Introduction to Algebraic Topology (Part-1)
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Lecture 56
Proof of Classification

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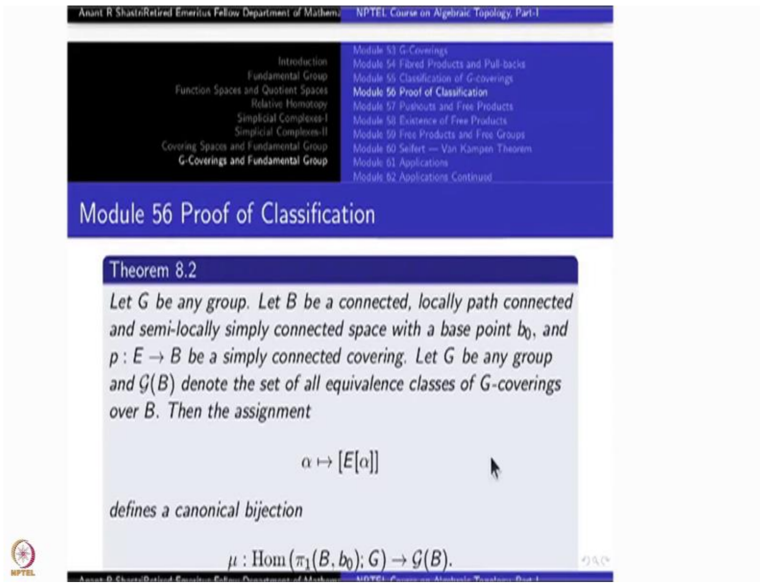

Today let us first begin with the statement of the classification theorem. We begin with a group G and a connected locally path connected semi locally simply connected space B with a base point b_0 and a simply connected covering $p : E \to B$. Of course that also has a base point e_0 which is mapped onto b_0. Let $\mathcal{G}(B)$ be denote the set of all equivalence classes of G coverings of B.

Then assignment $\alpha \mapsto E[\alpha]$, where $\alpha \in \mathrm{Hom}(\pi_1(B, b_0); G)$, that is a homomorphism from the fundamental group into the given group G to the extended action of G covering $p[\alpha]$; starting with the J covering $p : E \to B$, Then take this equivalence class $[E[\alpha]]$, that is a map $\alpha \mapsto [E[\alpha]]$ which we are denoted by $\mu : \mathrm{Hom}(\pi_1(B, b_0); G)\mathcal{G}(B)$. This is a canonical bijection.

Notice that the right-hand side here is a set of equivalence classes; on the left hand side they are just the set of all homomorphisms. In particular, if G is abelian, this set becomes a group otherwise may not be. So the equivalence classes of G coverings they themselves form a group in a strange way. This is a consequence of this bijection.

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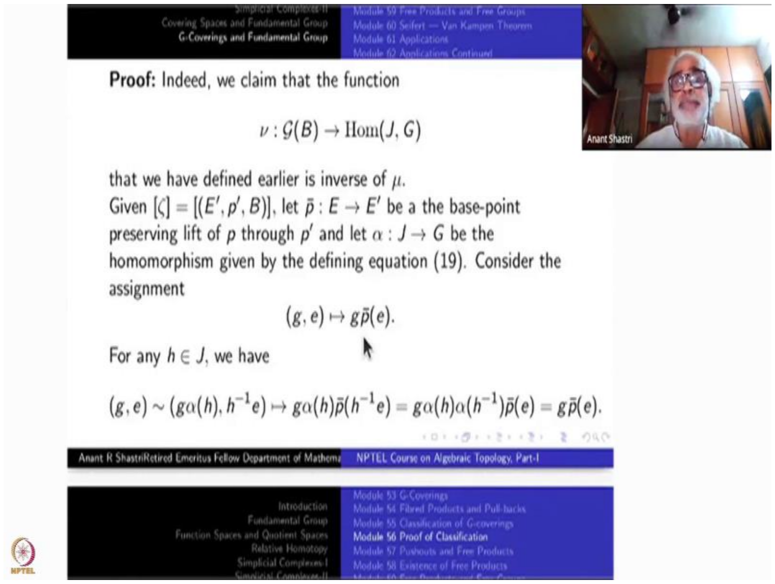

In fact what we want to do is we have already defined the inverse map, we have not verified it though $\nu : \mathcal{G}(B) \to \mathrm{Hom}(J; G)$, where $J = \pi_1(B, b_0)$. And we want to show that μ and ν inverses of each other. So, the first thing is to show that $\mu \circ \nu = Id;\ \&\ \nu \circ \mu = Id$. And then there is a part which referes to canonicalness. There are these three main parts of the theorem.

So, start with the equivalence class of G covering which I denote by $[\zeta]$, where $\zeta = (E', p', b)$. And then, because E is the universal covering, you will have a map $\bar{p} : E \to E'$ which is actually the lift of p through p'. If you take care to send the base to the base point, viz, $\bar{p}(e_0) = e'_0$, then $\bar{p}$ is unique map. So, take the homomorphism $\alpha : J \to G$ given by the equation 19. Whatever the equation is, let us have a look at equation 19 and come back here.

Here so you start with a loop ω representing the given element of π_1, lift it to a path in E beginning ar e_0, look at the endpoint. Then $\bar{p}$ of the endpoint and e'_0 are in the same fiber of p'. Therefore they are related by an element of G, that element g is $\alpha([\omega])$. This is a definition of αgiven by the equation (19). We will not remember this I will be using it again perhaps. So, we have this defining equation. Now, consider the assignment $(g, e) \mapsto \bar{p}(e)$. So, I want to say that this assignment respects the equivalence classes.

