

Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Indian Institute of Technology Bombay
Lecture 54
Pull-backs

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The screenshot shows a video lecture interface. On the left is a table of contents with 'Module 54 Fibred Products and Pull-backs' highlighted. On the right is a video feed of Professor Anant R. Shastri. The main slide area contains the following text:

Before we go to the central problem in this chapter, viz., the classification of G -coverings, let us discuss a method of obtaining new G -coverings out of the old, this time, on different base spaces. Since this method involves a new concept, which, on its own, is useful elsewhere, let us study this in a little more generality.

The footer of the slide includes the NPTEL logo and the text: 'Anant R Shastri/Retired Emeritus Fellow Department of Mathem. NPTEL Course on Algebraic Topology, Part-I'.

Before we take up the central problem in this chapter of classification of G -coverings, I would like to present a method of obtaining new G -coverings out of the old ones, this time, by a change in the base itself. Since this method involves a new concept which is important on its own, not only in algebraic topology but elsewhere also, let us study this one a little more carefully and in a little more generality, not full generality.

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Definition 8.4
 Let $f_i : E_i \rightarrow B$ be any two maps. Their fibred product is defined to be a triple (Z, π_1, π_2) , where Z is a space and maps $\pi_i : Z \rightarrow E_i$ are such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$, and they satisfy the following universal property:
 Given any two maps $\pi'_i : Z' \rightarrow E_i$ such that $f_1 \circ \pi'_1 = f_2 \circ \pi'_2$, there is unique map $g : Z' \rightarrow Z$ such

We start with a base space B and take 2 functions $f_i : E_i \rightarrow B$. Their fibred product is defined to be a triple consisting of a topological space and two functions $\pi_i : Z \rightarrow E_i, i = 1, 2$, with the condition that $f_1 \circ \pi_1 = f_2 \circ \pi_2$. So, the composite map from Z to E_1 to B or Z to E_2 to B , they are the same and this entire thing satisfying the so-called universal property.

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So, this is the picture, Z is here, $E_1 E_2$ are here, B is here ok. So, f_1 and f_2 are functions into B . Then π_1 and π_2 respectively are functions into E_1 and E_2 respectively. If you have another

commutative diagram like this, Z prime, π_1 prime, π_2 prime respectively maps into E_1 to E_2 such that when you compose f_2 here and f_1 there they are the same $f_1 \circ \pi_1' = f_2 \circ \pi_2'$ then we have a map here such and two commutative triangles.


This is the conclusion of the definition, there exist a map $g : Z' \rightarrow Z$, a unique map g , such that the whole diagram is commutative, namely $\pi_1 \circ g = \pi_1'$, $\pi_2 \circ g = \pi_2'$. For every Z' and pairs of maps which satisfies this property, there must be a unique map like this. But this is a universal property of this commutative square here. Then this Z together with these two maps is called the fibred product of f_1 and f_2 .

But this is the definition of the fibred product of two functions taking values in the same space B . The uniqueness of such a triple (Z, π_1, π_2) , $\pi_1 : Z \rightarrow E_1$, $\pi_2 : Z \rightarrow E_2$, up to a homeomorphism, follows easily by this universal property. Namely, suppose $(Z'; \pi_1', \pi_2')$ is another such triple which also has the same universal property, then it will admit a map from $Z \rightarrow Z'$ in the reverse direction here, again giving a similar diagram, all these diagrams are commutative. Then what you have is g prime let us say, g' going from Z to Z' , and coming back by g that will be another map from Z to Z itself fitting this diagram. Identity map fits this diagram from Z to Z also, So, there would be two of them. By the uniqueness property, this map composite map must be identity, $Id = g \circ g' : Z \rightarrow Z$. That means what? By the same argument, we get $Id_{Z'} = g' \circ g : Z' \rightarrow Z'$, that is $g : (Z', \pi_1', \pi_2') \rightarrow (Z, \pi_1, \pi_2)$ is a homomorphism. This is typical of all universal properties, definition given by universal properties. Existence of such objects is not guaranteed but uniqueness is guaranteed. So, let us look at the existence, the existence here is very easy compared to many such existence theorems.

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Indeed, the uniqueness of the triple (Z, π_1, π_2) upto homeomorphism follows easily as usual. Here is the existence by construction: Take

$$Z = \{(e_1, e_2) \in E_1 \times E_2 : f_1(e_1) = f_2(e_2)\},$$

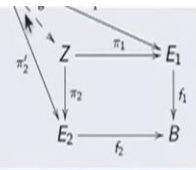
and $\pi_i : Z \rightarrow E_i$ be the restriction of the projections, $i = 1, 2$.

