Introduction to Algebraic Topology (Part-I)
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Lecture 6
Fundamental Group π_1

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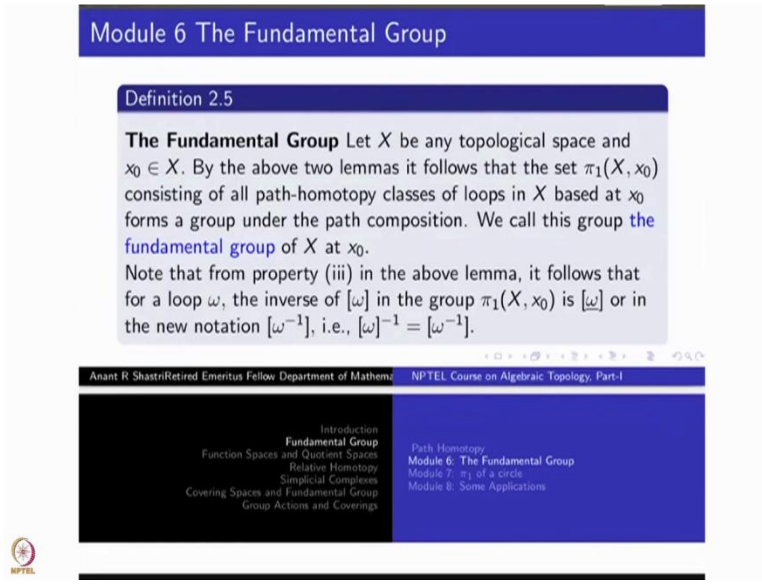

Last time we introduced paths, path homotopies and then examined or studied a number of homotopical properties of path composition. The path composition has two sided identities which are different as such because paths have different end points, it is associative and it has an inverse also. Each path if you take the reverse way of tracing it will be inverse only up to homotopy. This is what we have seen.

Today, we have specialized to the case when the end points are the same, $x_0 = x_1$. We start with a space X and fix a base point x_0. We are going to define what is the meaning of $\pi^{\Omega}_1(X, x_0)$ which is going to be a group. What is this group? It consists of homotopy classes, the path homotopy classes of loops. What is the meaning of a loop? A path which has both its end points at x_0.

Loops based at x_0. This is the set . The composition law which you have defined now becomes a binary operation on this set, because if two loops are homotopic to each other and another two loops are homotopic to each other, their compositions will be homotopic to each other, this homotopy is path homotopy. Therefore the composition law goes down to the set of homotopy classes of loops and that binary law we have seen becomes associative.

Now, the two sided identities which you have, namely, the constant loop at x_0, because both sides are the same now, i.e., the endpoints are the same- that is why. Also the inverse will trace the same loop in the opposite direction--- that will be the inverse. So, automatically we have got a group associated to a space X together with a specific point x_0.

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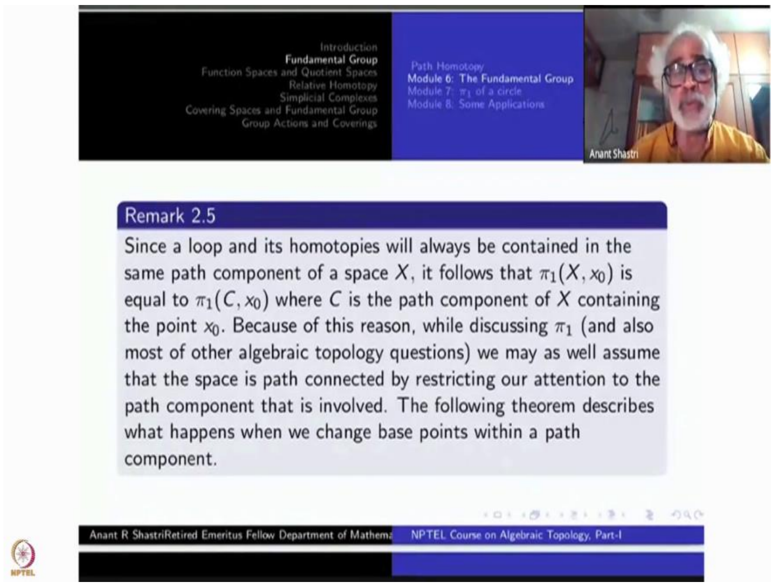

A loop and its homotopy will always be contained the same path component of X ,--it will never get out of the path component of X, which contains the point x_0. Because, the loops have to start there. Homotopies also have to respect that point x_0 and so on. Therefore, all the time we will be inside the component C of X which contains the point x_0. For each point there is a component.