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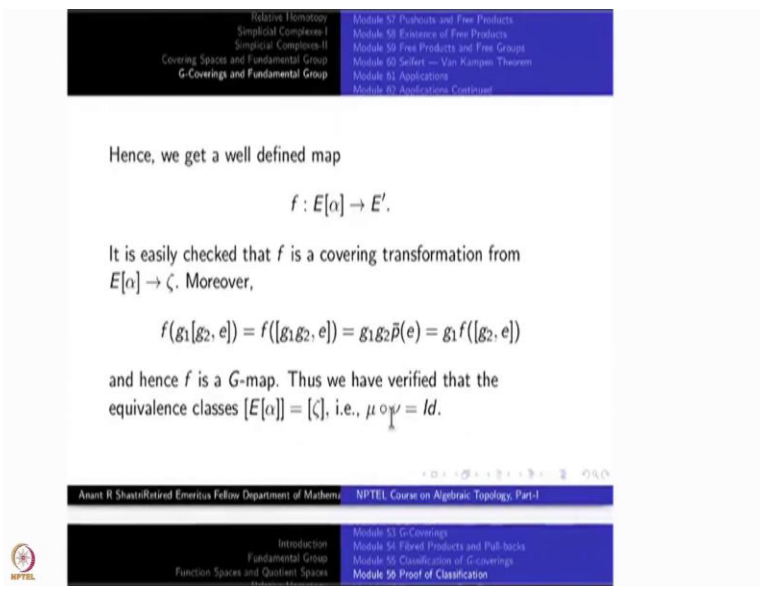

Finally, what I want define is a map from $E[\alpha] \to E'$ so, that this becomes a G-map so they are equivalent. So, before that to define a map like this I will define it on $G \times E$ into E' by $(g, e) \mapsto g\bar{p}(e)$. Then I only show that it respects the relations there the defining relation for $E[\alpha]$ So, what is that? an element (g, e) is equivalent to $(g\alpha(h), h^{-1}e)$ for some $h \in J$, the fundamental group J. That the equivalence relation. Now the second one is mapped, by the definition, to $g\alpha(h)\bar{p}(h^{-1}e)$. But this is same as $g\alpha(h)\alpha(h^{-1})\bar{p}(e) = g\bar{p}(e)$. This is just what the definition of alpha tells you just know what we equation 19.

Hence, we get a well-defined map from the quotient and that I am denoting by $f : E[\alpha] \to E'$. Now f commutes with the projection maps is obvious because of-- because of the definition here. So, it is a covering transformation from $E[\alpha]$ to E', that is the G covering ζ. Moreover, f is G-map also. By definition you see, $f[(g_1 g_2, e)]$ is $(g_1 g_2)\bar{p}(e)$. But that is same thing as $g_1(g_2\bar{p}(e))$ which is nothing but $g_1(f([g_2, e])$. So g_1 has come out. It just means that f is a G map between G coverings. Any G map is an isomorphism. That is what we have seen. So, therefore starting with E prime we looked at alpha and then we perform this E of alpha which is mu. And now we are showing that these equality to origin point. So, $\mu \circ \nu = Id$ ν first and then μ is identity. This is what we have got: $\mu \circ \nu = Id$.

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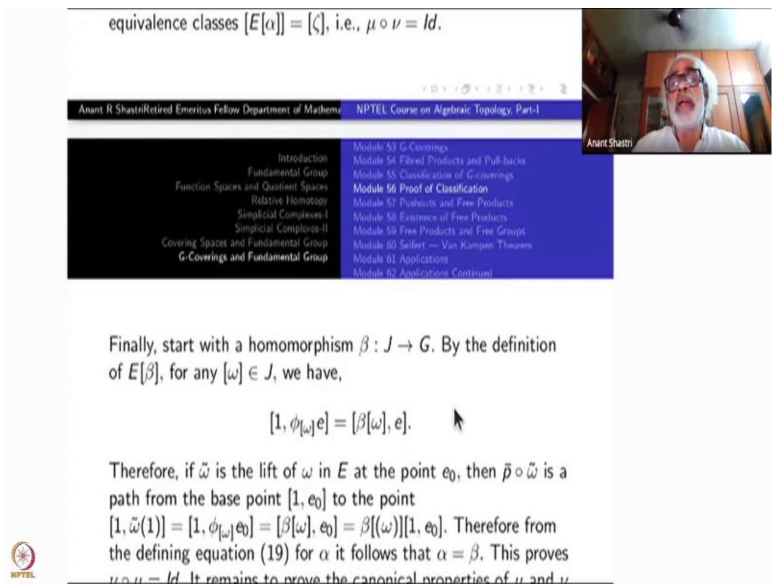

