Introduction to Algebraic Topology (Part-1)
Professor. Anant R. Shastri
Indian Institute of Technology, Bombay
Lecture 55
Classification of G-coverings

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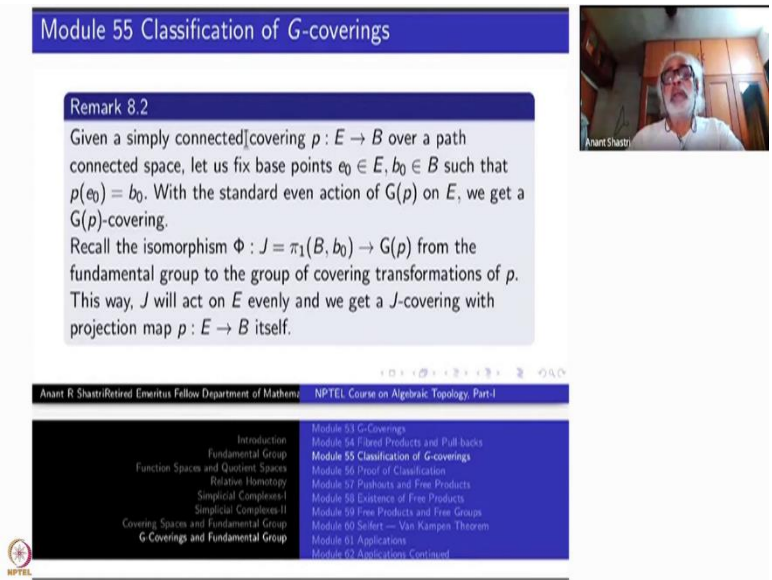

So, today we begin the classification of G-coverings. It will take some time before we complete it, complete things will not come today. The guiding principle here, is the simply connected covering of the base space B. That just means that we are first of all assuming that simply connect to covering exists.

So, B is a path connected space, $p : E \to B$ is a simply connected covering. So once for all this notation will be fixed. We are going to fix a base point $b_0 \in B$ and $e_0 \in E$ above that, that means, $p(e_0) = b_0$. Then this covering becomes $G(p)$ covering, where $G(p)$ is the Galois group of p, which is identified with the fundamental group $\pi_1(B, b_0)$.

Let us just recall how this identification is done. I am going to use a simpler notation

$J = \pi_1(B, b_0)$. Then there is this isomorphism $\phi : J \to G(p)$, where $G(p)$ is the Galois group, group of all covering transformations of p. How it is done? Recall this. Starting with a loop at b_0 lift it to a path in E at the point e_0, and look at its endpoint, endpoint is in the same fibre as e_0. Therefore, there is a unique covering transformation which takes e_0 to the endpoint of this path. So each path homotopy class of ω is sent to this covering transformation, this assignment is an isomorphism. This is what we have approved earlier.

And we know that under covering transformation, (this is the action of covering transformation, which is an an even action,) the quotient space is precisely B and quotient map is precisely p. These things we have already seen, while studying the universal covering or simply connected covering and so on.

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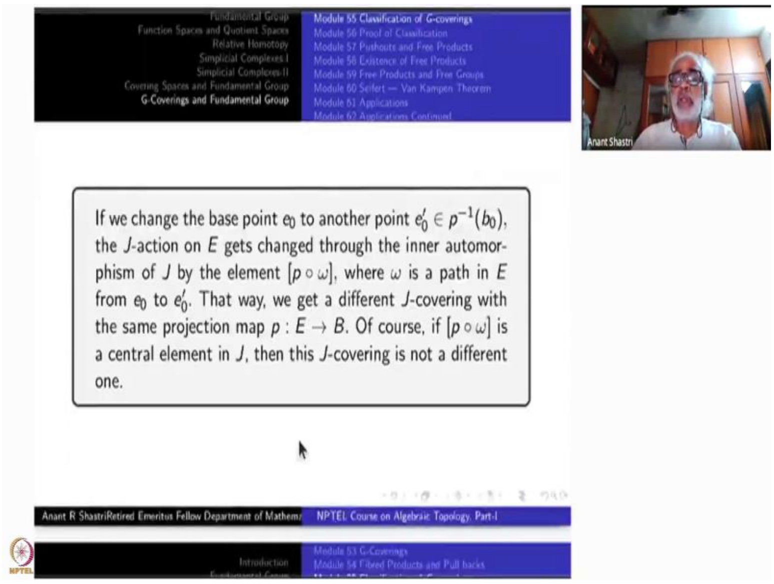

If we change the base point e_0 to some other point e_0' belonging to the same fibre, then the J-action on E gets changed and because this time you are lifting the loops not at but at e_0'. So, corresponding change is just got by taking any path ω from e_0 to e_0' inside E, i.e., $p \circ \omega$ will become a loop in B, at b_0. So, the entire action gets conjugated by this element and the inner conjugation is an automorphism. So, the new action is nothing but the one got by an automorphic change--- that is what we have studied last time.

