Introduction to Algebraic Topology (Part-I)
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Lecture 57
Pushouts and Free Product

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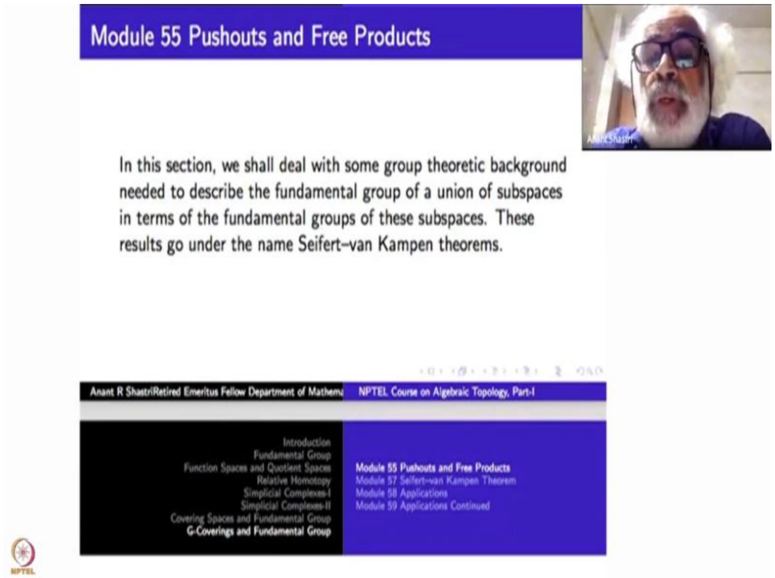

So, today's topic is Pushouts and Free Products. In order to study various versions of Seifert-van Kampen theorems which will tell you how to relate, how the fundamental groups of the total space is related with the fundamental group of some of its pieces, we need a certain group theoretic background. So, let us be familiar the final objects are going to be there. After all these are some groups, right? So, let us be familiar with them and then come back to the topology.

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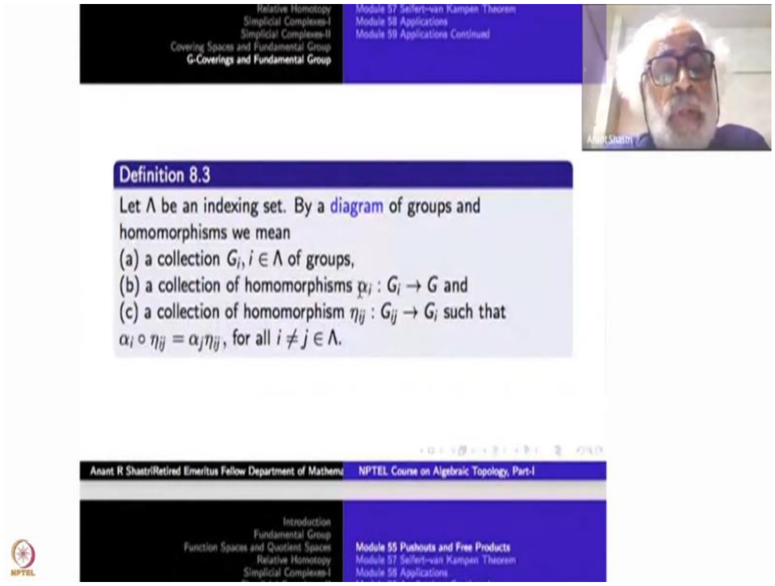

So, here is a formal definition. A diagram of groups, for the lack of any better naming, we are just calling them a diagram of groups and groups and homomorphisms. So, there is an indexed family of groups G_i and homomorphisms from $\alpha_i : G_i \to G$, $i \in \Lambda$. And another double-indexed family of homomorphism α_{ij} from another set of group G_{ij} to G_i such that the first set of group homomorphisms respect second one, i.e., $\alpha_i \circ \eta_{ij} = \alpha_j \circ \eta_{ji}$, for all $i \neq j \in \Lambda$. Interchanging i and j should be allowed for all $i \neq j$.

Note that there is no group G_{ii} and no homomorphisms α_{ii}. Whenever i and j are distinct, then only Gij's are some groups. Like pairs of topological spaces, pairs of groups and so on... These objects can be made into a category. Whatever it is, do not worry about the categorical language. I will explain what it is in simple terms. So, I take another set diagram with the same indexing sets, namely, $(G', G_i, G_{ij}, \alpha'_j, \alpha_{ij})$. Then by a morphism from the first object to the second object is simply a group homomrphsim $\gamma : G \to G'$ such that $\gamma \circ \alpha_i = \alpha'_i$ for all i.

The entire diagram must commute. So, that will be called a morphism between such diagrams so that is the categorical aspect. And then what we are looking for is some kind of a universal element what are called as initial objects in this category. So, I am going to describe that now.

