Introduction to Algebraic Topology (Part-I)
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Lecture 59
Free Products and free groups

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So, today, we shall first complete the proof of pushouts using the proof of the existence of free products and then discuss one or two special cases of pushouts. First, let us complete the existence of pushouts.

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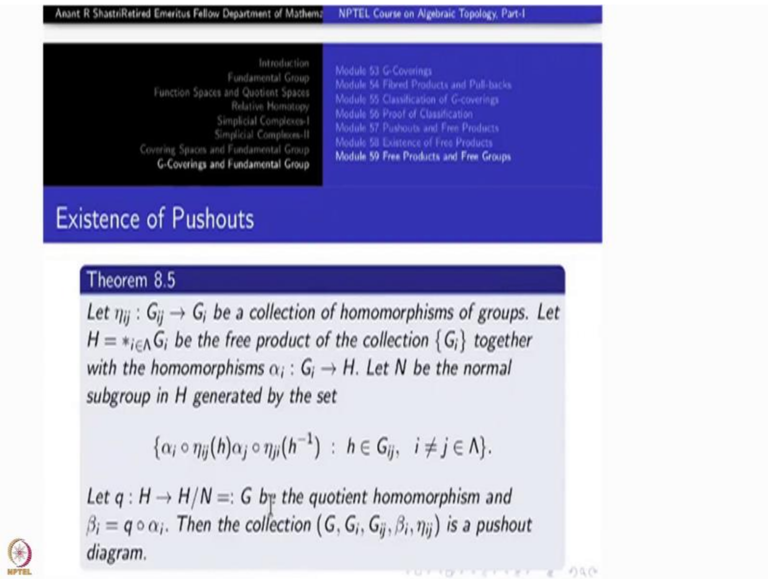

So, this is the statement of a theorem which will also recall, what is the definition of pushouts and so on. Let $\eta_{ij} : G_{ij} \to G_i$ be a collection of homomorphisms of groups. Let H be the free product of the codomains the free products of collection $\{G_i\}$ together with homomorphisms, a free product always comes with a collection of homomorphisms ofor for each factor group $\alpha_i : G_i \to H$. Let N be the normal subgroup in H generated by the set alpha i eta i j, $\{\alpha_i \circ \eta_{ij}(h)\alpha_j \circ \eta_{ji}(h^{-1}\ :\ h \in G_{ij}\}$, η_{ij} is from G_{ij} to G_i, then compose with α_i, so they are elements of H. The other one coming from another direction first α_{ji} from G_{ji} to G_j followed by α_j. I would like them to cancel out.

So, I take the normal subgroup generated by them, that is my N, for various i not equal to j, select all of them, h varies oover G_{ij} and i and j are different take all such pairs in Λ, take a normal subgroup generated by that, not just a subgroup. Go modulo that, call that G= H/N i and $q : H \to G$ be the quotient homomorphism. And put $\beta_i = q \circ \alpha_i, i \in \Lambda$.

But then I have this collection $(G, G_i, \beta_i, G_{ij}, \eta_{ij})$, G is this one, G i's are the given groups, beta i's are from where from G i, beta i is from alpha i composite q alpha i is from G i to H to G. So, this will be from G i to capital G, G i js are as before, eta i js are also as before. This collection is a pushout diagram, namely, if you have another such collection, $(G', G_i, \beta'_i, G_{ij}, \eta_{ij})$ then there will be a unique homomorphism $G \to G'$ making all these diagrams commutative.

Roughly speaking, what you want to say is the following, take the free product ignoring these η_{ij}'s, take the free product of G_i's, go modulo this normal subgroup that will give you the pushout. So, that is the essence of this theorem, let us prove this.

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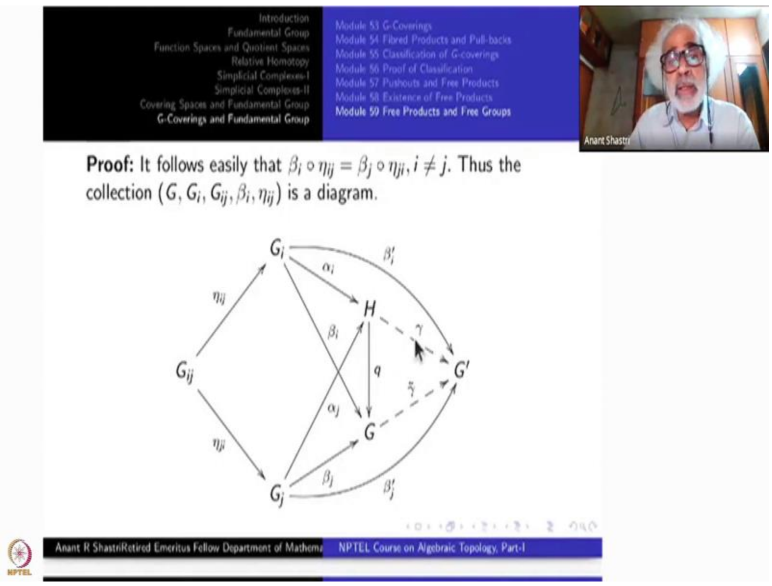