On the other hand, if this $p \circ \omega$ represents a central element, this ω is a path inside E, whereas $p \circ \omega$ is a loop in B at b_0 and so represents an element of J. So, if this is a central element in J, then the conjugation is trivial. In that case, the covering G covering structure does not change. Otherwise the covering G coverage structure will change.

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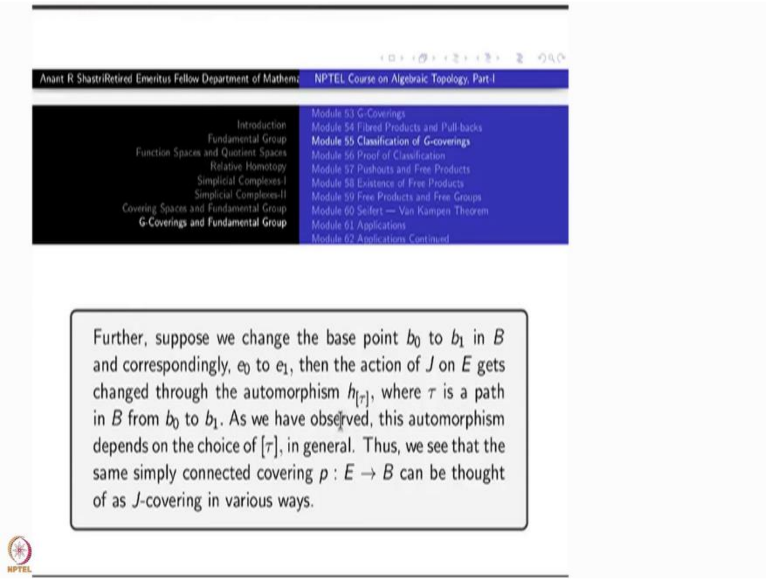

Suppose now, instead of just changing the base point at the top level, we change it at the bottom level, b_0 to b_1 both inside B. Then the corresponding expression for J itself will be different, it is no longer J but some other group which is again isomorphic to J. It is $\pi_1(B, b_1)$. And what is the automorphism here? It is isomorphism given by $h_{[\tau]}$. Here this τ is the path in B from b_0 to b_1.

So, we are conjugating by this τ which is a path, it is not a loop, so it is not an inner conjugation. So, under this conjugation, there is some automorphism of this group. So, the the G action will be different, once again this new action corresponds to some automorphism that is what we have seen.

So, there are all these change of base points etc, all these things are taken care in the concept of G covering, they may give you different G covering and theorems that we studied in the two modules before that tells you what exactly the changes can be, namely all the… ones the covering projection is the same, then the G actions are different only by an automorphism, so this is the theorem that we have already studied.

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Now, in the classification, this becomes our central idea and this idea get extended.

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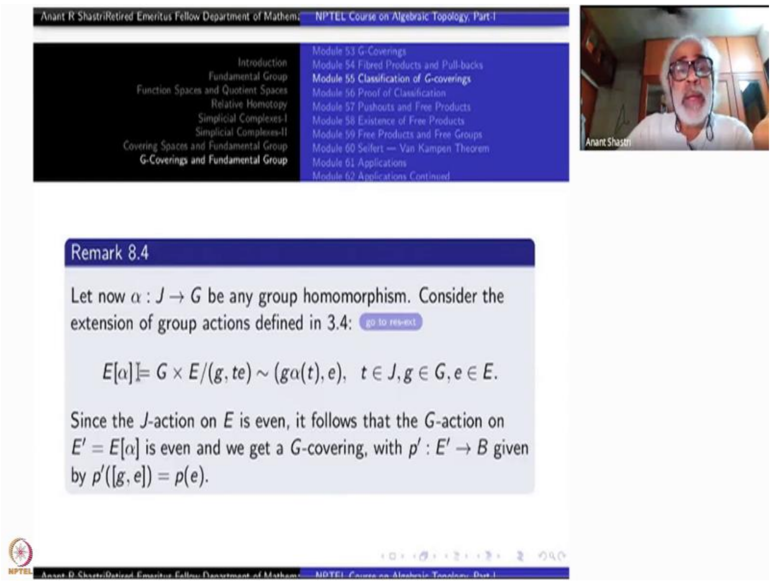

