{p_i\}$ of G-coverings, covering transformations here and t G -maps ϕ_{ij} from one to the other. So, perform the patch up to get a G-covering $p : \tilde{X} \to X$ which restricts to these coverings p_i over U_i that is the extension of this covering. Now again by the classification of G-coverings, p corresponds to a homomorphism from $\pi_1(X)$ into G, take that as gamma. Automatically it will follow that $\gamma \circ \phi_i = \alpha_i$ because the covering p restricts to the covering p_i for all i.

So, we have produced a gamma with the corresponding property. Why gamma is unique? Suppose there is another gamma prime which also fits this diagram, then what happens? If $p' : \tilde{X} \to X$ is the corresponding G-covering, when you pull it back over each U_i, since at the group level we have $\phi_i \circ \gamma' = \alpha_i$, it follows that $p'|_{U_i}$ is isomorphic to p_i for each i. Now by the patching up theorem, it follows that p' is ismorphic to p. Therefore, the homomorphism is the same, $\gamma' = \gamma$.

So you see that is the end of the proof. So the proof of Vam Kampen theorem has become complete tautology here, no problems at all. I repeat here. So, what we have done? Start with such a diagram, we have to produce gamma. And we have to show that this gamma is unique. Namely, when it commutes with these diagrams.

So, what we do? We take these homomorphisms and construct coverings corresponding to that, construct means we have this mu of this alpha, mu of that one I can take. When I put them back here, because the two homomorphisms are the same, they must be isomorphic here. To take those isomorphisms and patch up these all these coverings start with disjoint union of this and then identify them to get a covering on X.

Covering on X now you have to go back reverse gives you homomorphism, that homomorphism when you compose with this phi i its eta alpha i which just means that if you pull back this covering over this one, it must be this alpha. So, therefore this must be alpha i, I mean if this is alpha i same thing as saying that put back this covering must be the same thing.

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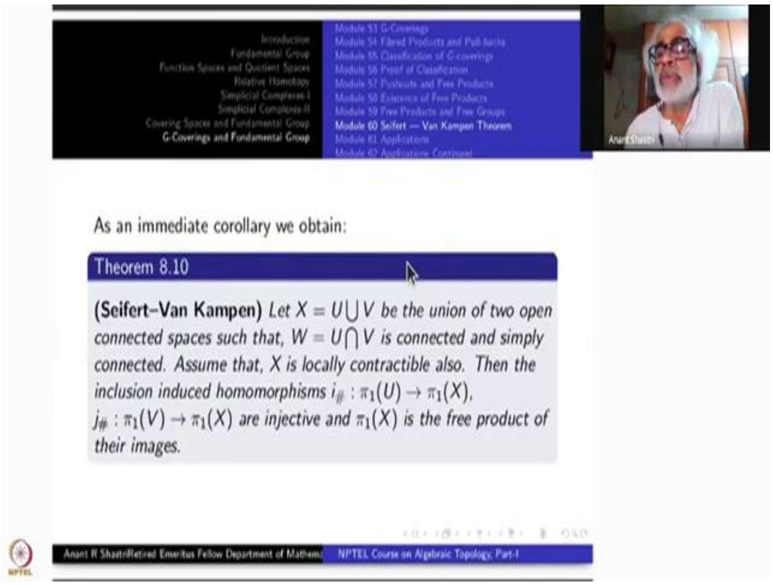

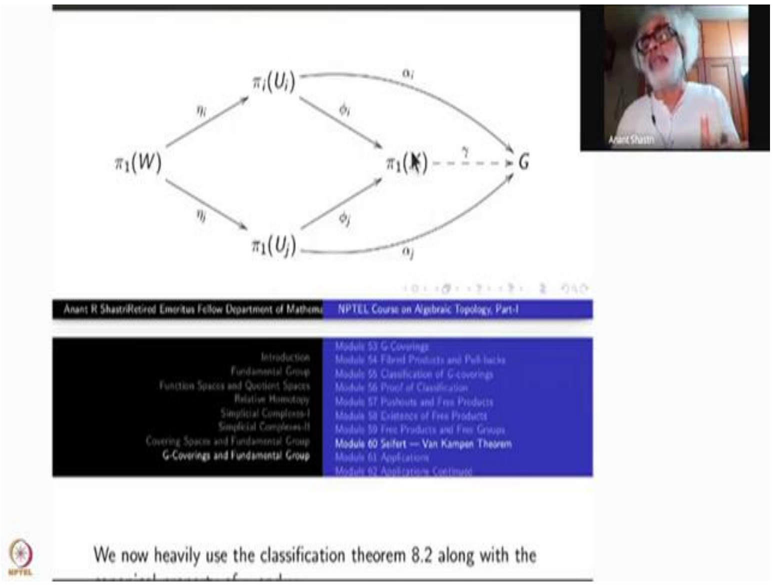

As an immediate corollary we will write down this one in a simpler case namely when X is the union of only two open sets, connected, path connected and intersection is simply connected, this was one of the cases that we are interested in. Assume that X is locally contractible or locally, semi locally simply connected and so on, so, that covering spaces makes sense and exist.