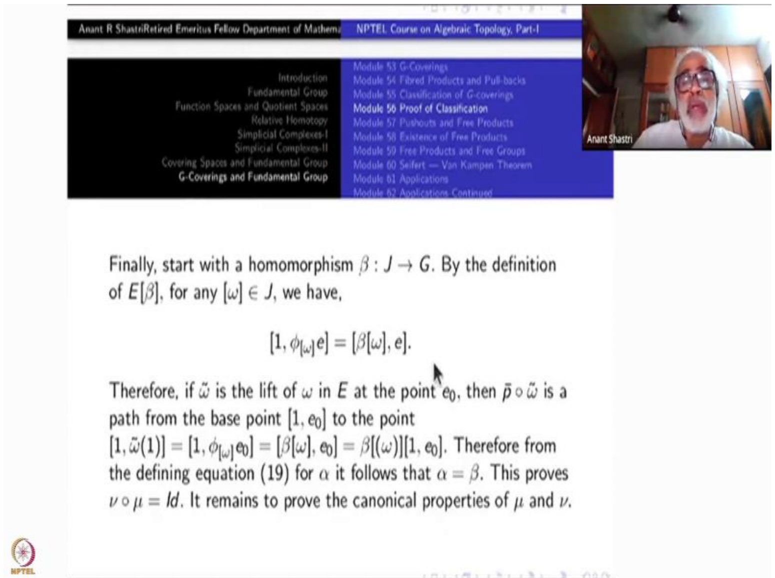

Now, let us look at it the other way round. Now, I start with a homomorphism instead of starting with a equivalence class of G or G covering that is the homomorphism $\beta : J \to G$. Then you perform the extension $E[\beta]$. By the very definition $E[\beta]$, is nothing but elements of the form $[1, \phi_{[\omega]}(e)] = [\beta([\omega]), e]$. This is for any $[\omega] \in J$. Therefore, if $\tilde{\omega}$ is the lift of ω in E at the point e_0, $\bar{p}(\tilde{\omega})$ is a path from the base point $[1, e_0]$ to $[1, \tilde{\omega}(1)]$.

Which is nothing but, by the definition, $[1, \phi_{[\omega]} e_0] = [\beta([\omega]), e_0] = \beta([\omega])[1, e_0]$. Therefore, by the defining equation 19 for alpha this means $\alpha = \beta$.

So, starting with beta you take E beta that is mu then the corresponding homomorphism is back to alpha itself whatever. So, this alpha must be this beta. So mu of nu is identity, $\nu \circ \mu = Id$. Remember that one way we get equality for the homomorphisms. Other way here we are only showing that the two G coverings are equivalent under f. They a may not be the same but they are equivalent under G maps. So, keep that in mind.

So, this shows that μ is bijection. The next thing is we have to verify that it is a canonical map canonical bijection what is the meanings of the canonical? I have to explain you and then prove it.

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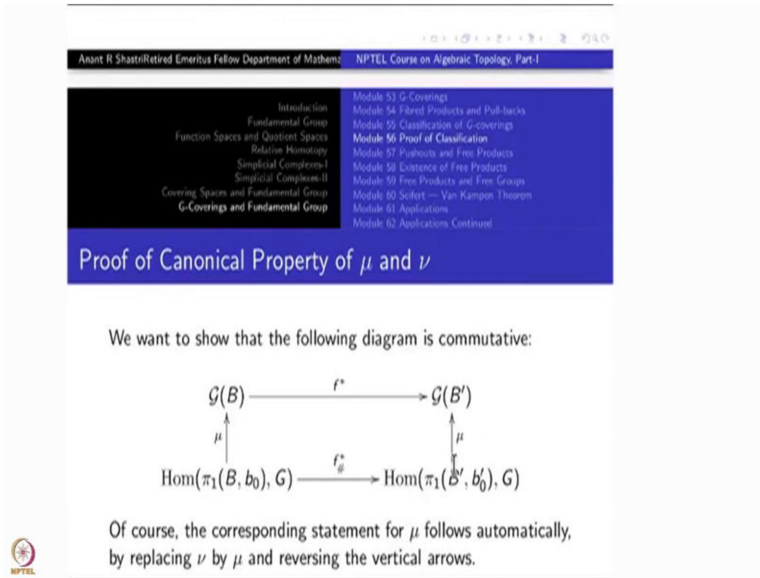

This is the meaning. Take any function $f : B' \to B$. Let us choose a base point $b_0' \in B'$ going to b_0 under f. That will induce a homomorphism $f_\# : \pi_1(B', b_0') \to \pi_1(B, b_0)$. I am going to denote it by $f_\# : J' \to J$. When you compose it with any homomorphism J to G what you get is a homorphism from J' to G. That is denoted by $f^*_\#$.

So, this is a homomorphism from Hom to Hom which under μ corresponds to a function from the set of all equivalence classes of G coverings over B to the set of all equivalence classes of G coverings over B'. So, whether you first take $f^*_\#$ and then go via μ or first take μ and go via f^* the effect is the same. f^* is what? It is the pullback of the G coverings over B to G covering over B'.

Or you first take $\nu[\zeta]$ here to come down here to a homomorphism $\alpha : J \to G$ and compose it with $f_\#$ to get an element of $\mathrm{Hom}(J', G)$ or you first take the pull back $f^*(\zeta)$ then take ν of that to get an element of $\mathrm{Hom}(J', G)$, the effect is the same.

So, this diagram commutes so this is totally independent of what f is you see the same f star you have to take though. Here homomorphisms at pi 1 level are induced by f. This oe is induced by f again here. It is a pullback of the bundle pullback of the coverings. You can put ν from $\mathcal{G}(B)$ to down here by reversing the arrow here. Because that is just the inverse of the if you reverse the arrows here and put ν hereand here, that will be also commutative, provided you prove this one.

Alternatively, if you prove that then this will get prove because we mu and nu are inverses of each other. So, the whole point is we have defined a mu not by one single set Hom. We have defined it for all possible different if we change a B if we change G the domain of mu will change. But the change if you corresponds under f under Hom under a continuous function then there is a commutativity of the diagrams this is the whole idea.