Now, let us go back to another construction of G coverings. Long back, we introduced what is called as extension of J-actions. So, we use that one, starting with the homomorphism, from this fundamental group J to any other group G, take some homomorphism α, group homomorphism. Extend the J action on $E[\alpha]$ (where E is the simply connected covering, that is fixed), to a G action by looking at this What is the definition of $E[\alpha]$? I am recalling it. It is $G \times E$, the total space but now we are going to have a quotient here, namely, by the equivalence relation $(g, te) \sim (g\alpha(t), e)$, where g is in G and t is in J. Remember α is a homomorphism, $\alpha(t) \in G$. So, this g and this $\alpha(t)$ are combined into an element of G-- there is a group operation on G.

So, this one element of g comma e, this is an equivalence relation and you know that modulo this equivalence relation, that is the space E alpha. So, this was the definition of E alpha. On E alpha, G will act from the first factor here. So, it becomes a G space. There is a G action on this quotient space now. So, we shall denote it by $E' = E[\alpha]$. Because the action of J on E is even, it will follow easily that the G action on E' is also even.

Therefore, we get a G covering again, $p' : E' \to B$. If you take the quotient of E' the quotient by the G action, then you get the space B, the original space B, $p' : E' \to B$ is given by $p'[g, e] = p(e)$.

The first factor totally disappears why? Because first you quotiented out by alpha, whatever overflow is still there, then G action will further take away all this, everything is identified. So, this is G covering starting with a J covering we are converted into a G covering via this map alpha.

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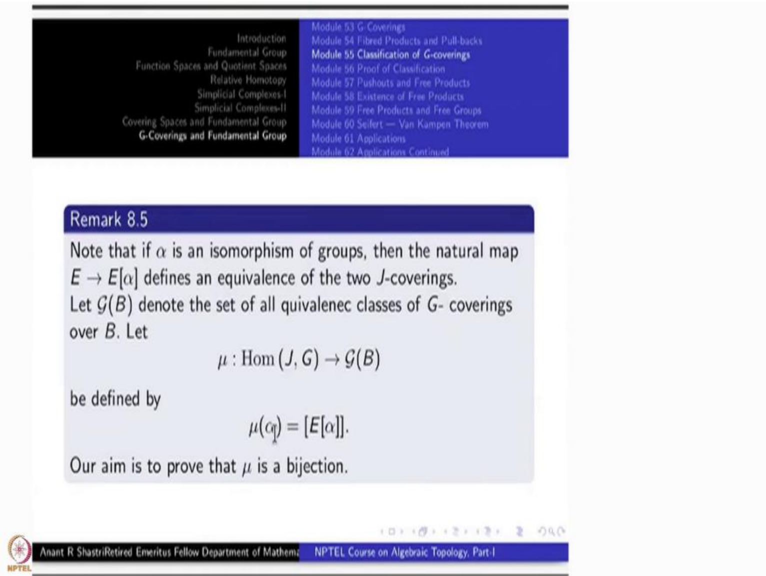

If alpha were an isomorphism, then you can treat this as a J covering itself. Both of them are J covering the usual map here, $e \mapsto [1, e]$, there is this canonical map, that will be a J map that becomes a J isomorphism. So, you understand this covering, this construction I have just recalled it, once we have extension of G action, at least like we have today. That is all. These things are rigorously stated under exercises.

Now, let this $\mathcal{G}(B)$ denote the set of all equivalence classes of G coverings over B, G is a group this $\mathcal{G}(B)$ denotes all the equivalence classes of G coverings. It has nothing to do with the fundamental group of B. Those would have been J coverings, that I would have denoted by $\mathcal{J}$.

Now, suppose you define a set theoretic map $\mu : \mathrm{Hom}(J, G) \to \mathcal{G}(B)$ by taking α to the equivalence class of the G-covering $E[\alpha]$. So, we get a set theoretic map here. Our aim is to prove that this itself is the bijection.

Starting with a homomorphism, we get a G covering. Now starting with a G covering, you must get a homomorphism such that, if you use that homomorphism and come back, you must get the same equivalence class and vice versa. So, this must be a bijection of equivalence classes here. Here just the set of all homomorphism, there is no equivalence classes here. Every homomorphism stands on its own.

So, how do you show that this is a bijection? By precisely constructing its inverse, directly constructing universe. Therefore, this proof is going to be completely constructive. We have constructed the map now and instead of saying there is a bijection, we are giving you its inverse. Of course we are giving the map and then showing that it is bijection, so it is a constructive proof.