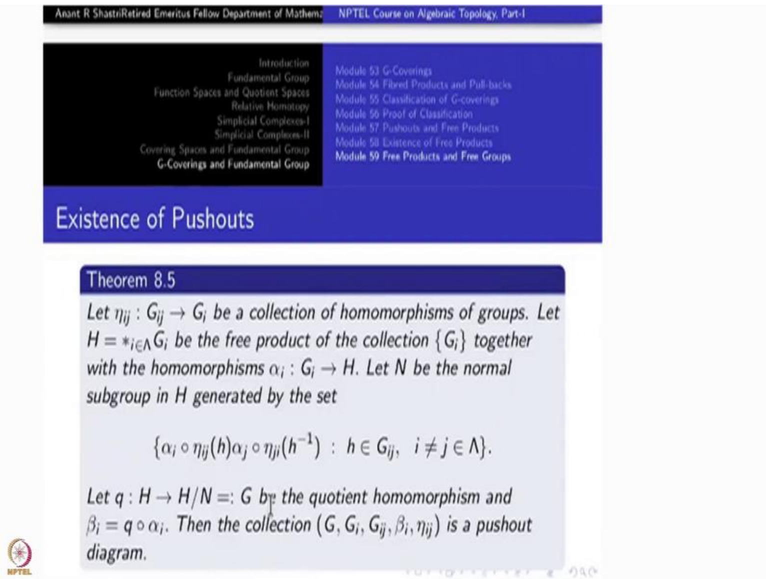

So, this is the picture, here we have. H is the free product ignoring these η_{ij}'s, just with the property that if you have any G' with homomorphisms $\beta_i' : G_i \to G'$ then there is a unique homomorphism γ here, fiting this diagram. That is the definition of H, I mean the universal property of H. Similar property for G we want to prove. What is G? G is sitting again, you see these are alpha i's by definition, these are the inclusion maps here.

This β_i is defined as α_i followed by q, $\beta_i = q \circ \alpha_i$. it is definition of this beta i's. These beta i's, this diagram becomes the pushout diagram is what I want to show. Namely, start with an arbitrary G prime and homomorphism beta j's, beta i's and so on, beta i prime such that if you start from here go like this or start from here and go like this, there must be, let say these diagrams are commutative, then I must produce a unique homomorphism which is I have denoted by dot dot dot here.

This is what I have to produce. Ignore this part apply the same property for H, I get a gamma here, it is a unique homomorphism. Making these diagrams commutative, forget about this part, this G part where all G is, forget about it, that is a commutative diagram. Now, look at this gamma, what is its property? If you take an element here, come here, an element here, come here, they are same but what I have done, this composite this inverse I have killed them, I have killed them here, because they are going to save element here. That is the meaning of this diagram.

If G i and beta i's are same this composite is the same. So, when you take this one, these all the, come here, this come here, they are same. Therefore, in the normal subgroups the generators here they are mapped to identity element by gamma. Therefore, the entire normal subgroup is mapped to an identity element under gamma. In other words, N is contained inside the kernel of gamma.

Therefore, gamma factors down by first isomorphism theorem, there is the diagram here, to a map gamma bar from G to P prime, because G is nothing but H by N by definition. So, there is a map here, there is a homomorphism here, automatically defined by the commutativity of this one, this gamma bar is defined by gamma, gamma bar, this q gamma bar is this gamma.

Therefore, once this diagram is commutative to all other diagrams will be automatically commutative, because starting with that those things are commutative, if you come from here to here, the same thing is going here. Therefore, come from here to here, here is same thing as this one is same thing as this one and so on. Therefore, it is one line proof essentially, this gamma bar makes this diagram commutative.

Why this is unique? The uniqueness follows because if it is fitting here then this composite this must be this one, but there is only one such map here and this is surjective map. Hence, it is surjective map, there are two maps you fitting here such as their composite is gamma, this is unique. So, if composite is gamma, then there must be, gamma bar must be gamma prime bar because this is surjective.