So, the group $\pi_1(X, x_0)$ is identical to the group $\pi_1(C, x_0)$, under the ordinary inclusion of C inside X. Because of this, I could have assumed that X itself is path connected. Wherever you have started, it will remain in that component. So, for this reason, in many many other things which will have to do all the time with continuous maps from connected spaces, path connected spaces, this will happen.

In algebraic topology, it is customary to assume that a space is path connected. What is the reason? That, each path component can be studied first and then you can put them together ---you get the study of the whole space. Because a space is always divided into its path components. Now, within a path component you may have taken different points for defining π $\pi_1(X, x_0)$. So, let us assume that X is path connected and suppose I take another point x_1

and look at $\pi_1(X, x_1)$. What is the relation between π $\pi_1(X, x_1)$ and $\pi_1(X, x_0)$? This is what we want to study. Namely, under change of base points, what happens to the group?

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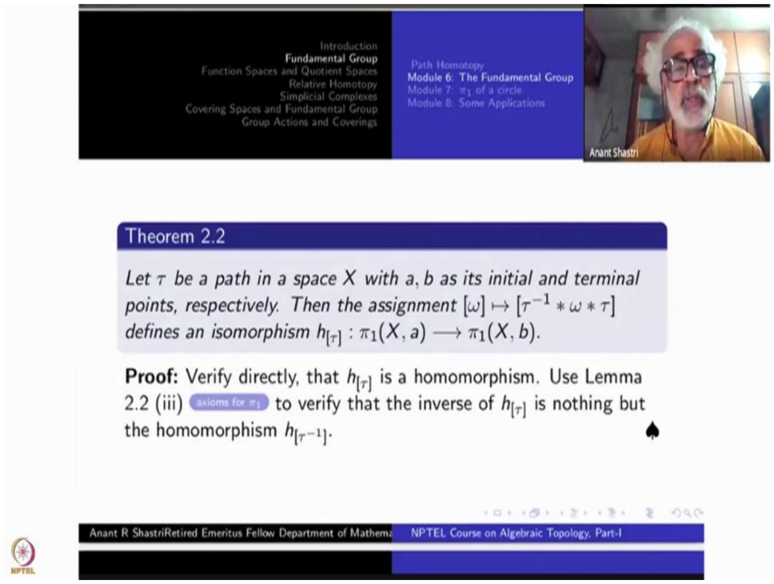

So, instead of x_0 and x_1, I have taken the notation a and b. (Changing notations like this is somewhat dangerous first of all, but it is also a good practice, so that, ultimately, you will start thinking without reference to actual notations.) Suppose you have two points inside X. You can take a path from one to the other. When you take a path from one to the other, suppose you have a loop at one point then you can view it as a loop with a tail at the other point.

A loop with a tail! It will be also a loop, but its base point will be the end of the tail. So, this is the picture you should keep in mind. Now, when I speak like this, I have not used any notation you see--that is the whole point of changing the notation.

So, suppose you have two points a, b-- these are initial and terminal points for this path $\tau\tau$ --- a is fixed, b is fixed, and $\tau\tau\tau$ is fixed. Then what happens? Take a loop ω at a. Pre and post compose it with τ^{-1} and τ respectively. To start with τ^{-1} , τ^{-1} will start from b and come to a. Then you trace ω, again you are at a. Now you go back by τ to b. So, you are starting with b and you are ending with b, so you get a loop at b. But ω is a loop at a, its class will go to the class of $\tau^{-1} * \omega * \tau\tau$. Why? Because if ω is homotopic to some ω_1 then pre and post composing by paths, $\tau^{-1} * \omega_1 * \tau\tau$ will be homotopic to $\tau^{-1} * \omega * \tau$.