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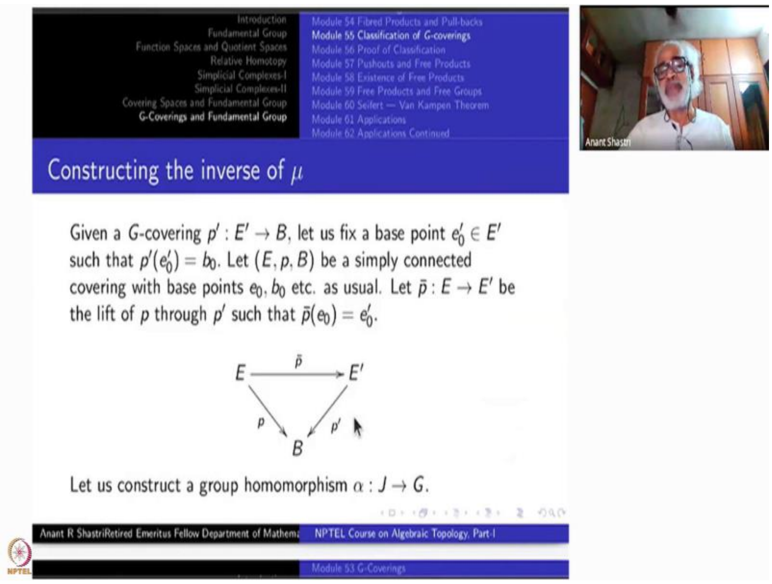

Constructing the universe of μ, let us do that. Given a G covering, $p' : E' \to B$ representing one class, I have to construct a homomorphism. Let us fix a base point $e_0' \in E'$

such that $p'(e'_0) = b_0$, which you can alsways do. So, (E, p, B) is the standard simply connected covering, with e_0, b_0 as ususal.

Since E is a simply connected covering, and this other one is a covering, by the lifting criteria, this p can be lifted. So, there will be a map $\bar{p} : E \to E'$ such that $p' \circ \bar{p} = p.$ If I specify where the base point e_0 goes then this p bar is uniquely defined. So, what I do is I demand p bar of e naught is equal to e naught prime. So, this p bar is completely determined by this property, namely, it is a lift of p, and it takes e naught to e naught prime.

So, these notations I am going to fix up now. Remember, this is some arbitrary G covering, this is the simply connected covering over p. I told you this is going to be our guide. So, out of this we will now a construct a homomorphism. From J, namely Gp of this one, to Gp of Gp prime of that. This is J equal to Gp, group of coverings transformations of p and G is a group of covering transformation in p prime. That is what we are going to construct now.

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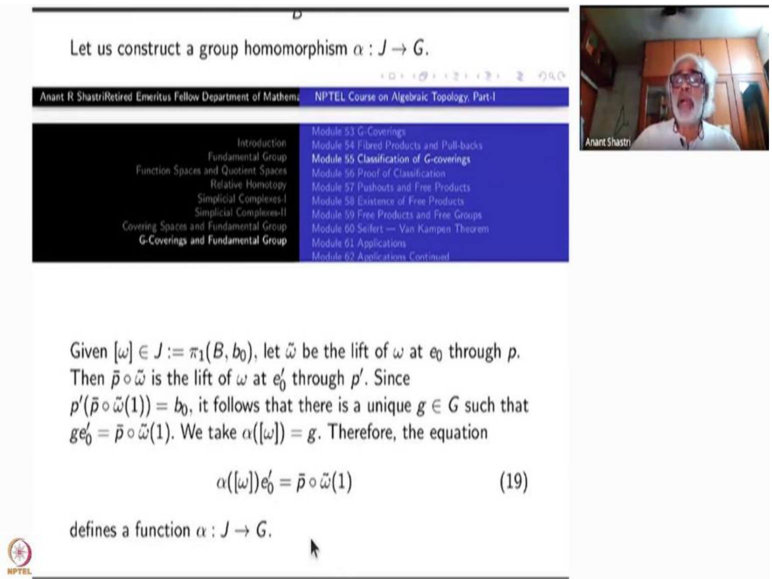

So, constructing this is obvious now. Starting with a class $[\omega]$ in J, which you can represent by a loop, lift this loop to a path $\tilde{\omega}$ at e naught through p in E. Take p bar of that lifted path, that will be a path at e naught prime. And if you take p prime of that, that will give you back omega. Therefore, you can think of this as a lift . There is a loop here. I do not know whether I can lift it here, but I can lift it here, and then I take the image of that, that will be a lift. That is why I am now doing this.