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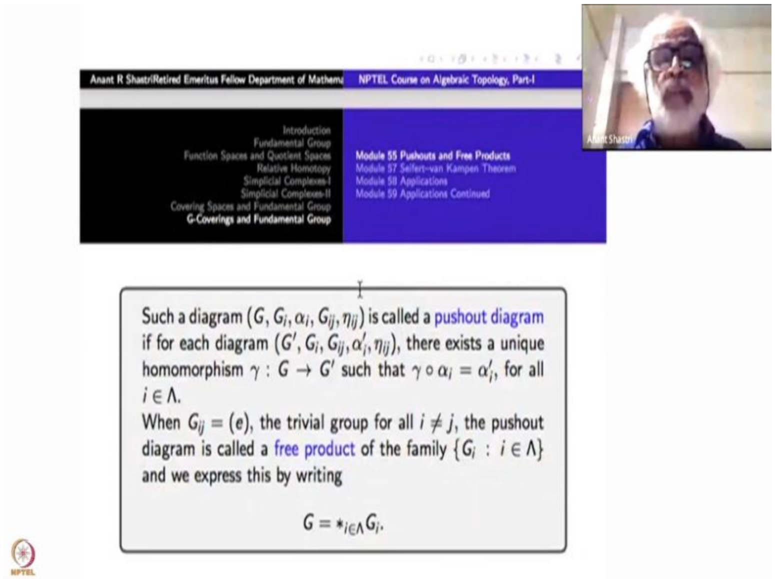

So, the first object will be called a pushout diagram if we have a unique homomorphism γ as above, whenever a second diagram is given.

So, in some sense, you see this γ is some kind of extension of all the homomorphisms $\alpha'_i : G_i \to G'$. An important special case is that when all these Gij's are trivial groups. Obviously, η_{ij} and α_i are trivial homomorphisms and then this condition $\alpha_i \circ \eta_{ij} = \alpha_j \circ \eta_{ji}$ is obviously satisfied because both sides will be trivial.

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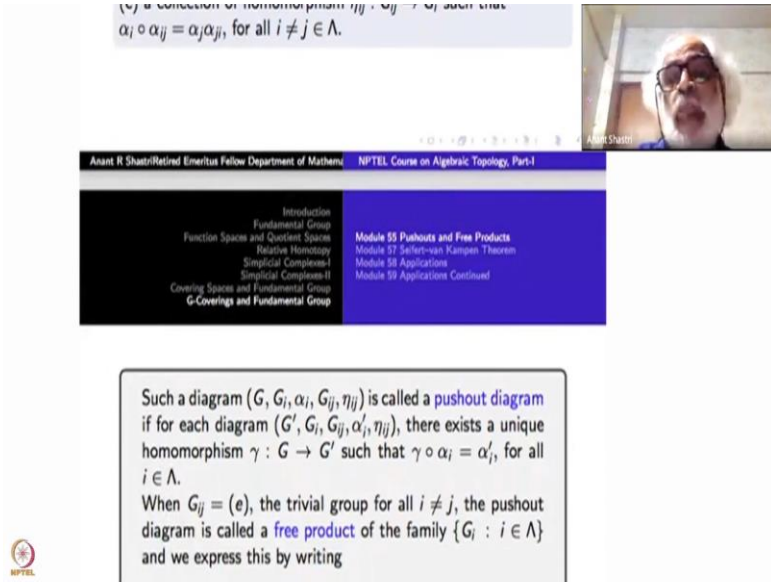

So, a set of homomorphisms alpha i from Gi to G which is the only data. So, this is a special case because all Gij's are 0 and this will be 0 and trivial homeomorphism and so on. But now given any other G prime and Gi's, the same Gi's and these alpha j primes will be there. Ok these alpha primes will be there alpha a prime. You must have a unique γ so as the diagram commutes. Then that G will be called the free product of Gi's.

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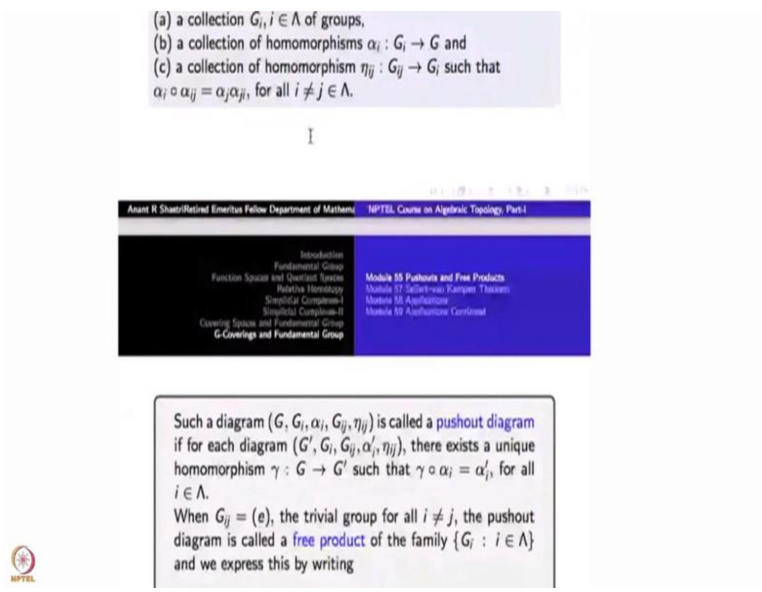

Student: Sir, in the previous diagram, what are α_{ij}'s?