This is what we have seen in yesterday's class and previous class. So, I shall denote this map by h_τ. This is a set theoretic function now, on the homotopy classes. But I claim that this is a homomorphism. Very easy to prove this one, using our earlier information, namely, what happens under compositions? Associativity law can be used here. Insert $\tau\tau * \tau^{-1}$ in between because that is homotopic to identity. h_τ

The entire class does not change. If you use that trick, then you can show that h_τ is a homomorphism. What is the meaning of h_τ a homomorphism? h_τ of a class ω $\omega_1 * \omega_2$ must be the class of $h_\tau(\omega_1) * h_\tau(\omega_2)$. How to get it? You have ω $\omega_1 * \omega_{2,}$ you can write as ω $\omega_1 * \tau\tau *$ $\tau\tau^{-1}$ and then $\omega * \omega_2$.

$\tau\tau * \tau^{-1}$ in between can be introduced because it is null homotopic. Now, if you apply h_τ, $\tau\tau^{-1}$ and $\tau\tau$ will come again on both sides. The entire thing you can break it into two groups, put brackets that will become $h_\tau(\omega_1) * h_\tau(\omega_2)$.

In exactly same way, you can see that $h_{\tau^{-1}}$ namely, using the path $\tau\tau^{-1}$ which will be from b to a. Therefore, you will get a map from $\pi\pi_1(X, b)$ to $\pi_1(X, a)$. If you take first h_τ and then take $h_{\tau^{-1}}$ that is the same thing as composing with $\tau * \tau^{-1}$ on the left as well as on the right. But $\tau * \tau^{-1}$ is homotopic to identity, the constant loop. Therefore, it is nothing but identity map. This just means that $h_{\tau^{-1}}$ is same thing as $(h_\tau)^{-1}$.

That means, h_τ is an isomorphism. So, we have proved that changing-base-point is given by a group isomorphism. If we are interested only in the isomorphism class of the group, then there is no problem. It will be displayed in a slightly different way. Isomorphism of a group gives a different copy-- isomorphic copy. So, you must understand that the groups need not be the same, but they may be isomorphic, like two equilateral triangles of the same side length you draw. They are different triangles after all. For example, one may be containing the origin, another may be containing some other point say, (100, 100). They will be different, but as triangles they are isometric. Similarly, in group theory it is important to understand that groups may be isomorphic yet they would be different groups. So, sometimes this difference does cause problems, you have to be careful with them and then you would like to see what is that isomorphism that we are taking. So that the isomorphism becomes important. So, this point you know, is not just an imagination, it actually happens in algebraic topology itself. Understanding of this is needed much later. Right now, it will not come in our way in this course.

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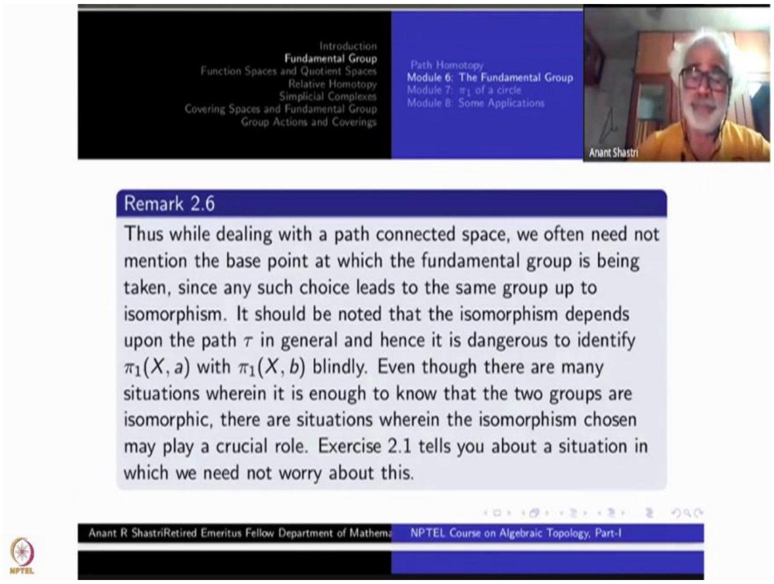

So, this is what the sum this remark 2.6. While dealing with a path connected space, we often need not mention the base point at which the fundamental group is being taken. Why? Because they are all isomorphic. If your interest is only knowing the group up to isomorphism. It should be noted that the isomorphism itself will depend upon what path you have taken. Within X, joining a to b, there may be several paths.