So, p bar composite $\tilde{\omega}$ is a lift of ω through p' at the base point e_0'. Its end point is mapped to b naught, it follows that the end point of $\bar{p}\circ\tilde{\omega}$ is in the same fibre. Therefore, there is a unique $g \in G$ such that $ge_0' = \bar{p}\circ\tilde{\omega}(1)$.

So, starting with omega we have got a element g, this g is element of capital G, the relation is precisely given by this equation (19), I am calling this g as alpha omega, alpha omega operating upon e naught prime gives you the endpoint of the lift of omega through p prime, which I read and write as p bar of omega tilda.

So, this alpha is the function from J to G, there is no ambiguity in the definition because the endpoints depend only on the homotropy class. Endpoints always the same, no matter which loop you take to represent the homotropy class.

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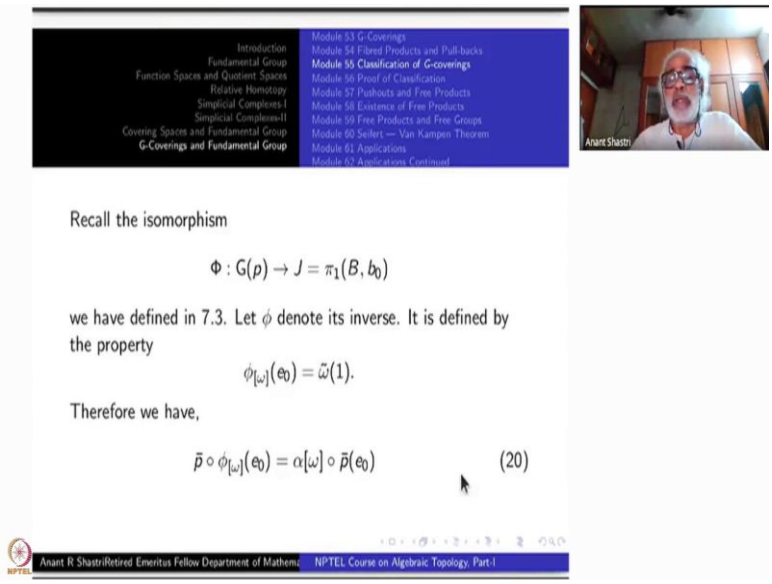

So, I am recalling the isomorphism $\Phi : G(p) \to J$ here. I recalled it just now. If ϕ is the inverse of Φ, then $\phi[\omega]$is the unique element in $G(p)$ which maps the base point e_0 to the end point of $\tilde{\omega}$. This is the correspondence of the inverse of that map.

So, if this is the case, you apply $\bar{p}$ on both sides, we get

$$\bar{p}(\phi([\omega])e_0) = \bar{p}(\tilde{\omega}(1)) = \alpha([\omega])\circ\bar{p}(e_0) = \alpha[\omega])e_0'.$$

So, this will tell you what is the relation between alpha and this phi. Phi of omega p bar of that is alpha omega p bar of e naught. This you can take as the definition of alpha, either this one or this one or this (19).

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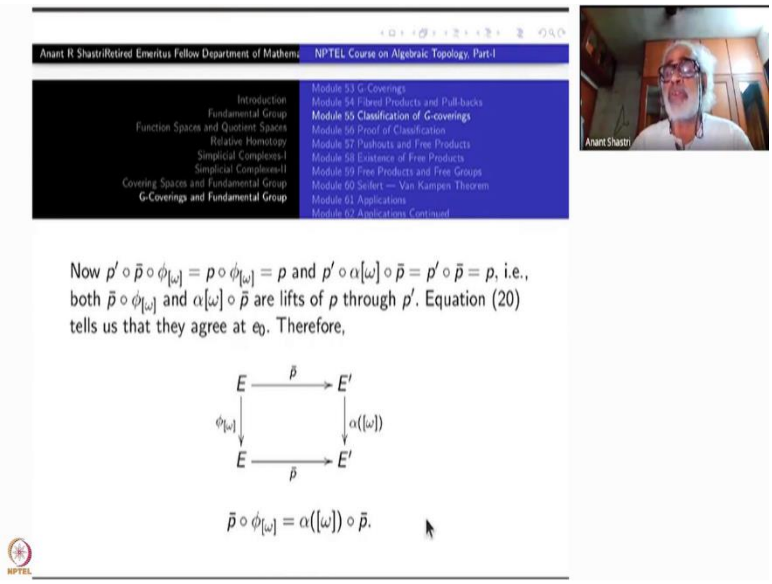

Now, you take p prime on both sides p prime of p bar of phi omega, what is it p prime p bar is always p, is p of phi omega, but phi omega is a covering transformation so, it is p. So, this is just p same thing if you take p prime of alpha omega of p bar p prime of alpha omega, alpha omega is an element of g p prime, p prime is a covering projection which same thing as p prime, p prime of p bar is p.