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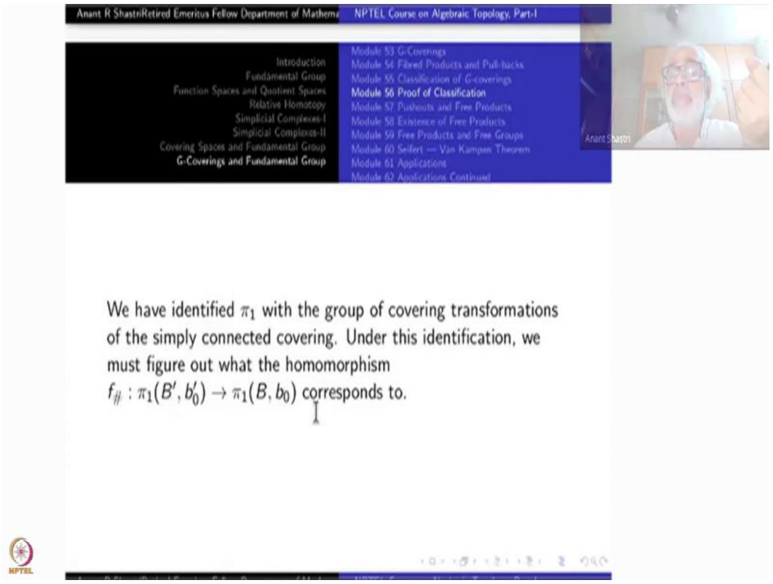

So, once again the important thing here is to understand what is happening at the universal cover level, namely, at the simply connected cover level under this f check your map from f from B prime to b it induce a homomorphisms at the pi 1 level. It also gives you a map from E prime to E the lift of f. Because E prime is simply connected I am thinking simply connected covering E prime over B prime. That f check has f bar whatever the lift of f.

We will have something to say about the group g p, g p prime on one side and g p on the other side that is the thing that you have to understand. Once you understand that the canonical property falls out very quickly. So, once again you have to identify π_1 with the group of covering transformations to understand this again and again. So, under this identification we must figure out what the homomorphism f check from pi 1 of B prime to pi 1 of B b naught corresponds to.

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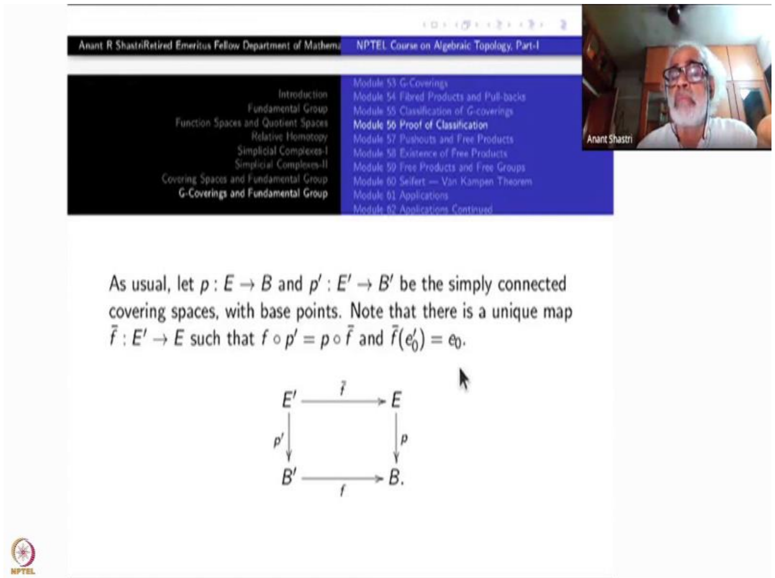

So, as usual let $p' : E' \to B', pE \to B$ these are simply connected covering spaces with base point preserving maps etc. Note that there is a unique map $\bar{f} : E' \to E$ such that $f \circ p' = p \circ \bar{f}$. I am just taking this composite this, this is simply connected so it is a lift. If you specify the base points, the lift is unique. That is all has been used. So, $\bar{f}(e_0') = e_0$.

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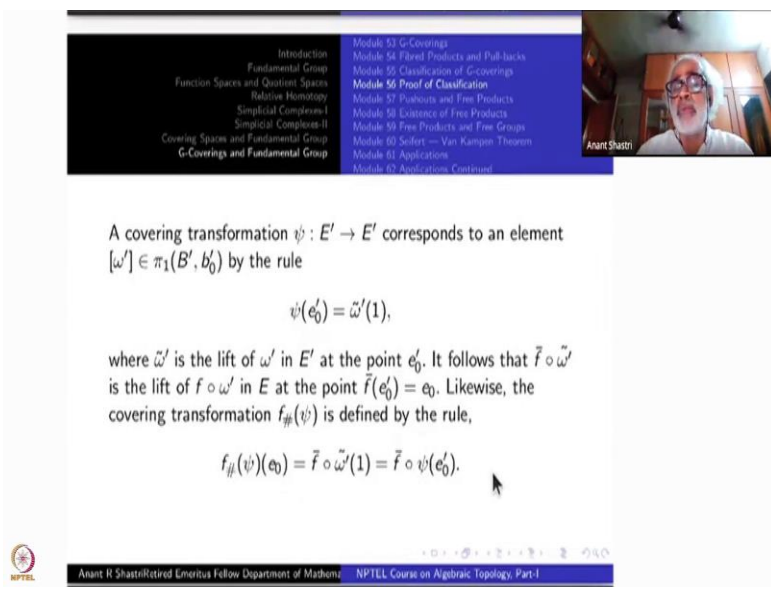