For instance, if the two paths are homotopic- path homotopic. Then the isomorphisms should be the same. Yet they may not be identity isomorphism, there is no identity isomorphism between two different groups. Identity isomorphism makes sense only when the groups are the same. But they may be same isomorphisms if the two maps are ---two paths are path homotopic. If they are not, then the isomorphisms may be different. So, you should keep this in mind. In this course, we will never meet this aspect at all.

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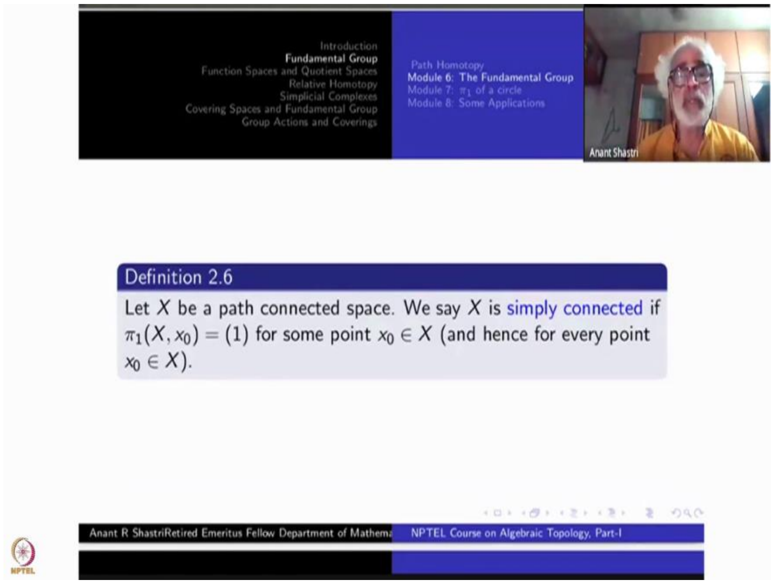

Let X be a path connected space. We will make a definition now, the modern definition of simply connectedness. So, the space X is said to be simply connected if the fundamental group $\pi\pi_1(X, x_0)$ is the trivial group consisting of one single element. This may happen at one point but then it will happen at all the points, because at all other points also it is isomorphic to the trivial group.

So, this definition is independent of what point you take for a path connected space and if the fundamental group is non-zero at some point it will not be, it will be non-zero at all other points, because all of them are isomorphic--that is all I want to say. This definition of simply connectedness is the most useful and the strongest definition. You might have come across with many other definitions of simply connectivity-- for instance, when you were doing complex analysis.

In complex analysis, you can have something like 10 definitions of simply connectivity if you want or even more. But when you come out of that to arbitrary spaces, most of those definitions will not work at all. Even if they work, some of them, they will be quite different than just simply connectivity. This definition is the strongest of them all.

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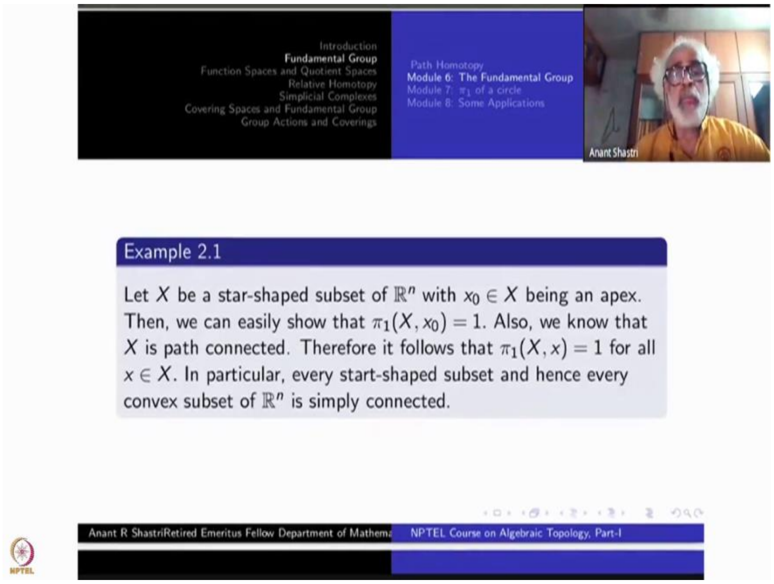

So, going back to examples, we again come to the star shaped regions inside $\mathbb{R}^n$. Take a star shape region, starred at a point with apex point x_0 . What does that mean? That means that if you take any other point in X, then the line joining that point and x_0 is completely contained inside X. Using this line segment, you can easily show that every loop based at x_0 is null homotopic, namely, homotopic to the constant map at x_0. Take any $\alpha\omega$, which is a loop at x_0. $\alpha\omega(t)$ can be directly joined to the constant loop, constant loop at x_0. What is the map? -- s $\omega(t) + (1 - s)\, x_0$. Does this line segment make sense? Because the entire line segment is inside X. This we have seen before right? I am just repeating this.