Professor: They are also

Student: I think it will,

Professor: Eta ij’s

Student: Yeah, it will be eta ij’s right.

Professor: At some stage, I might have changed it to apha ij’s yes. So, good to clarify that, these are alpha eta ij’s are here, eta ij’s. This is ij, everywhere it is eta ij. In any case the special case when it is a free product eta ij’s are all trivial Gij’s are also trivial.

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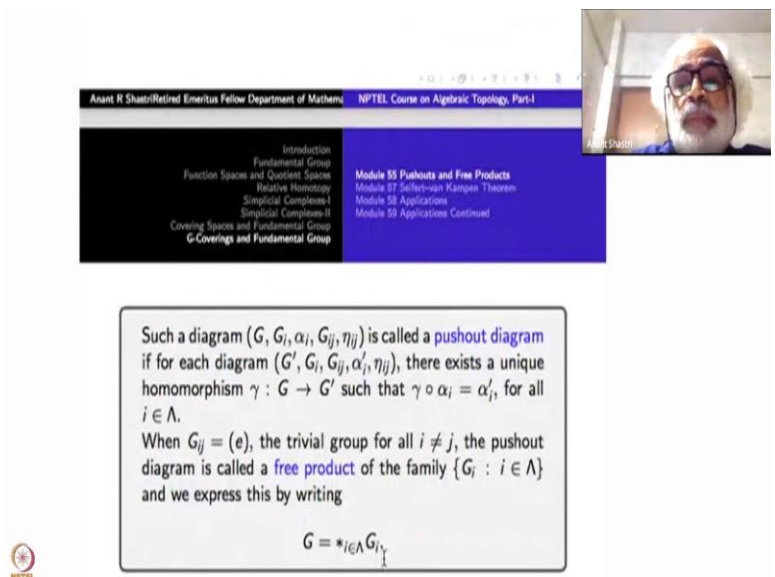

Then we have a special notation also that G, namely, $*_{i\in\Lambda} G_i$. star taken over all i belonging to lambda. In particular suppose there are only rwo of them, then we may write it in the form $G_1 * G_2$. If there are finitely many of them, then we may write $G_1 * \cdots * G_n$ G1 and so on. This notation is not convenient when there are infinite families, even when Λ is countable. So, we better write like this. And by the very nature, this is commutative G1 star G2 and G2 star G1 if you look at there is no problem. They are, they must be isomorphic. So, notation is justified. There is no problem about the order, there is no order on Λ here.

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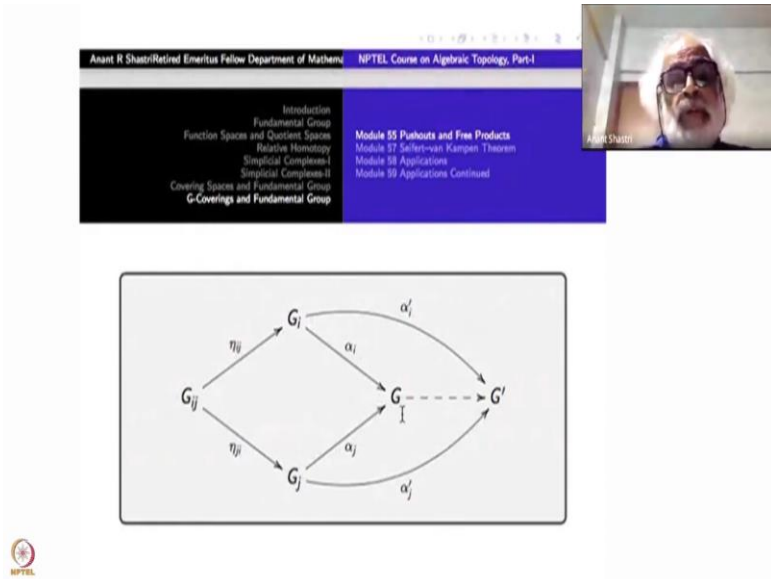

So, a diagram like that is presented like this, where in only i and j have been picked up. But you have to keep on varying these things, writing all of them infinitely many of them is not possible. So, you have to just write like this. Which is this part is a diagram. If we ignore this one the outer thing that is also a diagram, with the same eta ij's, same eta ij alpha ij's alpha ij are different such that this composite this must be equal to this composite.

That is the composite that is the assumption. Then if there is a unique homeomorphism here. For every possible G prime like this, the bigger diagram like this such that this diagram is converted to this one is also commutative and so on. The entire diagram commutative means this composite this is this arrow, this composite this is this arrow. This part is already given to be commutative.