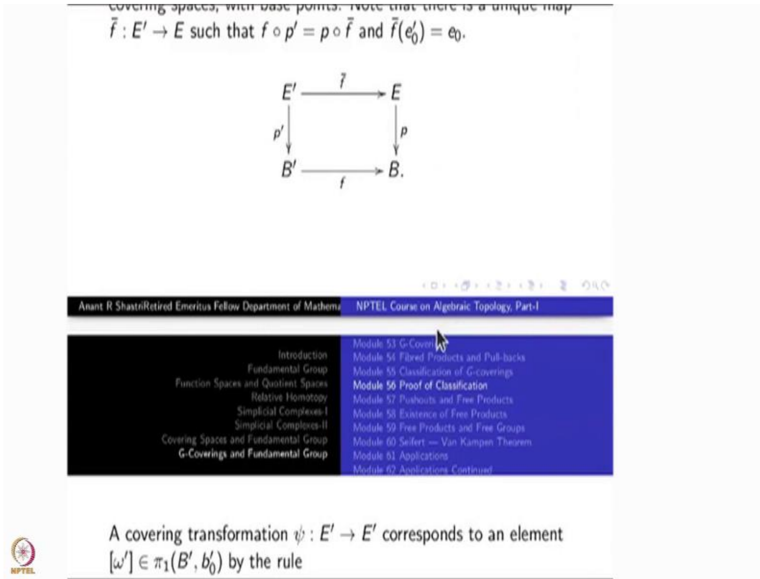

Now, a covering transformation $\psi : E' \to E'$ corresponds to an element $[\omega'] \in \pi_1(B', b_0')$. What is the rule? The rule is: the endpoint $\tilde{\omega}'(1)$ of the lift of ω' at e_0' is equal to $\psi(e_0')$. Same story for E to E also which we have being using.

Now, the point is that if you take a loop ω' at b_0' in B', $f_\#([\omega'] = [f \circ \omega]$ at the π_1 level. That is the way f check is defined at pi 1 level by just take composition with f. So, it follows that $\bar{f} \circ \tilde{\omega}'$ is equla to the lift of $f \circ \omega'$ at the point e_0 in E. If you first lift in E' and then push it to E via $\bar{f}$ is the same as first push the loop into B via f and then lift in E by the uniqueness of lifts. So, that will tell you what is the relation between these covering transformations under $\bar{f}$

So, it follows that we have $f_\#(\psi)(e_0)$, where ψ is an element of the group of covering transformations of p' being now thought of as an element of $\pi_1(B', b_0')$, is equal to $\bar{f}(\omega'(1))$ by definition, but this is turn is same thing as $\bar{f} \circ \psi(e_0')$. So, this covering is defined by this is also another lift. So, once you fix the effect of base point it is uniquely defined that is what I am telling.

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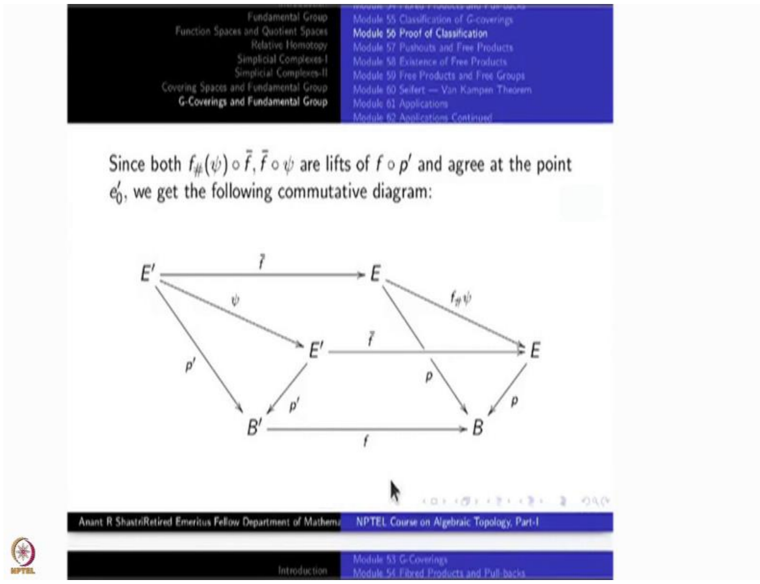

Since both $\bar{f} \circ \psi, f_{\#}\psi \circ \bar{f}$ are lifts of the same map $f \circ p'$ through p, and they agree at one point, they are equal. So, this is the commutative diagram that we arrive at. Your covering transformation here there will be covering transformation there psi of f of f check of psi. If this corresponds to the fundamental group element f check of that will be the this element that is why have it written f check of psi that the covering transformations is precisely there.

So, the f check can be thought of as a map from the covering transformation $f_{\#} : G(p') \to G(p)$ under these identifications. So that is also a canonical property. Start with any continuous function here base point preserving then these identifications of universal covering and they are the covering transformations in the fundamental row that itself is canonical. So, this picture tells you that.

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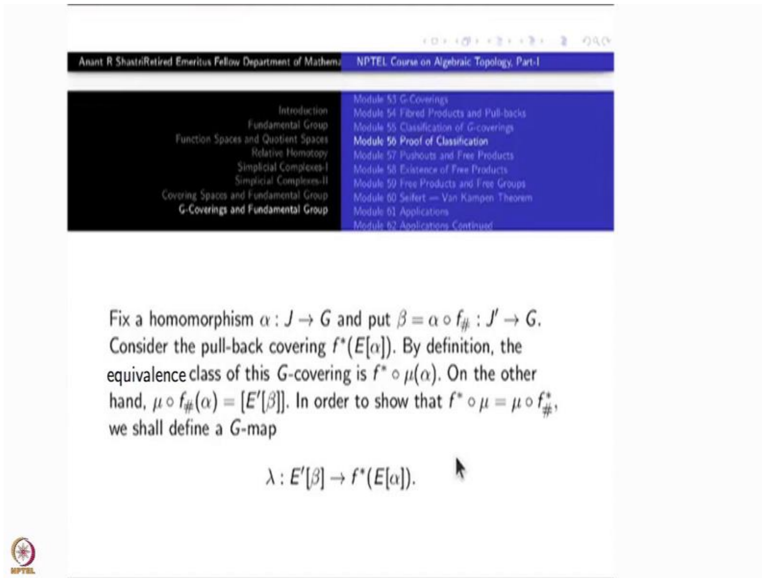