So, that completes the proof of pushout diagrams very easily by using similar property for free product. So, we have taken a middle course namely free products the special case, which immediately gave you the general case, but there are other more special cases and so on, all those things are also quite easy. Once we have proved this middle thing, so, let us do those things now.

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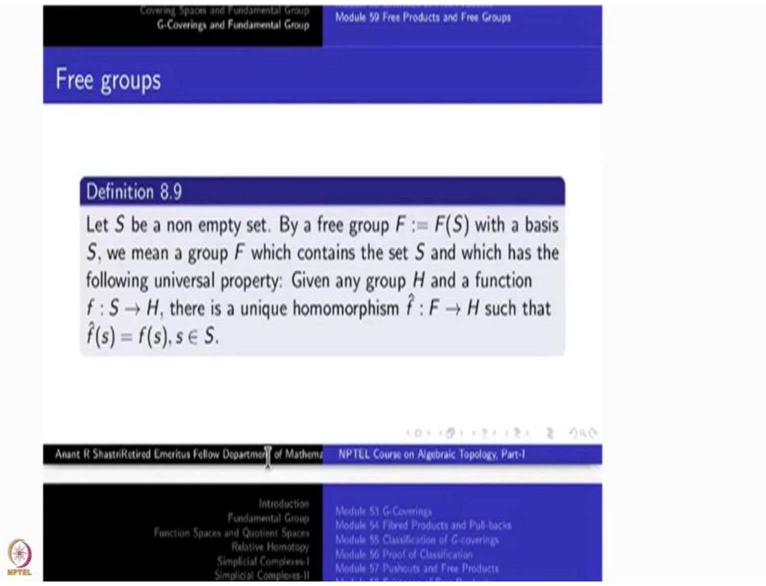

So, we come to a notion called free groups. So, let us first make a definition which is similar to the definition of these pushouts and so on, but much simpler. Once again it is given by a universal property. Start with a non-empty set S. By a free group which we denote by F_S because it depends on this S, with S as a basis, we mean the following.

So, free group or S comes with a basis, it is a part of its definition with basis S, it is like a space X comes with topology that is underlined topology has to be there. So, with a basis S, we need a group F which contains S as a subset and which has the following universal property. What is the property? Take any group H and a set theoretic function $f : S \to H$, function from S to H; that function has a unique extension $\hat{f} : F \to H$ which is a homomorphism from the whole group F to H, f hat is a homomorphism, F is just a set theoretic map and f hat is a unique there is no two homomorphisms.

$\hat{f}(s) = f(s), s \in S$ means f hat is an extension of s. Such a thing is called a free group. The simplest case is one element, suppose S contains one element. Then the infinite cyclic group is the free group over that one element. Namely, the generator there, an infinite cyclic group has precisely two generators, I mean one of them, if t is a generator t inverse is also generator, like in the additive group of integers which written additivity 1 is a generator minus 1 is also a generator, to write it multiplicatively t is a generator t inverse is also generator.

So, that t or t inverse, one of them you take then the infinite cyclic group becomes a free group over that one single element $\{s\}$ Why? Because of this property, I have just checked this property. Namely, take any group H, and send the generator s to any given element h in H that is a function, function on the singleton set $\{s\}$ that extends to a unique homomorphism from the entire infinite cyclic group.

How? All elements of this cyclic group are t power n, t goes to little h in H then where do you send t power n, to h power n. That is the only way you can extend it and you can extend it. So, this is the first simplest case, it has this property, that property has been extracted to define the free group over any set.