So, all star shaped subsets have trivial fundamental group, so they are simply connected. In particular every convex subset is simply connected. In particular, the whole $\mathbb{R}^n$ is simply connected. $\mathbb{R}$, $\mathbb{R}^2$, they are all simply connected. All closed discs are simply connected. Even regions inside an ellipse or ellipsoid and so on-- they are all simply connected because they are all convex subsets.

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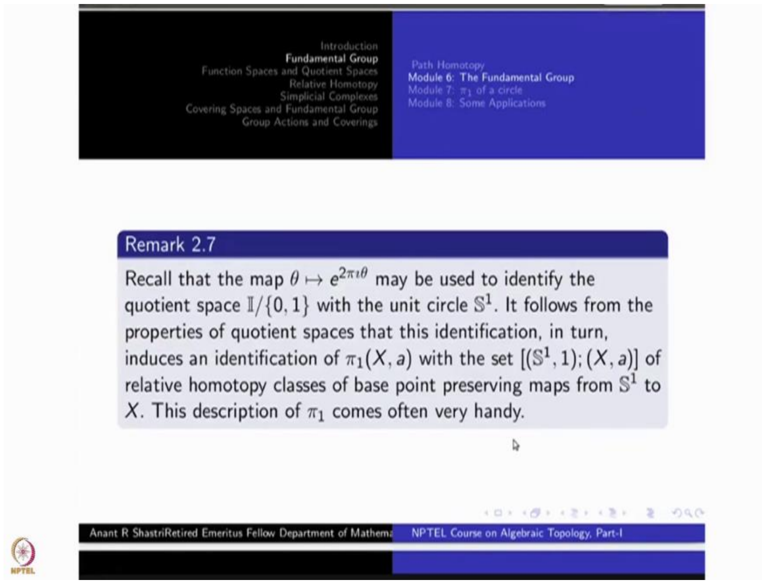

Now, we will give you some different way of looking at $\pi\pi_1$(X, x_0). Look at the map $\theta \mapsto e^{2\pi\imath\theta}$ defined on the closed interval [0, 1]. It is injective except at point 0 and 1, they were the same point, namely, the unit vector 1, (1, 0) in the complex plane, in $\mathbb{R}^2$. That is the property of the map $\theta \mapsto e^{2\pi\theta}$. We need this map very much.

So, this means that the endpoints are identified and all other things are kept as they are under a one-one mapping. So, when you take the quotient space of the interval, wherein endpoints are identified, interval modulo 0, 1 that is my notation here. This will be homeomorphic to the circle $\mathbb{S}^1$ via this map . So, it follows that, when you have a loop namely, a map $\omega\mathbb{I} \to$ X, wherein 0 and 1 go to same point, that map will factor down through this quotient space $\mathbb{I}$ modulo 0 identified with 1.

Which is the same thing as having a map from the circle into X. A very specific point here is the image of 0 and 1, namely, the unit complex number 1 in $\mathbb{S}^1$. Therefore, every loop can be thought of as a function from $\mathbb{S}^1$ to X and its base point being the image of 1. Therefore, instead of looking at the way we have done ---paths from a closed interval, we can take this set namely set of all continuous functions from $\mathbb{S}^1$ to X which takes the base point 1 to the base point x_0.

Conversely, if you have such a map, you can compose it with $\theta \mapsto e^{2\pi\imath\theta}$ and get a map from interval $\mathbb{I}$ into X which sends both 0 and 1 to the point x_0. Therefore, under this identification, what you get is a new way of looking at $\pi\pi_1$(X), which is nothing but homotopy classes of maps from $\mathbb{S}^1$ to X, wherein the homotopy is taken with respect to the base point, this base point never moves, $\omega^{\omega_s(1)}$ = x_0 always, the entire homotopy always fixed at this point x_0.