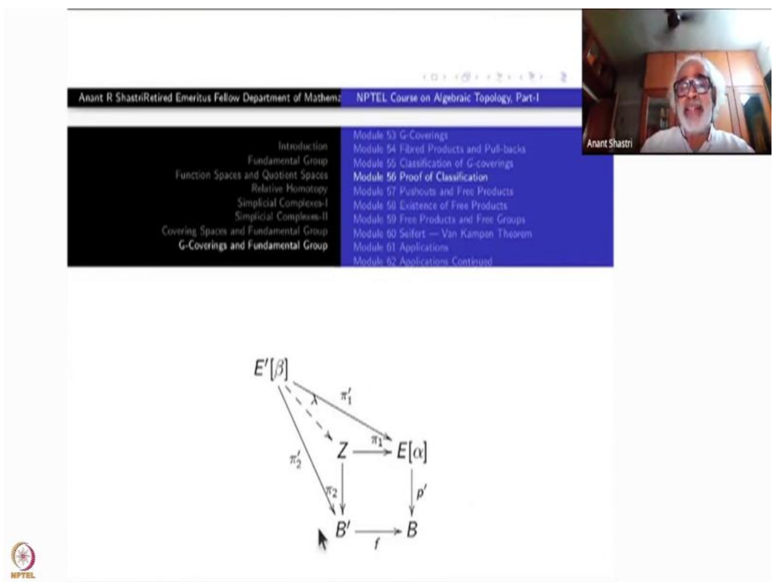

Now, we can complete the proof of canonical property, viz. Commutativity of this square. Start with a homomorphism from $\alpha : J \to G$, compose it with $f_\#$ to get a homomorphism J' to G, remember $f_\# : J' \to J$. So, let us denote it by $\beta = f_\# \circ \alpha$.

Now, consider the pullback covering $f^*(E[\alpha])$. By definition, the equivalence class of this one is $f^*\mu(\alpha)$. On the other hand, you can take f check of alpha that is beta. And then for the extension that is $\mu(\beta) = [E[\beta]]$. What you have to show is that these two are in the same class namely this equivalence classes are the same. That is what we have to do in order to show $f^* \circ \mu = \mu \circ f^*_\#$.

We should define a map $\lambda : E'[\beta] \to f^*(E[\alpha])$, here which is a G map. That would mean that the equivalence class of these two G coverings are the same. This lambda should be a G map that is all we want. Alright? a continuous function which which respects the G actions on either side. Then you are done.

So, to define the map λ from here to here, I will use the universal property of the pullback $f^*(E[\alpha])$. Remember what $f^*(E[\alpha])$ is: $E[\alpha]$ is a G covering right? It is a covering projection, there is a projection map from $E[\alpha]$ to B. We will denote by p'. This is our notation anyway f is a map from $B' \to B$. WE are taking the pullback. Let us denote the total space by Z. We know that it is a subspace of $E[\alpha] \times B'$ and so on. That we know already. There are maps π_1, π_2 projection maps here. So, this map $\pi_2 : Z \to B'$ is the pullback G- cover that we want to show tis equivalent to $E'[\beta]$. That means I must find a λ here such that this triangle is commutative and that this lambda must be a G-map. There is a unique way, there is an easy way of doing all this, using the universal property Z.

What I have to do is to get a map from here to $E[\alpha]$ and another here to B' such that when you compose with p' and f respectively, and come to B, they are the same. This compose this one is same to thing as this composite this map here. This map is already given to you do want this one to fit here. Will choose this one so that this composite this is this one, this one that is all. What is this map? Well if you just look at what are those elements of $E'[\beta]$, this map comes automatically.

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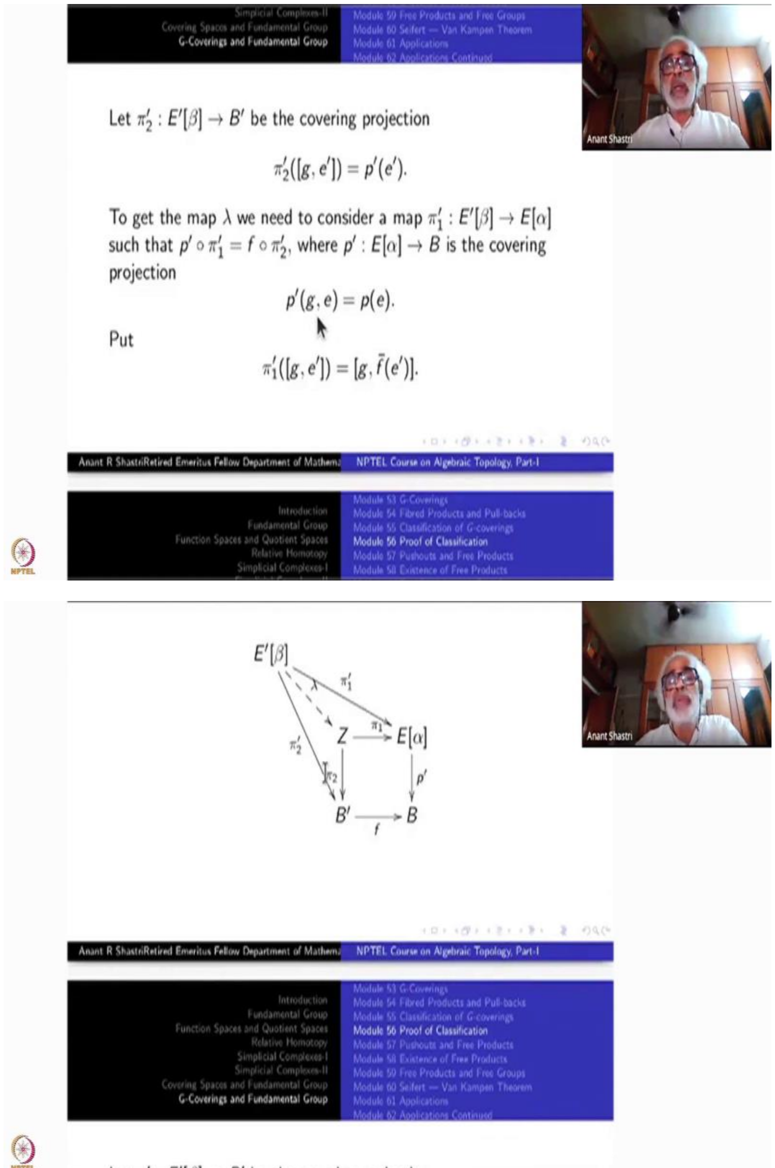