That means, these two are both lifts of p and they agreed one point because of this definition, at e naught they are equal what does it mean they are equal everywhere. So, what I get is this is to lift stitch one equation tells you they agree at e naught they are equal everywhere. Alpha omega composites p bar is the same thing as p bar composite p omega. So, this is the equation we have got, you see it is equation for the covering transformations and p bar and so on this is valid for all points of E.

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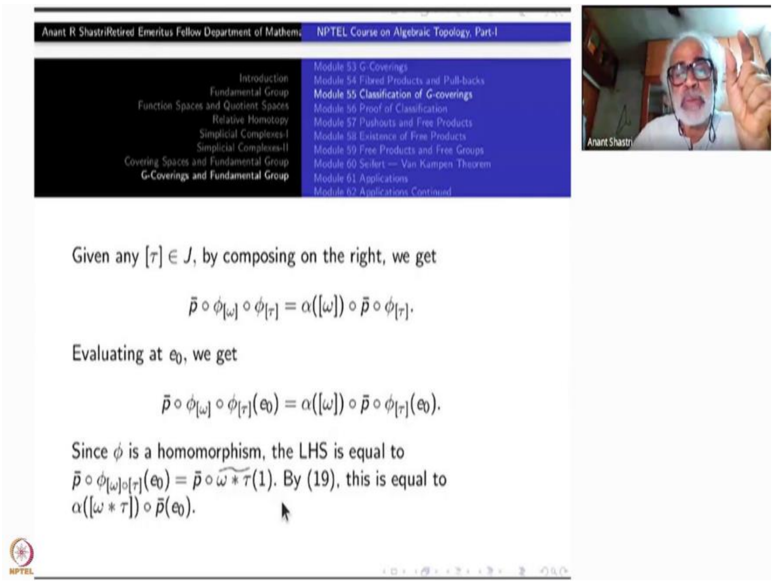

Now, given another element τ, let us say τ is an element of J, i.e., $[\tau] \in J$ you can compose on the right, i.e., starting with the equation $p \circ \phi[\omega] = \alpha([\omega]) \circ \bar{p}$, compose both sides on the right with $\phi[\tau])$. I get this equation, you evaluate it on e_0, you get this equation.

But ϕ is a homomorphism, $\phi([\omega]) \circ \phi([\tau]) = \phi([\omega * \tau])$ this is nothing but p bar of you lift omega star tau as you know this is loop composition lift it and take the endpoint. So, that is, by the definition in (19) this is nothing but alpha of this element now, alpha of omega star tau operating of p bar of e naught, or e naught prime. So, this is the LHS. what is RHS? This part is alpha tau, alpha tau of this one by very definition I just got.

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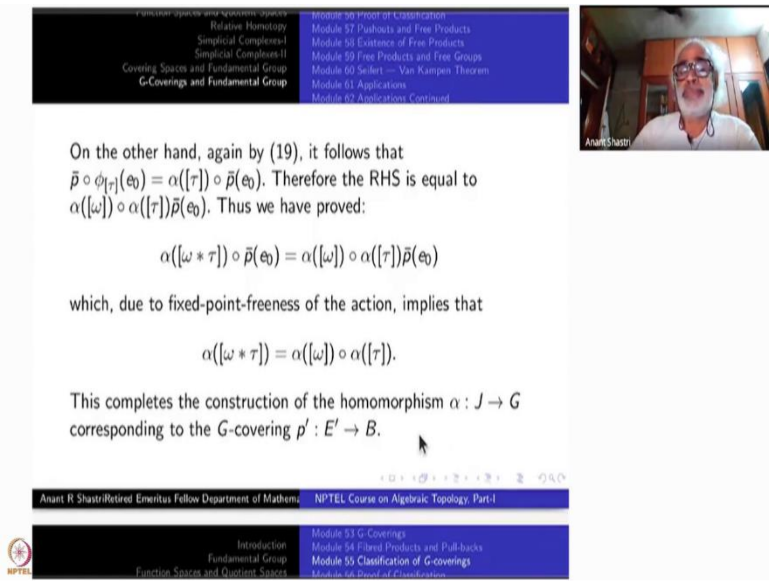

On the other hand, again by (19) p bar of phi tau is alpha tau of p bar of e naught. Then I have left multiplication alpha omega that whole thing I get alpha omega composite alpha tau, alpha is a homomorphism, that is what we want to prove, you do not know that yet, but this composite is equal p bar of e naught, one hand it is equal to this one here the other hand is equal to this. So, these two are equal means alpha omega star tau operating on p bar e naught is equal to alpha omega into alpha tau operating on p bar e naught.