If there is a unique this dot, dot, dot then, for all possible G prime that is a universal property. Then G is called the pushout of the original diagram and it is called a pushout. Then the special case, all Gij's here trivial groups and obviously these eta ij's are trivial homeomorphisms which just means that this part, there is nothing no conditions, so starting with arbitrary family of groups and homeomorphisms into another group. That another group G will be called the free product. If it is satisfied it is a universal property that is what I am trying, telling that in categorical terminology this is called a pushout diagram is nothing but the initial object in the category of this diagrams.

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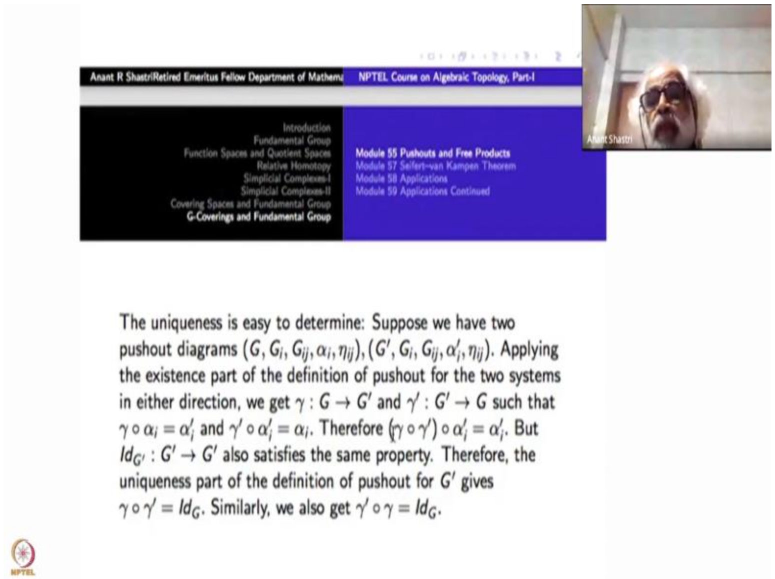

The definition by nature, namely universal property has one advantage namely uniqueness up to isomorphism is built-in. You have to study one or two of them to used to the idea. After that you will see a lot of advantage. You do not have to checked the uniqueness at all, you know it. So, it is like defining a vector space over a set as a generator by universal property. The vector space up to isomorphism is unique.

So, what is the uniqueness? Let us check this one here. Suppose G, Gi, Gij, alpha i, eta ij is one system G prime, Gi again Gij's, alpha i prime is different G prime sequence Gi, Gij's, eta ij's are the same. The homeomorphism alpha i primes and G prime these are the structures built over the Gij's and Gi's over eta ij's. If there are 2 such objects, then go back here.

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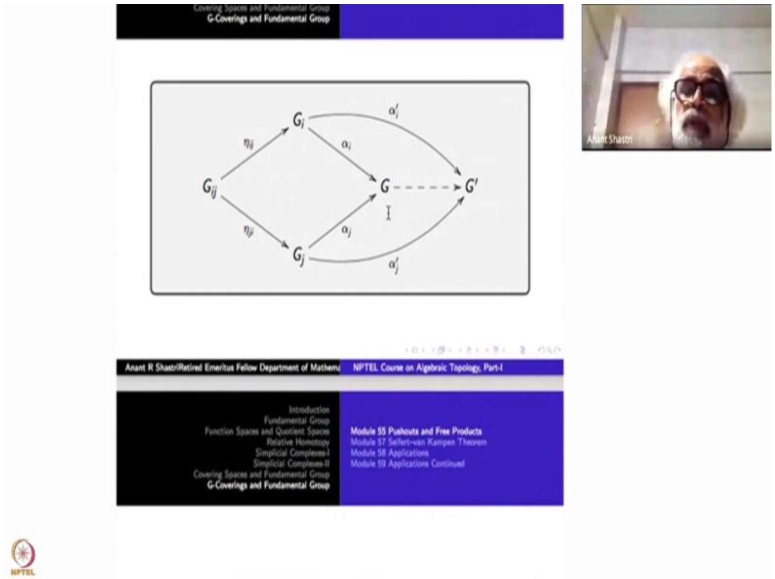

By definition of pushout for G, there is a unique homeomorphism like this. But by definition of G prime as a pushout, there is a unique homeomorphism like this. Now, if you take see this gamma, this is gamma prime let us take gamma composite gamma prime is a self-map homomorphism. By the uniqueness for the same G, take both the places G and G the alpha i's and alpha i, the uniqueness says the map whatever the map here is unique.

What is that map? I can always tell it as identity map. But now I have got 2 different maps gamma and then followed by gamma prime. So, gamma composite gamma prime must be identity morphism. Exactly for the same reason gamma prime composite gamma must be also identity of G prime just does not mean gamma is an isomorphism such that these structures are triggered already by the varied nature. That is the meaning of uniqueness here.