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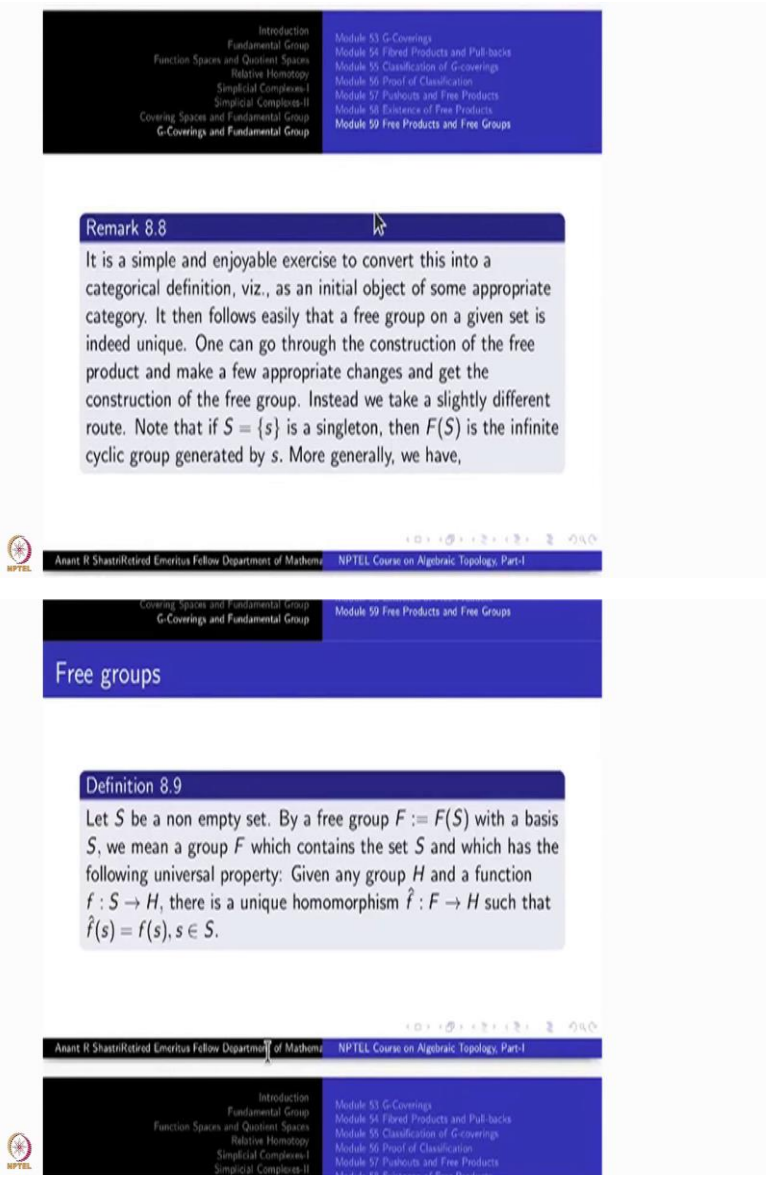

So, I repeat a few things, the simple and enjoyable exercise to convert this into a categorical definition. Namely, that F is an initial object in appropriate category. Let us not do that, it then follows that a free group on a given set is indeed unique. The uniqueness as usual up to isomorphism of the set, so that the set goes to the set bijectivity. That is a consequence of this definition of the uniqueness there is a unique map as usual I will not go to that one, as usual.

This homomorphism is a unique, this part will take care of F itself is unique. Now, one can go through the construction of the free product and make appropriate changes and then get the definition of the construction of free group. Instead of that, we will deduce it in a simple step from the construction of free groups. The key is the infinite cyclic group is generated by one single element there, so this is what it is.

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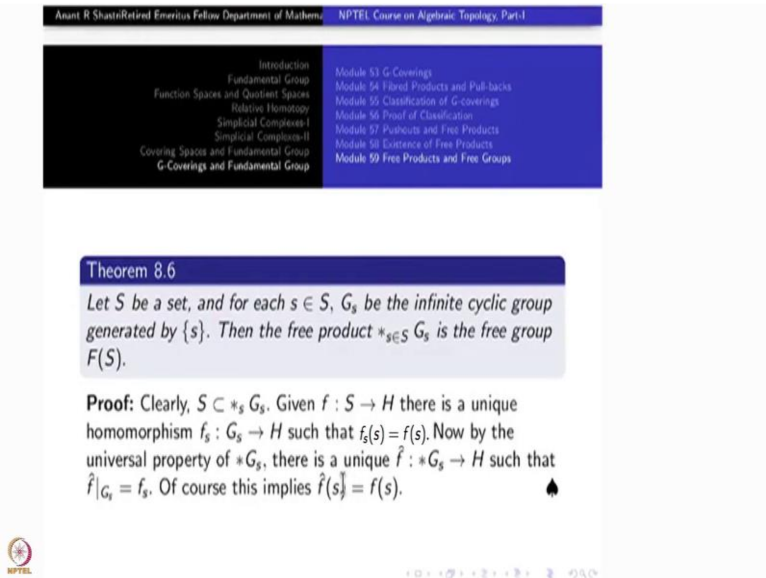

Let S be a set. For each $s \in S$, let us denote the infinite cyclic group generated by s by G_s. What is G s? $G_s = \{s^n \ : \ n \in \mathbb{Z}\}$ so that is a group. So, take G_s to be that group, this is just a notation. Then the free product of the collection $\{G_s \ : \ s \in S\}$ indexed over the given set S, is the free group F_S . We have the definition of F_S and we have the construction of $*_{s\in S}G_s$, and the properties of this free product, we will verify that this satisfies the properties of F_S. That is all we have to do.