So, this way loops are nothing but images of $\mathbb{S}^1$ that is what you have to think about. If you need a base point there, you know on a circle you could have thought of any point as a base point, it is still a circle. So, that freedom is there, but you have to fix a base point before taking homotopy classes. that will be the base point for arbitrary loops inside X.

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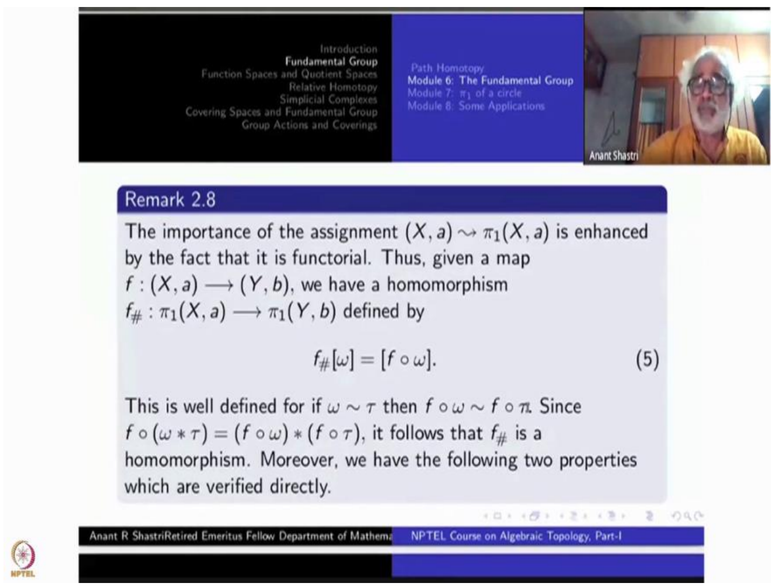

Now, what we have is starting with a pointed space, (X, a), we have given a group. This assignment has what I keep calling functorial properties. So, let me repeat: What are these functorial properties. Suppose, you have a map from X to Y such that f(a) = b, then what happens? If you take a loop at a, composing with f you get a loop at b.

If two loops are homotopic in X. f of this and f of that are homotopic to each other inside Y. Therefore, a map like this induces a homomorphism $f_\# : \pi\pi_1(X, a) \to \pi\pi_1(Y, b)$. The definition is: $f_\#([\omega])$ is nothing but the class of $f \circ \omega$. Because of the way we defined concatenation it follows easily that $f_\#([\omega] * [\tau])$ is $f_\#([\omega]) * f_\#([\tau])$. So, everywhere we are taking composition with f so the homotopy will be $f \circ (\omega_s * \tau_s) = (f \circ \omega_s) * (f \circ \tau_s)$.

(This is not because of associativity sorry.) So, this makes it a homomorphism:

$f_\#([\omega] * [\tau])$ is $f_\#([\omega])$ $=f_\#([\tau])$. This is more like distributivity. So, any continuous function induces a homomorphism of the corresponding fundamental groups.

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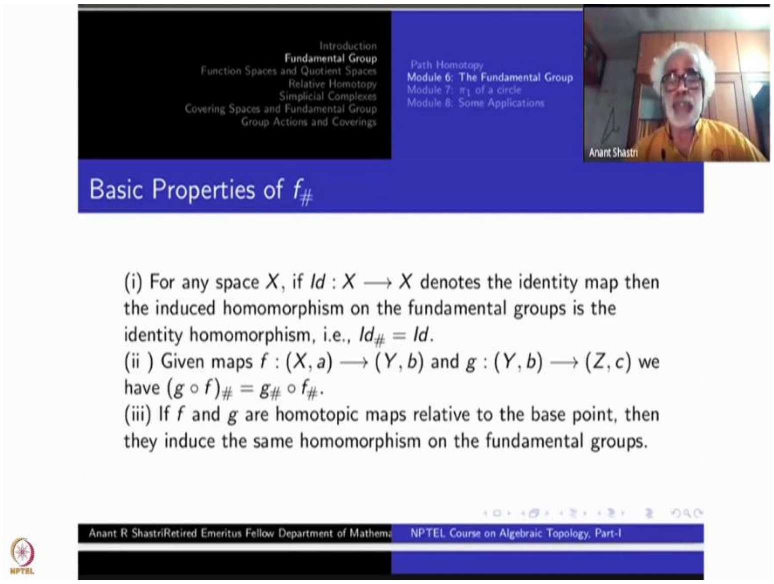

With the additional property that if you have identity map, what does identity induce? --induced homomorphism is also identity. Suppose you have another map from (Y, b) → (Z, c),

f : (X,a) → (Y,b) and g :(Y,b) → (Z,c). Then I can talk about g composite f : (X, a) → (Z, c). Under this g ○ f, what happens?