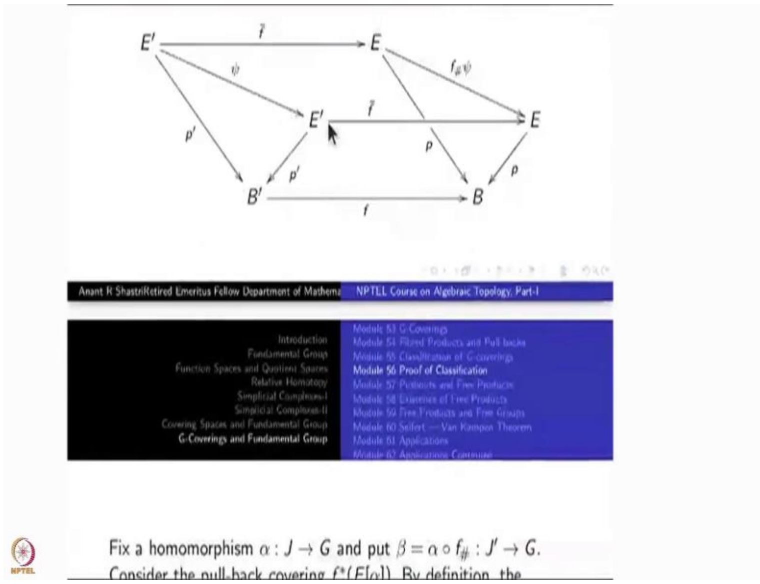

So let $\pi_2' : E'[\beta] \to B'$ be the covering projection defining $E'[\beta]$. What is it? It is, by very definition $\pi_2'([g, e'] = p'(e')$. Sorry. There is some overlapping of notation.

We know that p' was used for covering projection $E' \to B'$. So, maybe I should change the notation for the covering $E[\alpha] \to B$ to $q : E[\alpha] \to B$ or some other thing. there is a typo here. But this p' is the canonical notation. $p : E \to B$ and $p' : E' \to B'$. So, that is a nice notation let us not change that one. So, to get a map lambda, we need to define a map $\pi_1 : E'[\beta] \to E[\alpha]$ such that $q \circ \pi_1' = f \circ \pi_2'$.

So, we recall what the covering projection $q : E[\alpha] \to B$ is. What is the definition? $q[g, e] = p(e)$. This is the definition of q. So, I just take $\pi_1'[g, e'] = \bar{f}(e')$, do not disturb the G-action!

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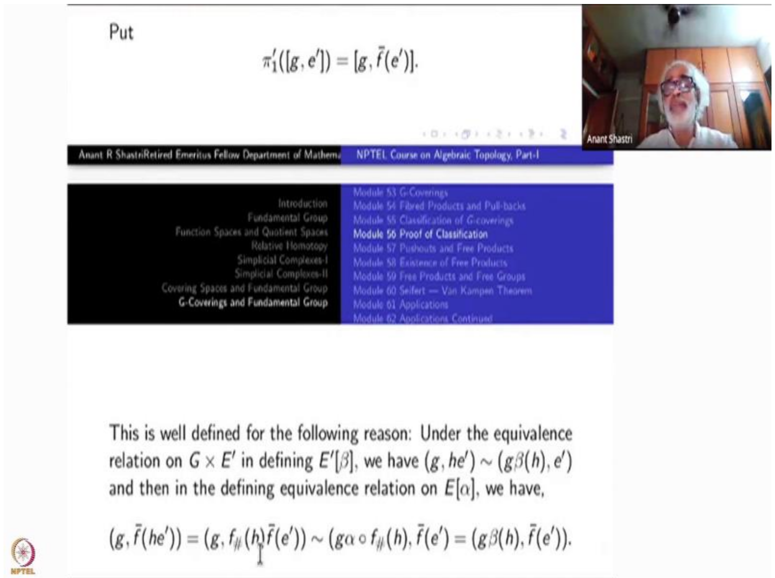