Notice that each G_s is contained inside this free group free product, that we know. Therefore, each singleton $\{s\}$ is also contained in the free product. So the set S itself is a subset of this group, the free product. Now, take any function from S to any group H. Just now we have seen that each element S belonging to H, so there is the inclusion map $S \subset *G_s.$ Also, for each $s \in S,$ there is a unique homomorphism $f_s : G_s \to H$ this infinite cyclic group G_s to H such that $f_s(s) = s.$ This is the first stage.

Now, by the universal property of this free product, all these f_ss get extended to a unique $\hat{f} : *_s G_s \to H$ such that, when you restrict it to, restriction means under this inclusion map from G_s into $*_s G_s$, we have $\hat{f}|_{G_s} = f_s$. That is for all s belonging to S. The uniqueness follows because free product has uniqueness problem. So, this proves that free product over an arbitrary set, free product of what, of infinite cyclic groups over an arbitrary set S is the free group over set S, this is one thing.

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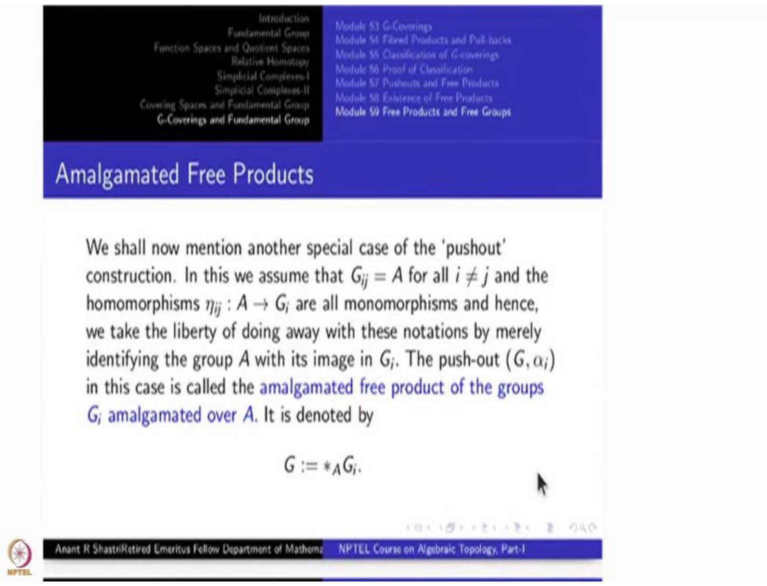

The second thing of importance is what is called an Amalgamated free products. Even if you do not use it, if you understand this one, your understanding of pushouts and also of free products and free groups will be better. Therefore, I am introducing this one. Amalgamated free product itself is useful in 3-dimensional topology.

Now, we are not doing that kind of topology here. So, this special case arises when all the G_{ij}'s in the definition of the pushout, they are the same group A, one single group A, all G i js are equal to A and all the homomorphisms $\eta_{ij} : G_{ij} \to G_i$ these are all monomorphisms. Now, there is no need to have double suffixes here because they are all same A, A to G i, you can put an eta i or you can even ignore that and identify A with some particular subgroup of G i for each i.

You can think of the collection of groups G_i and A is a common subgroup to all them. So, notation becomes very simple. Now, you have to do pushouts. This is not a free product now, but pushout makes sense, take the pushout for this special case that is called Amalgamated free product. Free product is there but there is some identification. The A part is common to all of them. In the free product, only identity element was common.

All the identities were collapsed to a single identity empty set, empty word. So, there is only one element after all, in any group, there is only one element which is the identity element. But we have so many groups, each of them has an identity element to begin with. So, they are all collapsed to a single empty word. Here, you have to do a lot of collapsing, but A part is not collapsed. A will be a subgroup of the whole thing.

So, let us see how the structure of these Amalgamated free product looks like. This is not difficult because we have understood the free product completely. So, let us do that. So, notation for Amalgamated free product over A, you use a free product notation, $*_A G_i$, but instead of writing all G i's and so on, now i is an indexing set. You write that A there, that shows that A is common to all of them, this is the notation for Amalgamated. If you have only two of them for example, G 1 and G 2 then you write $G_1 *_A G_2$ G . That is the notation for Amalgamated free products.