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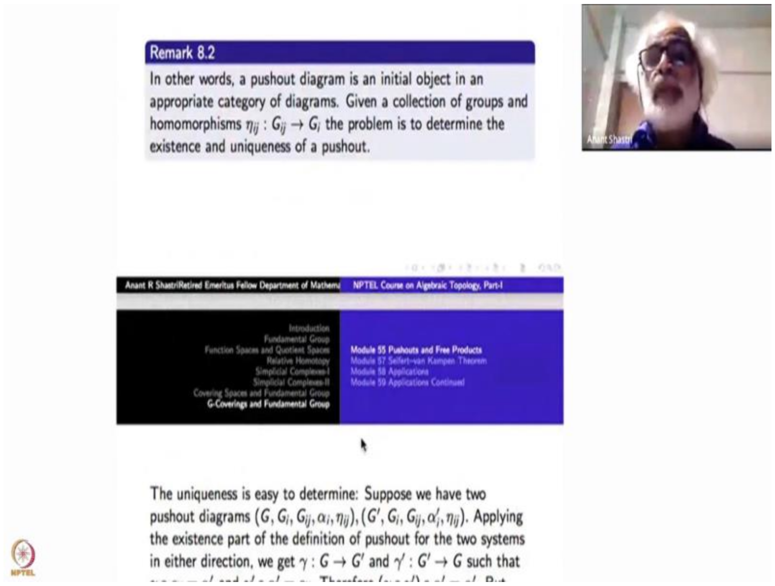

The problem is that in many categories making this definition is easy but sometimes the object does not exist. Then you have to make a subcategory where it may exist. Add extra conditions and so on, so showing that the existence of these things is the important part of this kind of definitions.

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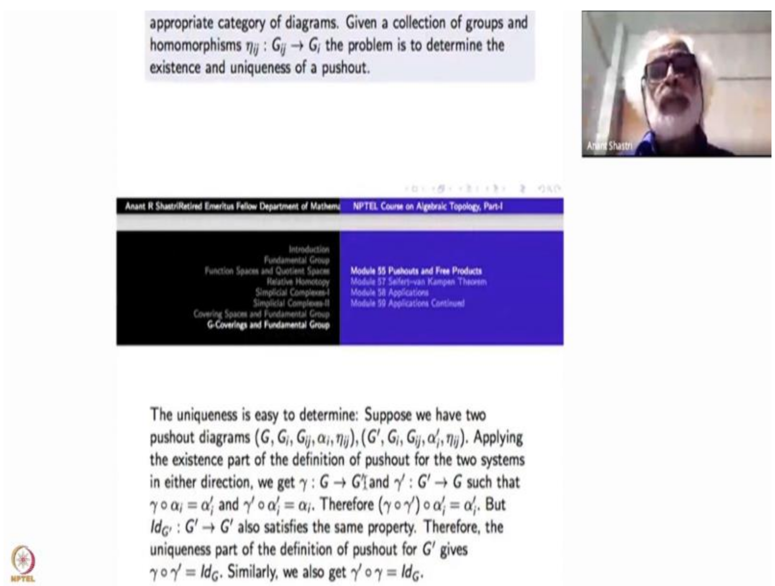

After that many, many properties and results are obtained by just by going back to this universal property of this definition rather than the construction. Unfortunately, this is not the case with these pushouts and the special case free products. There are some very hard results which could not be just proved by using universal properties. Luckily, we are not going to do such deep results. So, that is just a warning that is all, though there are some such cases but most of the time, the, you know the economy of this definition is that you do supply everything you want.

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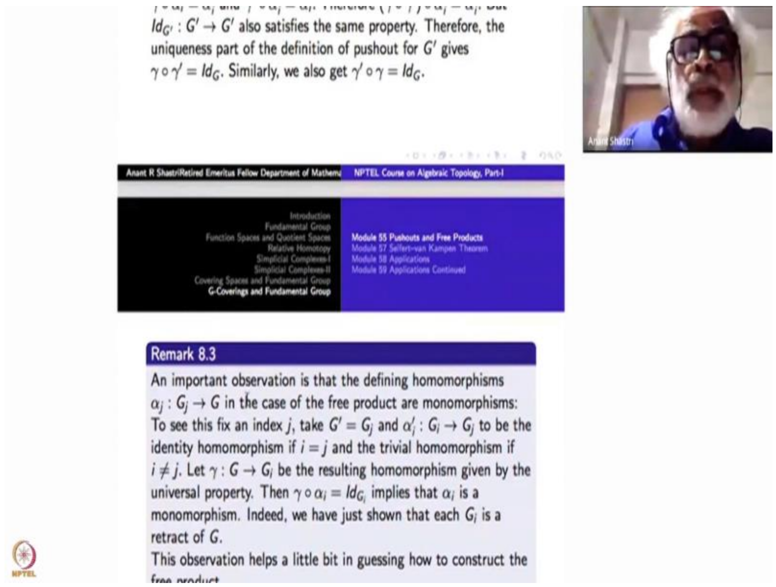

So, let us make one single observation just by using the universal property. The defining homeomorphisms alpha j from Gj to G or I can write it as alpha i from Gi to G. In the case of free products, free products means what now, Gij’s are trivial, alpha ij’s are eta ij’s are also trivial so they are not mention. In that case these alpha j’s automatically become monomorphisms injective homeomorphisms. Let us see how. All that I do is I go back here.