Then we have the inclusion induced homomorphism $i_{\#}:\pi_1(U)\rightarrow\pi_1(X)$ and $j_{\#}:\pi_1(V)\rightarrow\pi_1(X)$. Why I am not writing the base points? I am assuming that we have one single common common base point of all the three spaces involved. That is why I am not writing. The intesection is simply connected and so, it follows that $\pi_1(X)=\pi_1(U)*\pi_1(V)$ is the free product. Moreover, $i_{\#},j_{\#}$ are injective.

This is a consequence of the pushout diagram when here, when this is trivial group is simply connected, in this trivial group the push out diagram is same free product of this and you know that these are monomorphisms. So, you can identify each group Gi with the free product of them. So, here I am only applying two of them at a time in this theorem, you could have had any number of them.

So, that is a simple version, so all these are when there are several versions of Vam Kampen theorem, unfortunately some of them are very, very complicated, they cannot be handled directly by our classification theorem.

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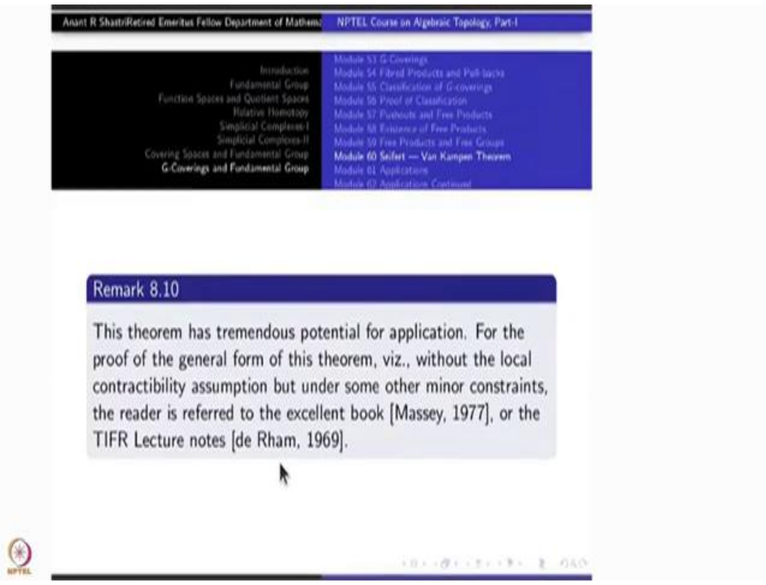

However, whatever we have proved itself is of tremendous potential, tremendous application potential. The general form of this theorem without local contractibility or etc, but slightly other kinds of assumptions are there, you can read an excellent book of Massey's "An Algebraic Topology".

But even this one will not give you all the versions, there is another prototype version available in TIFR lecture notes by de Rham, Algebraic Topology lecture notes, these things are available on our website also, downloadable. So, you can read them and if you are working in 3 dimensional topology you need to have many other versions also.

But once we have this training (())(23:24) the fundamental relations and how things are working here and what is the relation between covering spaces and fundamental group and so on, those things when you read they will become easy for you. That is all.

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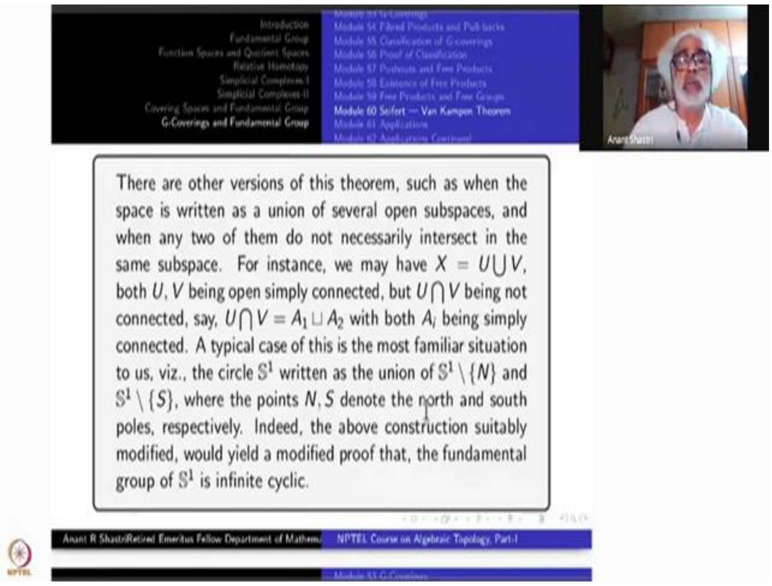

Just for instance just look at the circle S1, you can circle S1, you can write it as disjoint union of, sorry overlapping union of say S1 minus 1 and S1 minus minus 1, but the intersection is not connected, it will contain two open arcs. All of them are trivial now. So, you can think of this as U and V they are arcs, so they are simply connected, pi 1 U is trivial pi 2 is trivial, but the intersection is not simply connected.