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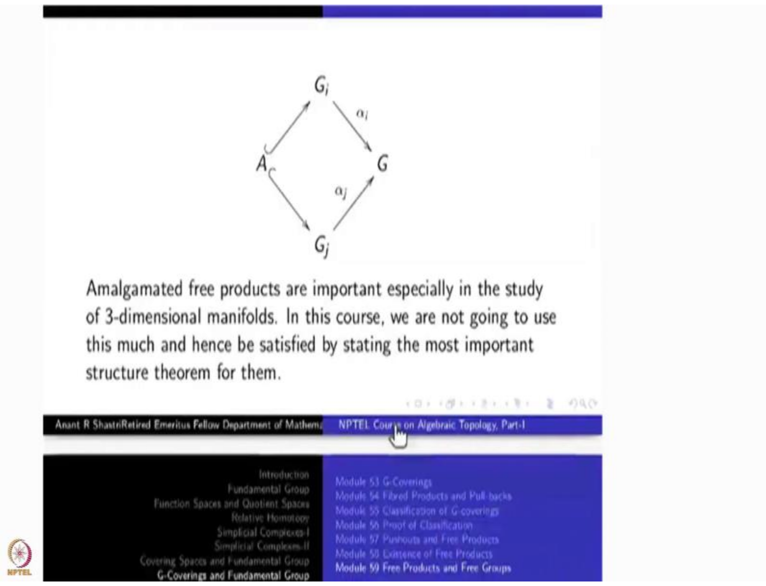

So, this diagram is a simple diagram now. Of course, G i and G j vary. Many of them dot dot dot here. Inclusion maps $\alpha_i : A \to G_i$, this is a pushout diagram, means what? If we have another of G prime here and similar diagrams here, then there is a unique map from G to that G prime. So, that is the definition of pushout, the special case being, this A being a one single group and all these things are inclusion maps.

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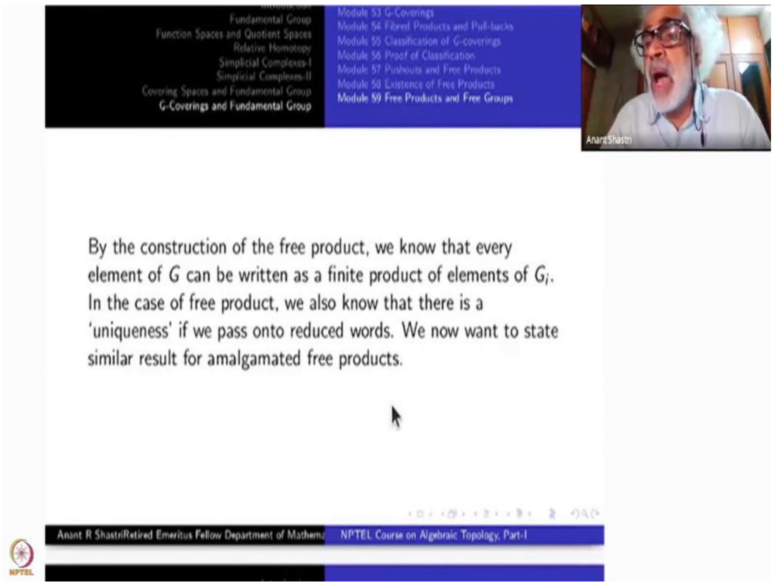

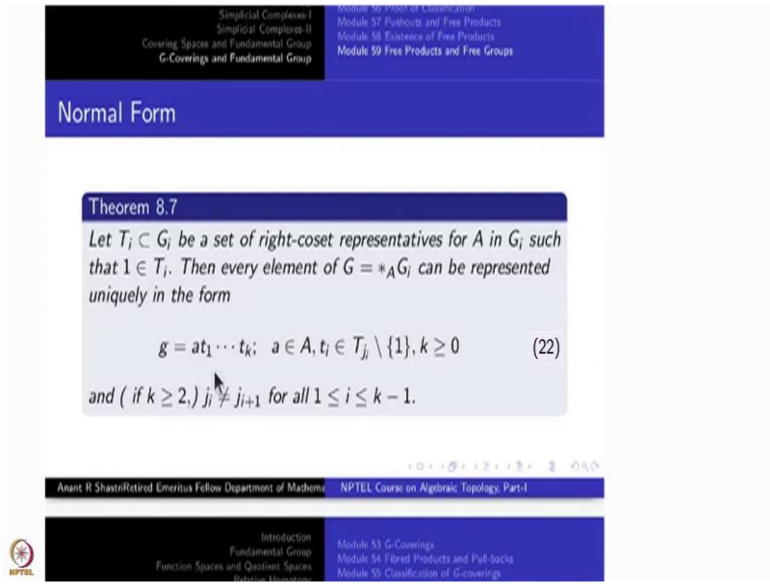

So, let us state a result similar to the reduced words inside a free group. What is the equivalent of reduced words inside Amalgamated free product? Let us go through the whole thing in the construction of free product and indicate only the changes that we hav eto do here. I will not repeat all those things which you have done for free groups again then I will say only, this is similar. So, this is the statement. Start with a right-coset representative for A inside each G_i So, let T_i be a subset of G_i, each element of T_i represents a right-coset of A and I would like to include the right-coset A itself represented by the identity element.