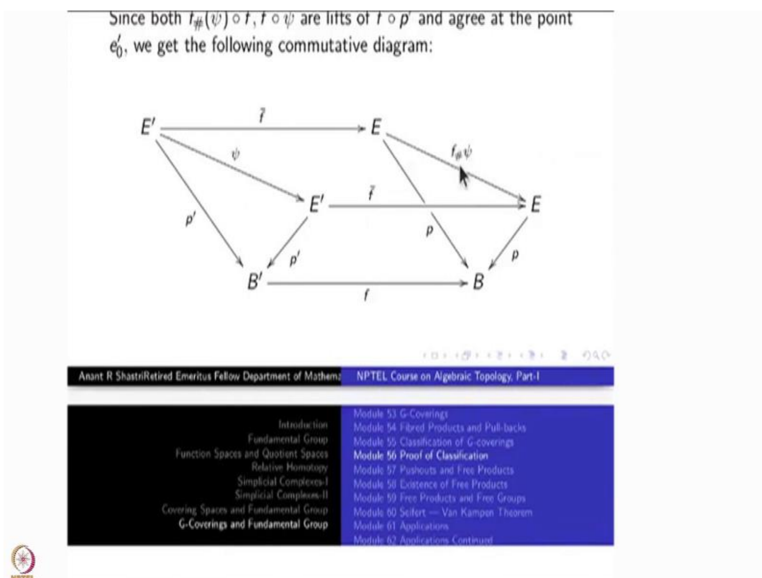

Why this is well defined? If there is an equivalence class, here why all of the elements of that class go to the same class under π_1'? That is what you have to verify. So, that is what I have checked. Under the equivalence relation on $G \times E'$, remember this $E'[\beta]$, the extension of the covering $p' : E' \to B'$, by the homomorphism β. So, it is a quotient of $G \times E'$. What is the relation? Relation is $(g, he') \sim (g\beta(h), e')$, this h is transferred to the other side with a beta h g of beta h time e prime. So, I have to show that these two elements go to the same element under π_1'. But what is the defining relations here on in this side in $E[\alpha]$? We have $(g, \bar{f}(he')) = (g, f_\#(h)\bar{f}(e')) \sim (g\alpha \circ f_\#(h), \bar{f}(e')) = (g\beta(h), \bar{f}(e'))$.

Therefore $\pi_1'(g, he') = (g, \bar{f}(he')) \sim (g\beta(h), \bar{f}(e')) = \pi_1'(g\beta(h), e')$. Therefore, π_1' is well defined.

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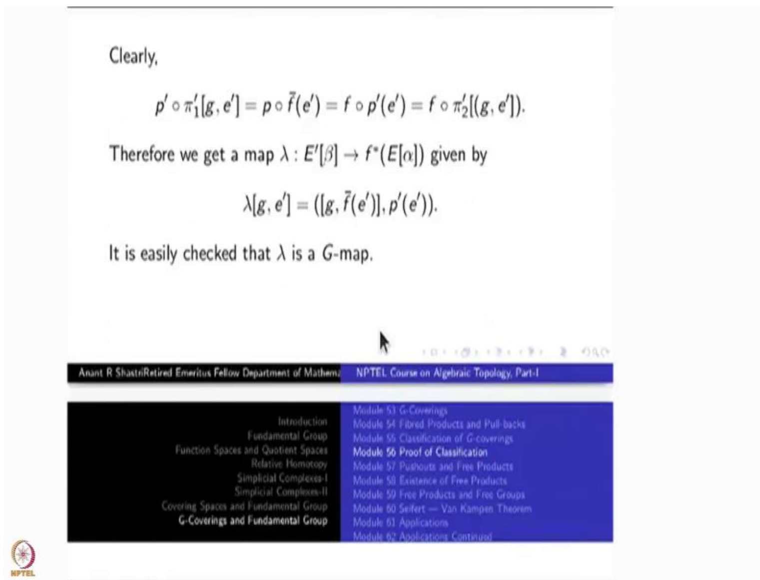

It is easy to say that $q \circ \pi_1' = f \circ \pi_2'$. You compose first with with q and the second one with f you the two are equal. Why I am writing p' here again? Change it to q. $q \circ \pi_1'(g, e')$ is by definition $p \circ \bar{f}(e') = f(e')$ which is clearly equal to $f \circ \pi_2'(g, e')$.

Therefore, we get a unique map lambda here and you know how it is define. It is define $\lambda = (\pi_1', \pi_2')$ the ordered pair of the two functions. So, that is lambda. So, $\lambda([g, e'])$ has its first coordinate is just $[g, \bar{f}(e')]$ and the second coordinate is $p'(e')$.

Thus, this λ is a G-map is very obvious because if g 1 g 2 if you put here. So, that is g 1 g 2 will come here nothing happens here. Then g 1, g 2 g 1 will come out here this is same thing is as you can put it all the way on set by the definition of this action. Action on the pullback. So, it is a G map.

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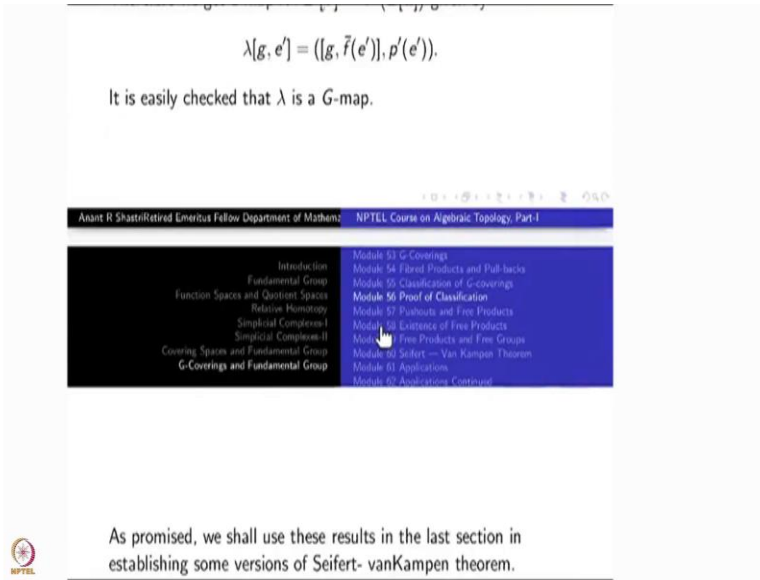

So, this completes the prof of the entire theorem. As promised, we shall use this one in the final result of various forms of Van-Kampen's Theorem. Thank you.