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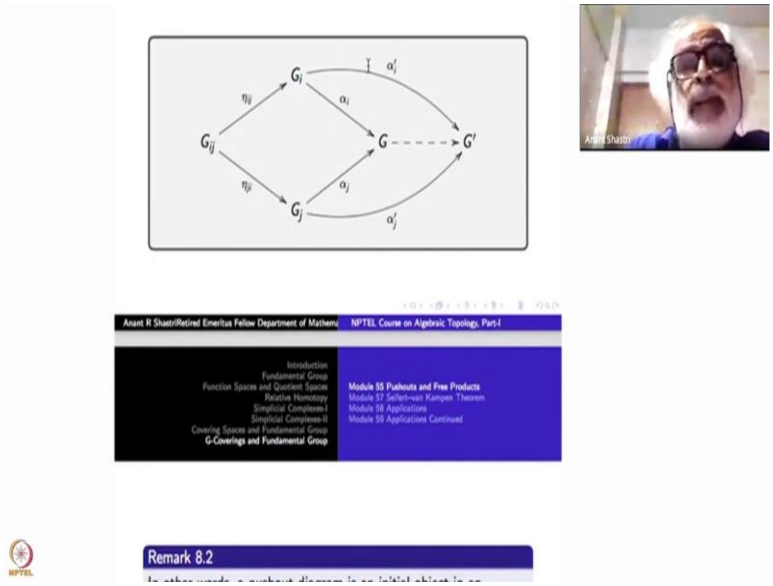

Here I take G prime itself to be Gi, one of the Gi's, fix an i, take Gi here. This alpha i prime you take it as identity map from your Gi to Gi. All Gj to Gi you can take any homeomorphism but take trivial homeomorphism you know that. Given any 2 groups that is a trivial homeomorphism. Every element here going to trivial element of this group take that. That is this part is not there. I am talking about free products.

With this hypothesis because G is a pushout, G is a free product it should, there must be homeomorphism for here to here which makes these diagrams commutated. Forget about this one. This also tells you something but what about this one? This says alpha i composite followed by gamma is identity map. That just means that this gamma has a right inverse on alpha i, so in particular alpha i must be injective. So, there is a proof.

In fact, so this is true for all i because I have no specific, I have chosen i to be arbitrary anyway. So, all the groups Gi's are included through alpha I inside the B group G. What happens to this gamma which is the inverse? That defines this actual interaction of this is stronger than they are saying this is an homeomorphism which is a split monomorphisms. Our Gi's are actually retracts of G. Retract means what? right inverse is there.

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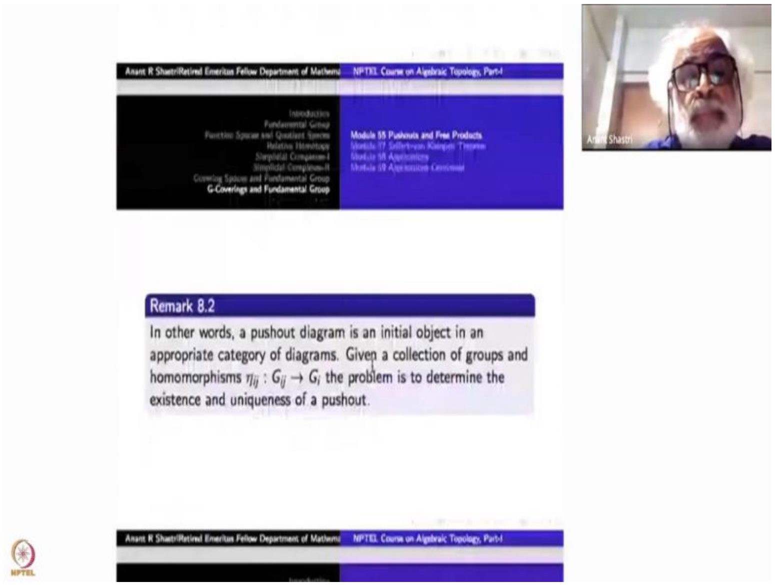

Of course here, there we have continuous retractions in topology. There are continuous maps here there must be homomorphisms that is all.

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So, this is what I have written down here, existence and uniqueness are built in the homomorphism. The uniqueness part means what there exists a unique homomorphism so you have to use the uniqueness part here, most of the time. Just now what we used is only there exist one, all right.

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So, but sorry this is not this is uniqueness part. Here what I wrote down already the retraction, so all the groups Gi's are subgroups of G. If you remember or if you have studied direct sum of a abelian group. The same definition can be taken, except that everywhere you should put abelian group, abelian group, abelian group. That is the meaning of working inside the category of abelian groups.

Of course, homomorphisms are homomorphisms. There is no special homomorphism in abelian groups. Then what you get is instead of free product you get what is called the direct sum. So, in the categorical language, free product is a misnomer. They should have called it a sum. The product actually corresponds to the dual notion, namely the cartesian product which is more familiar to you.