Intersection consists of two components and both of them are simply connected. So, what is the push out diagram, everything is identity here, everything is identity group. But you know, the fundamental group of S1 is non-trivial. Where did it come from? So, if you cleverly answer this which is what de Rham does, you get a different kind of push out diagrams, because lack of time I cannot do all that now, because it will go beyond course.

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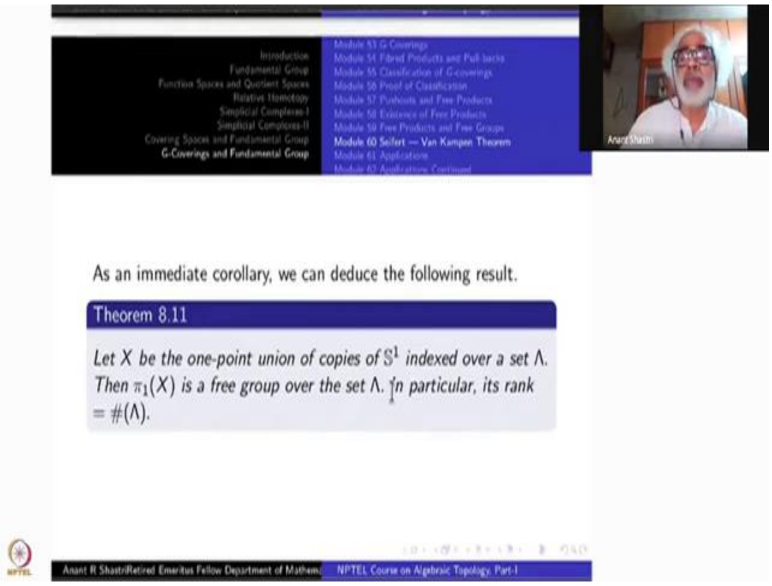

However, let us, whatever we are done let us use that in as cleverly as possible way and carry out quite a few interesting results here. So, let X be one-point union of copies of S1 index over a set lambda, one-point union, what do you call it?- a wedge of circles, then the fundamental group is a free group over the set lambda, in particular its rank will be equal to the cardinality of the set lambda, how many circles you have taken.

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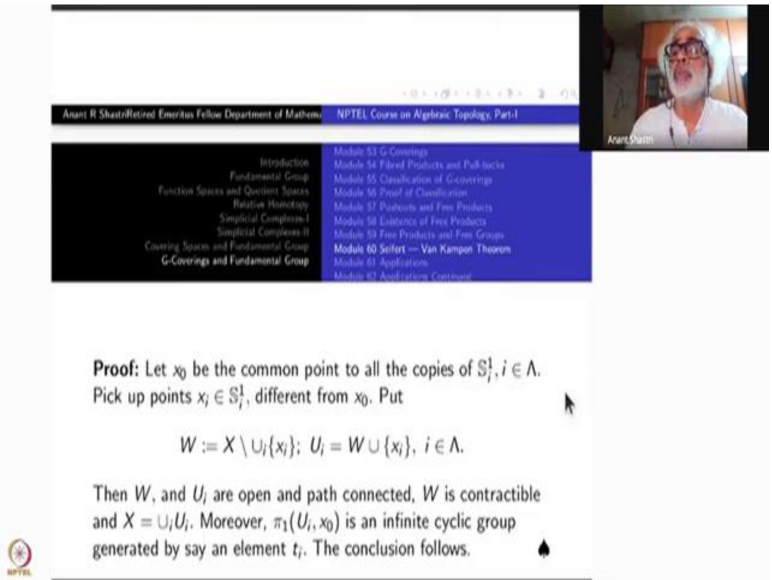

So, this is an immediate consequence. All that you have to do is take the common point wherein you have identified all these circles, one point from each circle, take that as x_0, the base point. Pick up a point $x_i \neq x_0$ in the copy $\mathbb{S}^1_i$ throw away all these points from X, X is the union of all these circles, throw away all these points to get an open set, call it W.

Put U_i equal to W union one of the element, one of the points that you have thrown out, call this $U_i = W \cup \{x_i\}$. that is also an open set in X. So, for each I, I have defined these sets. Clearly union of all the Ui's will be definitely equal to X and if you take Ui, all of them will contain this W, but the common intersection all Ui's is precisely W, that is an open set.

So, now, what is W? W you have removed all the points, one point from each of this $\mathbb{S}^1_i$ when you remove that point you are left with the two open arcs, which you can collapse to the single x_0, so this you can do for each circle, which just means that W is contractible. In particular W is simply connected. So, you have the nice picture inclusion maps from W various Ui's and then into X. pi 1 of W is the trivial group. So, the push out diagram becomes a free product.

So, what are $\pi_1(U_i)$? In U_i what you have done? In W which is contractible, in one of the circles you have put back the point you have removed. so that circle reappears, the rest of them contract to the basepoint, so U_i is homotopic type of a single circle.

Therefore, pi 1 of Ui is infinite cyclic. What is the conclusion? Conclusion is that the free product of these infinite cyclic group is the pi 1 of X which is the push out. So, we get a free product of infinite cyclic groups, one for each i in lambda. That is why you can think of this as free group over lambda.

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So, more applications we will see next time.