Do not take any other element of A. Any element of A would have been a representative but there you do not have freedom. You have to take identity element 1 belonging to T_i for each I. So, for other representaives, you have a choice you are free to choose, except that for A itself you have to choose 1 as a representative. Suppose, you have chosen such a representative, then every element of which is the Amalgamated free product of G i's can be represented uniquely in the form and this form is called normal form, in the following way.

How? Take an element $g \in G.$ \There will be some element of A this may be trivial, but I have to write it, this a is inside $a \in A$. Then it is a product $t_1 t_2 \cdots t_k$ a reduced word inside these T i’s because I want it reduced, they are inside this T j i’s but not 1. The element 1 should be thrown away, that is why we have taken 1 in A here, so that you can throw it away If it is an arbitrary element of A, we can throw it away, that is our idea.

So, t_i are strictly non-identity elements of capital T_{j_i}’s, these are representatives of right co-sets remember that. So, take the reduced word. Every element $g \in G$ can be written like this. Take a reduced word and multiply it on the left by a, but a is element of A. Including this empty word, after all elements of A are inside G, A itself is contained inside G as a sub group, so that is also allowed. So, this is solved.

So here, if k is greater than or equal to 2 then consecutive elements here are coming from different groups, $t_i \in T_{j_i}, j_{i-1} \neq j_i.$ (Of course that clearly implies $t_i \neq t_{i+1}$.) Since consecutive t_i’s are coming from different $G_i's$ therefore, this entire thing will be reduced. So, that is the meaning of this one.

The first part itself is not reduced representation because if t 1 may be inside any G_{j_1} then a and t_1 can be combined. But that is the only way to get all the elements of G_j’s. Suppose, $t_1 \in G_1$ t 1 then all the elements of $At_1 \subset G_1$ are represented. Let us check how to get such an expression for every element $g \in *_A G.$

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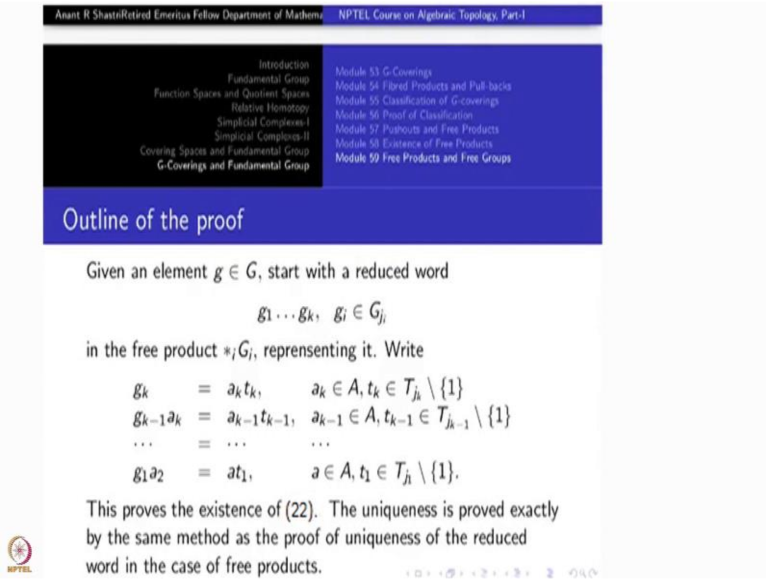

This very straightforward. Start with a reduced word in $*_i G_i$. Say $g = g_1 \cdots g_k$ Look at the right most element g_k belonging to G_{j_k}. It must be of the form $g_k = a_k t_{j_k}$ for some $a_k \in A$ and $t_{j_k} \in T_{j_k}$. It may turn out that $t_{jk} = 1$. But then $g_k = a_k \in A$ and hence could be combined with g_{k-1} and we would have started with a reduced work of shorter length. So, you may assume that $t_{j_k} \neq 1$. Now, I will take this a_k multiply on the right with g_{k-1} inside $G_{j_{k-1}}$ and rewrite it as $g_{k-1} a_k = a_{k-1} t_{j_{k-1}}$, for some $a_{k-1} \in A$ and $t_{j_{k-1}} \in T_{j_{k-1}} \subset G_{j_{k-1}}$.

Like this, you keep shifting this portion of A, all the way back to the frst place in the sequence on the left side, left most. Automatically, what you get is $t_{j_1} t_{j_2} \cdots t_{j_k}$ the consecutive ones will be in different G_j's. So, automatically this will be a normal form. So, once you know how to reduce a word, you know how to make it into normal form. Question remains, why this is unique.