$(g \circ f)_\#$ will be $g_\# \circ f_\#$. Because all that I have to do is take ω,ω compose with f on the left anfd then with g. So, g composed with f (ω) is nothing but g (f (ω)). So, this is associativity. The g of f of something means $g_\#$ of f of that. So, it is $g_\#$ ($f_\#$ ($[\omega]$)).

Suppose now, f and g are homotopic themselves as maps from (X, a) → (Y, b) where the base point a does not move ---this is the extra hypothesis we have to put, not just homotopic maps. They are homotopic maps relative to the base point (which we have not yet defined but I am telling you now what is the meaning of this,) this just means that a does not move during the homotopy --f_t of a is always b, all the homotopies must have f_t(a) = b, g(a) is also b or some other, g here was in item 2 g or something different here, I am taking f and g are homotopic maps but they are from

$(X, a) \to (Y, b)$ both of them. Then the induced homomorphisms on the fundamental groups are the same. That means $f_\#$ is equal to $g_\#$.

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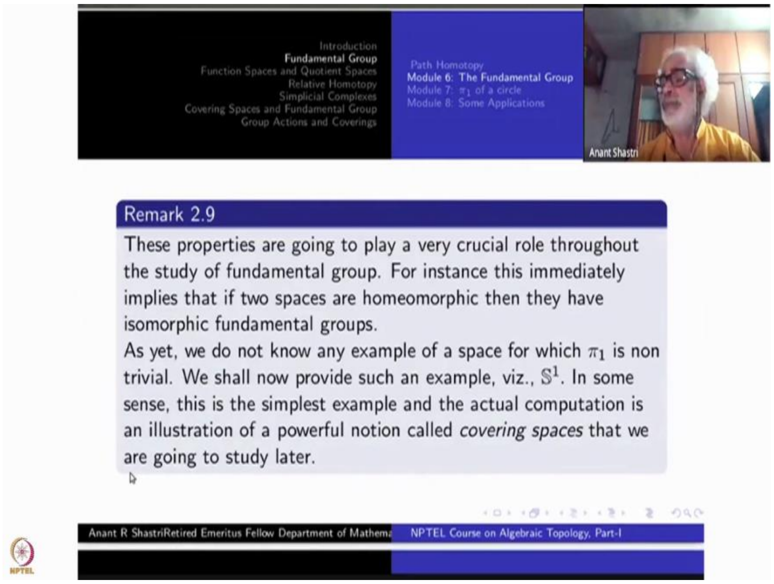

These properties are going to play a very crucial role throughout the study of fundamental group. For instance what we get is this: suppose two spaces are homeomorphic, pick up a homeomorphism, let us call f is a homeomorphism from X to Y. Take a base point a here, and put $b = f(a)$. Then I have already an isomorphism from $\pi\pi_1(X, a)$ to $\pi\pi_1(X, b)$. Why? Because f is a homeomorphism, f inverse will give you the inverse of $f_\#$; $(f_\#)^{-1}$ is the same thing as $(f^{-1})_\#$.

Therefore homeomorphic spaces must have isomorphic fundamental groups. So, if you take two spaces and conclude that their fundamental groups are not isomorphic, then you have solved a big problem, the spaces that you are given are not homeomorphic to each other. So, this is the way it is used ---already in our introduction I have told you,-- how this was used to solve a big problem in topology, namely, that the classification problem cannot be solved. However, we do not know any space yet for which π_1 is non-trivial. All our examples were convex sets and star shaped sets and so on. We do not know any example of X where $\pi\pi_1(X)$ is non-trivial. So, we will do that by computing $\pi\pi_1$ of a circle. In some sense, this is the simplest example. The actual computation is an illustration of a powerful notion that we are going to study later, namely the covering space. That will be done in the next module. Thank you!