The beauty is, if X and Y are two topological spaces, the union is also a sum. So, Van Kampen theorems state that fundamental groups of the sums are some sort of `sums' of fundamental groups. That is the way to remember it. That is too simplistic one but there are lots of conditions that we need as inputs. So, these Gi's being subgroups of G will tell you where to look for the construction, what kind of groups I should take, namely in the abelian case, we had taken restricted cartesian, sum of the restricted cartesian product. Remember that direct sum is that one. So, what do we do here? So, this is non-abelian case.

So, to more or less gives you some hint for how to go about the construction. Existence have to be proved constructive here that is one thing good about you know being so much abstract algebra but we are not just seeing there exist one and be done with it. We actually construct it, algorithmic. So, that is the beauty of most of these construction methods for many of these existence theorems here.

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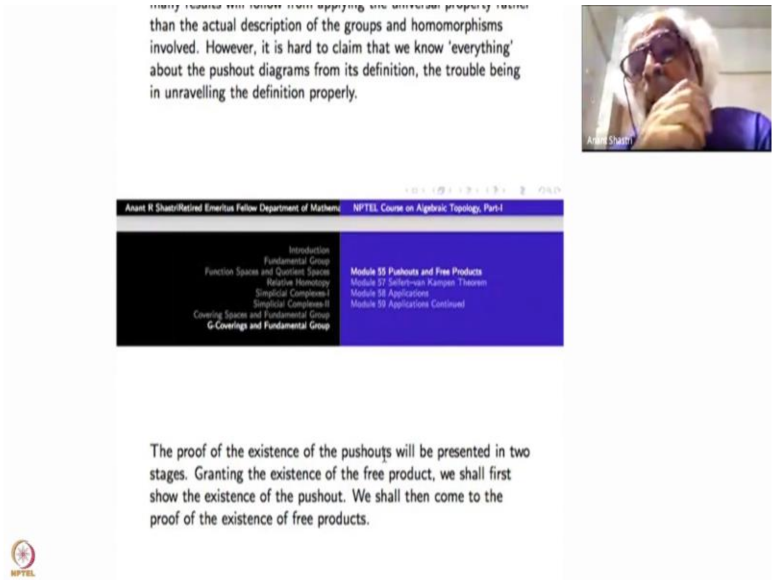

Well, nevertheless we have to work harder here. So, the pushout general case will be done after doing the special case. I am talking about the construction now. The special case itself is complicated, first let us understand that. Then one can do the general case by a small trick. So, let us assume that free products exist and then be done with the general pushout diagrams. How to get that?

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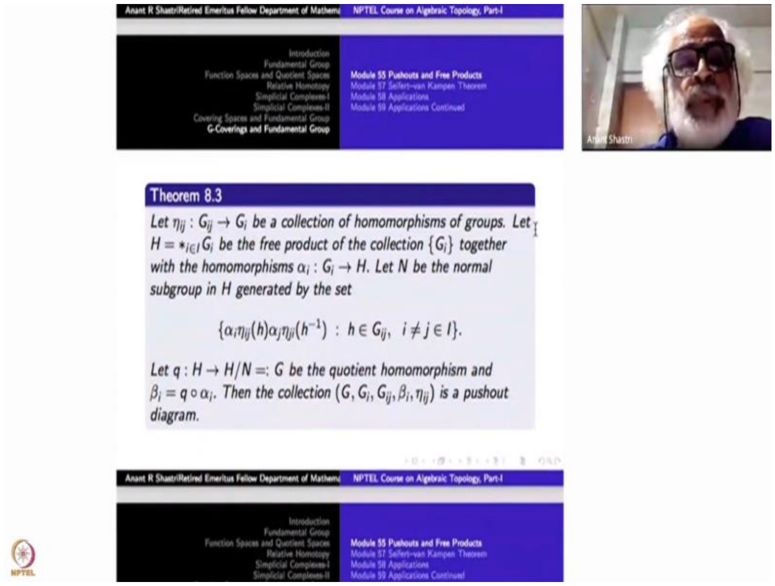

Here is the way. So you have full data now $\eta_{ij} : G_{ij} \to G_i$ collection of homomorphism. You have to define these $\alpha_i : G_i \to G$. In particular, you have to define what is G now. Then $\alpha_i : G_i \to G$ such that $\alpha_i \circ \eta_{ij} = \alpha_j \circ \eta_{ji}.$ so on.

Take H to be the free product $*_{i \in \Lambda} G_i$, The free product comes with this comes with homomorphism from Gi into H. Right now, you can call them as α_i's all right. Then you are forced to, you have to modify these α_i for this part ok? These α_i's a will not do for the pushout in the general case. The free product ignores this η_{ij}'s ok? So, it is the direct free product of Gi and it comes with these $\alpha_i : G_i \to H,$ and with the universal property.

Now you look at N, the normal subgroup of H generated by all elements of a particular form.