Now, you use the fact that the actions are fixed-point-free. Therefore, if they agree at one point these elements must be the same, therefore alpha is a homomorphism, the proof is somewhat similar to what we have done earlier, due to fixed-point-freeness. So the construction of the homomorphism is over.

Corresponding to any G covering we have got a homomorphism, but what we want is corresponding to an equivalence class here there must be one homomorphism. So, what we have to show, we must show that this alpha is independent of the class for whole class whatever you choose you should get the same alpha, it is same for the entire class. So, that remains to be shown. Let us show that.

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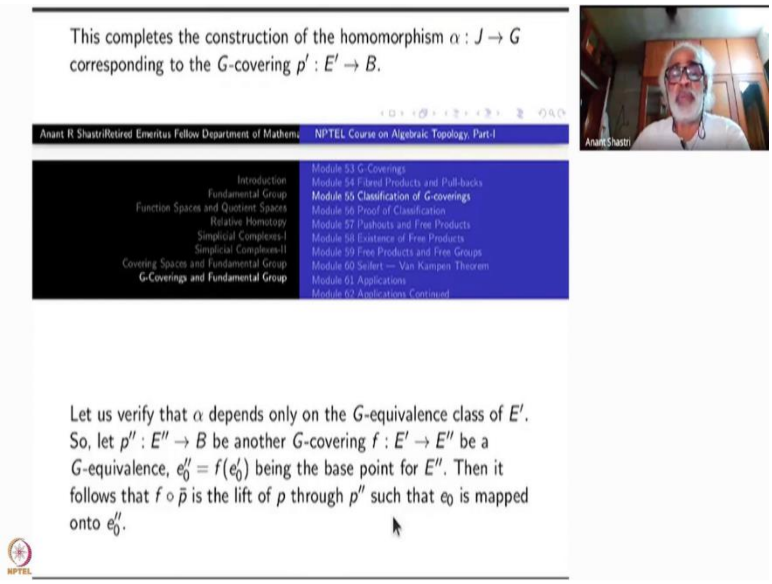

So, let us verify that alpha is independent of the G equivalence class. Suppose $p'' : E'' \to B$ is another G covering and $f : E \to E''$, G equivalence. G equivalence means that f is a G-map of the two coverings. So, you have a G map f from E prime to E double prime. I can assume that $e''_0 = f(e_0)$, this is the base point I am choosing, base points I have to choose. So, that is a base point for E''.

Then what happens, p bar is there you compose it with f that will play the role of p bar for the p double prime, remember we have used the notation p bar for all these from E to E prime. Now, you compose with f you get E to E double prime, this f composite p bar will play the role of p bar for E double prime. And it maps e naught to the base point, base point (())(28:09) base point.

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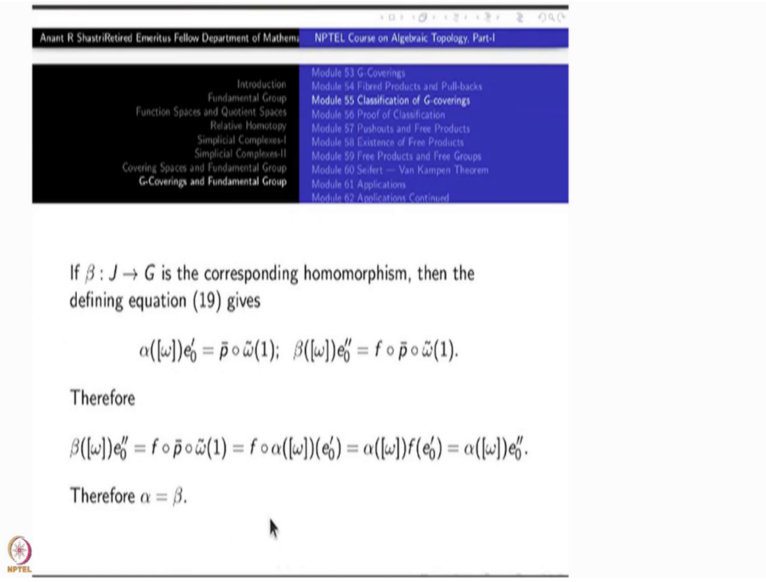

Now, let us assume that whatever you have constructed for E'' is β, i.e., $\beta : J \to G$ is the corresponding homomorphism for E''. I have to show that alpha is equal to beta. What is the equation for beta? By definition (19), what I have got? $\beta[\omega](e_0'') = f \circ \bar{p}(\tilde{\omega}(1))$.