And people say it is obvious because there is only one way to get it, if you are satisfied, it is okay. Otherwise, the same proof like what you have to do, take normal forms and define these L_i's from each G_i into the permutations of all these elements. Do the same thing for here, then you get the uniqueness of this one. Actually, once you have done the uniqueness of free groups, and this this algorithm to get the normal form it has to be unique.

So, I am accepting this also as a proof. Choice is of, t i's you have to choose after that there is no choice. All g k is given, t k and a k are well defined, there is no ambiguity in that. Getting the reduced word itself, you did not know what it is, but now we have proved that one. So, this is a proof.

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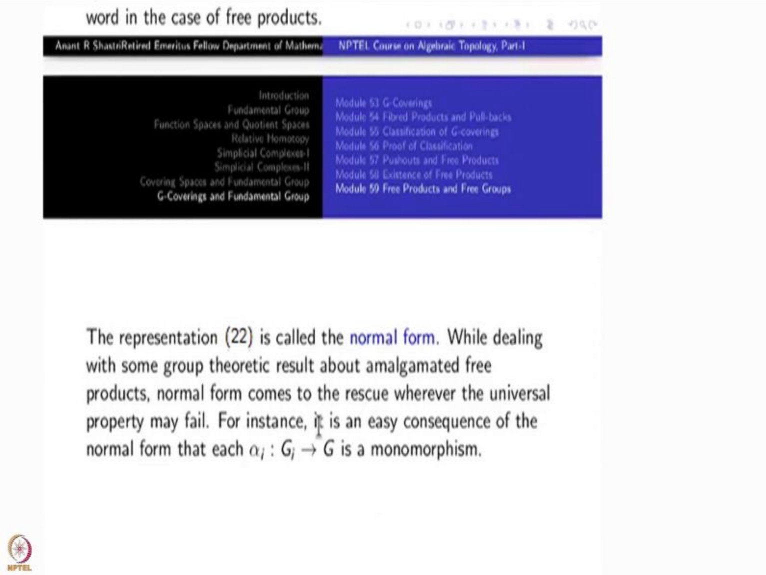

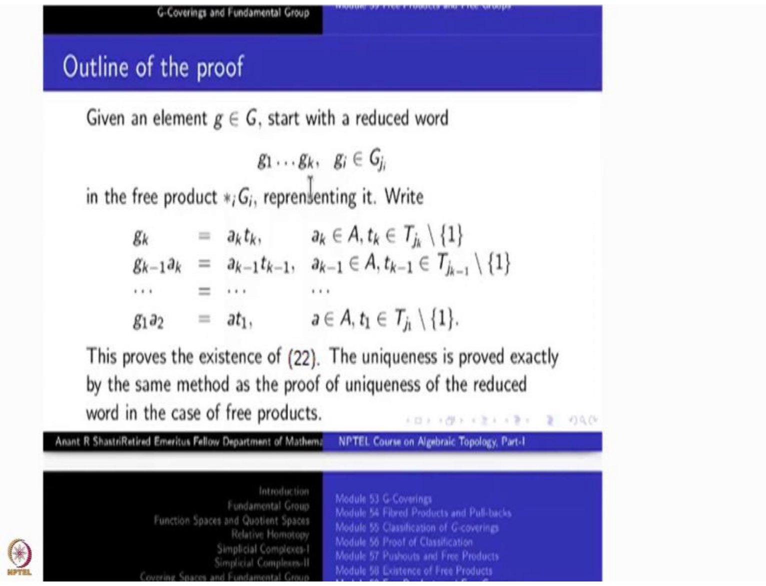

So, final remark is that you can use the universal property of the pushouts to obtain many results. But often, what may happen is, you may get stuck up; you do not know what to do. Then this normal form will help you, it is like a mother, it will help you out, just like the reduced words will help you out. It has the the same role as reduced words in the case of Amalgamated free product, the normal form. For example, using this one you can immediately see that, each G i is a subgroup of G.

Go back here, if this starting element is already inside one of G_i's, then what will you have done? Look at the normal form, it is just where you write inside G_i it is a times some co-set representative, nothing else. And once you have uniqueness inside G_i going inside $*_A G_i$ this one is injective. Let us stop here.

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The next thing is putting all these preparations to produce and we will do some topology now. So, we have done a lot of algebra and regression of these things, pushout various deals and classification of G coverings and so on. Now, we will reduce a lot of things about fundamental of group itself. That is our next topic. Thank you.