For each $i \neq j \in \Lambda$, take an element $h \in G_{ij}$. Apply η_{ij} and then α_i to get an element $\alpha_i \circ \eta_i j(h) \in H.$ Similarly, interchange i, j, to get another element $\alpha_j \circ \eta_{ji}(h)$ in H. I wanted that these two elements msut be the same for all possible $h \in G_{ij}$ and for all $i \neq j$ so that the commutativity of the diagram is valid.

In the free product H this may not hold, because it has ignored these extra structures. So, we define a relation in H which will make these elements `same'. Namely, take N to be the normal subgroup generated by all elements of the form $\alpha_i \circ \eta_{ij}(h) \alpha_j \circ \eta_{ji}(h^{-1}),$ where $h \in G_{ij},$

$i \neq j \in \Lambda.$ Take G be the quotient group and let $q : H \to H/N =: G$ be the quotient homomorphism.

Now let β_i, not to confuse with α_i 's, I am making another notation here, put $\beta_i = q \circ \alpha_i : G_i \to G.$ You may call it alpha i bar. I do not want to do that, so I put it to beta i. So, now you have got the full diagram. We have only to verify its commutativity, viz., $\beta_i \circ \eta_{ij} = \beta_j \circ \eta_{ji} : G_{ij} \to G, i \neq j \in \Lambda.$ That follows because $q(\alpha_i \circ \eta_{ij}(h)) = q(\alpha_j \circ \eta_{ji}(h))$ for all $h \in G_{ij}.$

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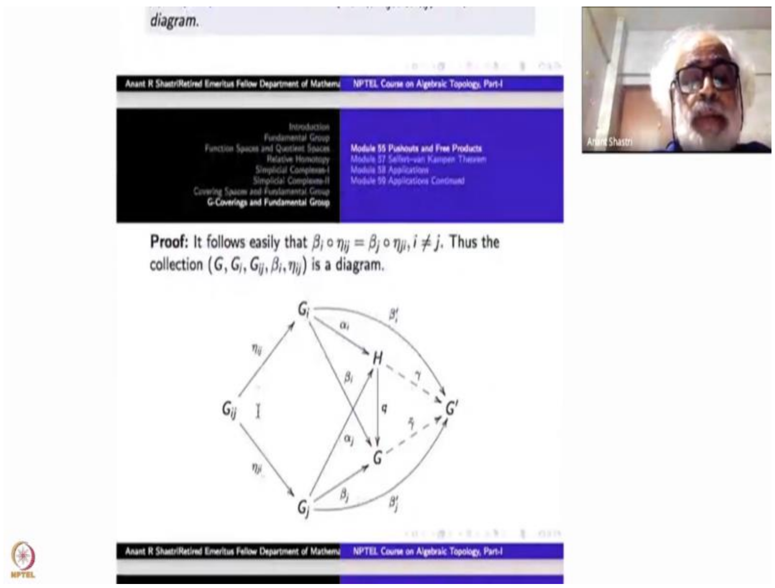

So, here is the picture. Starting with eta ij's, Gij's and all this, you ignore this part and construct H which is the free product of Gi's. That will come with the alpha i's and alpha j. This beta is going down beta i prime is going to come here like this, beta j is here. That means these are alpha i's. What are beta i's? Beta i's are after composing with the quotient map q. What is this quotient? It is $G = H/N$, by the normal subgroup N that we have defined.

Take this quotient. Now, I want to show that this lower diagram here, this part is a pushout. This part is a free product This square, the all these rectangles together, they will be define G as a pushout. So, what I have to do? Take another diagram like this outer one. I have to produce $\bar{\gamma}$ here a unique homomorphism, which makes these diagram commute, viz., $\bar{\gamma} \circ \beta_i = \beta'_i$ for all $i \in \Lambda.$

So, how to get this $\bar{\gamma}$? That is very straightforward. All that I have to do is use the same property for H, (H is the free product), let us say we have the unique homomorphism $\gamma : H \to G'$ here such that $\gamma \circ \alpha_i = \beta'_i, i \in \Lambda.$ That is a property of free product.

Now, you take this normal subgroup generated by this composite this H of that and H inverse of this one and so on right. Then you go down by that normal subsgroup. What does it mean? When it come here, these 2 elements would have gone to the same element right? Those two elements are now identified here. x is equal to y in the quotient is same thing as saying that $xy^{-1} \in N$ the normal subgroup. So, that is what has happened here. Therefore, by first isomorphism theorem, this $\gamma : H \to G'$ factors down through q to another homomorphism $\bar{\gamma}$here such that $\gamma = \bar{\gamma} \circ q.$ Now it is clear that $\bar{\gamma} \circ \beta_i = \bar{\gamma} \circ q \circ \alpha_i = \gamma \circ \alpha_i = \beta'_i.$

The uniqueness of this one is also clear because if there is another map which fits them then this map should be such that this diagram is also commutative. But then that this map is unique there is there is map 2 of them and this is surjective map. Therefore, if there are 2 of them here they must be same. This completes the proof of the existence of pushouts provided we have proved the existence of free products. The next time we will try to complete the proof of existence of free products. Thank you.