Because $f \circ \bar{p} \circ \tilde{\omega}$ is the lift of omega E'', its endpoint defines beta omega. For alpha, inside E', we have the old equation $\alpha([\omega])(e_0') = \bar{p} \circ \tilde{\omega}(1)$. So, this is what we have in E' and this is what we have in E''. Therefore, $\beta[\omega](e_0'') = f(\alpha([\omega])(e_0'))$.

Now, use the fact that f is a G map and alpha is an element of G. So, this alpha omega comes out f of e naught prime, but what is f of e naught prime? It is e naught double prime. So, you have $\beta[\omega](e_0'') = \alpha[\omega](f(e_0')) = \alpha[\omega](e_0'')$ Again, by the G action which is even action, so fixed point free therefore, this alpha omega must be equal to beta omega. But this is true for any omega, therefore alpha equal to beta.

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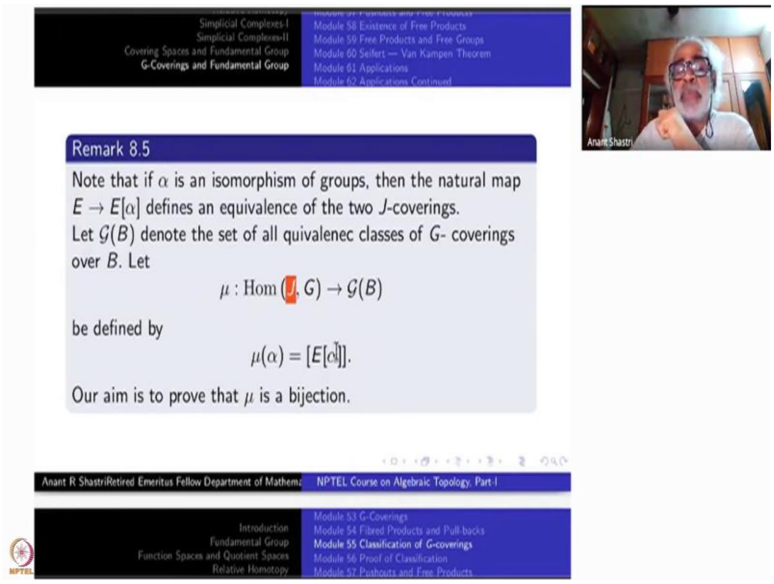

So, what we have done is that we have a map $\mu : \mathrm{Hom}(J,G) \to \mathcal{G}(B)$. Now, we have a map $\nu : \mathcal{G}(B) \to \mathrm{Hom}(J,G)$. We have constructed a map like this. What we want to show is that, mu and nu are inverses of each other.

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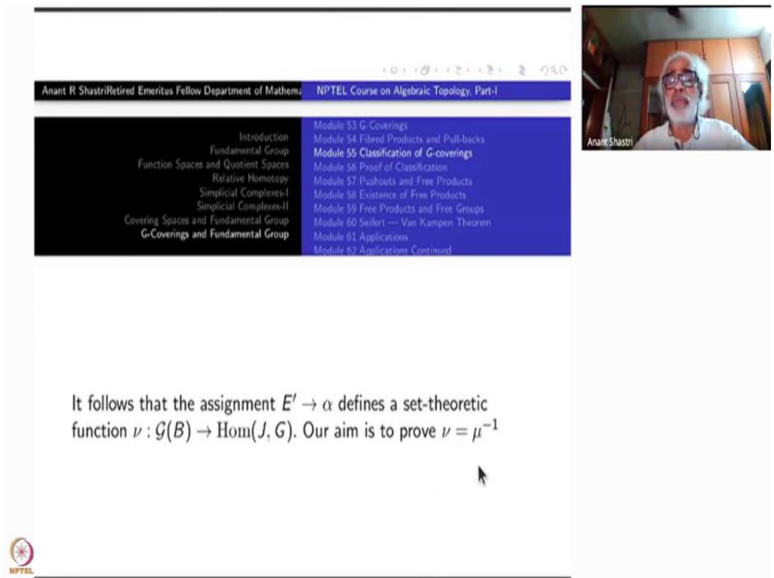

So, we have constructed this map, you want to say that these are inverses of each other, that we will do next time. Today, this enough. Thank you.