So, for the existence, what I do is I define Z to be the subspace of $E_1 \times E_2$ consisting of pairs (e_1, e_2) such that under f_i 's, we have $f_1(e_1) = f_2(e_2)$. Take the product and take the, this subspace. On this subspace we already have the restriction of the projection maps given by this product, $\pi_i : E_1 \times E_2 \rightarrow E_i, i = 1, 2$, just take restrictions on Z . So, these new maps, I am using the same notation but I am thinking them as maps from $Z \rightarrow E_i$. Obviously, $f_1 \circ \pi_1(e_1, e_2) = f_1(e_1) = f_2(e_2) = f_2 \circ \pi_2(e_1, e_2)$.

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Given any two maps $\pi_i : Z \rightarrow E_i$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$, there is unique map $g : Z' \rightarrow Z$ such that $\pi_i \circ g = \pi'_i, i = 1, 2$.

Indeed, the uniqueness of the triple (Z, π_1, π_2) upto homeomorphism follows easily as usual. Here is the existence by construction: Take

$$Z = \{(e_1, e_2) \in E_1 \times E_2 : f_1(e_1) = f_2(e_2)\},$$

So, the commutativity of this diagram is obvious. You have got, you have got a commutative diagram like this, all right? That we have constructed inside the product space.

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Anant Shastri

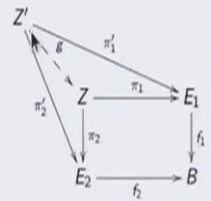
Clearly, $f_1 \circ \pi_1 = f_2 \circ \pi_2$. If (Z', f'_1, f'_2) is another triple with the same property, then we take $g(z') = (f'_1(z'), f'_2(z'))$. This g fits the diagram and this is the only map to do so.



Definition 8.4

Let $f_i : E_i \rightarrow B$ be any two maps. Their fibred product is defined to be a triple (Z, π_1, π_2) , where Z is a space and maps $\pi_i : Z \rightarrow E_i$ are such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$, and they satisfy the following universal property:

Given any two maps $\pi'_i : Z' \rightarrow E_i$ such that $f_1 \circ \pi'_1 = f_2 \circ \pi'_2$, there is unique map $g : Z' \rightarrow Z$ such that $\pi_i \circ g = \pi'_i, i = 1, 2$.



But now I have to show this Z has the universal property right? That is also easy, for if (Z', π'_1, π'_2) is another triple with the same property, then I am taking $g : Z' \rightarrow Z$ to be $g(z') = (\pi'_1(z'), \pi'_2(z'))$ which is an element in $E_1 \times E_2$ as such. But when you go here, they agree that is why, by the very definition of Z the pair $(\pi'_1(z'), \pi'_2(z'))$ is actually inside Z .

So that gives you a map into Z . So, that is what I am doing here. f_1 prime Z_1 and f_2 prime Z_2 . It is not, this definition in this notational output π_1 prime here. So, I should take π_1 prime, π_2 prime ok not f_1 prime f_2 prime. So, that g fits the diagram in the only way a map can fit there.

Because any map into a product, first of all, is determined by the two projection maps π_2 and π_1 the two coordinates. So therefore, if there is a map it has to be this map. The first projection first coordinate has to be this one, this map. The second coordinate as this map and that defines an element of Z because that is taken as those points were in when you go to f_1 they agree and f_1 and f_2 they agree. So, that is the construction and the construction and the uniqueness are over, all right. So, this is a very simple minded thing to begin with but it has wonderful properties.

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The fibred product is symmetric in f_1, f_2 . It has some wonderful properties: If f_1 is surjective, homeomorphism, local homeomorphism or satisfies many such topological properties, so does π_2 . Just to get familiar with this concept, let us check some of these claims.

The fibred product by the very definition is symmetric in (f_1, f_2) . You can say, you can write here $f_1 \times_B f_2$ or $f_2 \times_B f_1$. So I just omit having a notation for this one. It has some wonderful properties---I will use this diagram.

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Given any two maps $\pi'_i : Z' \rightarrow E_i$ such that $f_1 \circ \pi'_1 = f_2 \circ \pi'_2$, there is unique map $g : Z' \rightarrow Z$ such that $\pi_i \circ g = \pi'_i, i = 1, 2$.

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Indeed, the uniqueness of the triple (Z, π_1, π_2) upto homeomorphism follows easily as usual. Here is the existence by construction: Take

If f_1 is surjective, then π_2 is surjective. If f_1 is injective and π_2 is injective. If f_1 is a homeomorphism then π_2 is a homeomorphism. If f_1 is a covering projection then π_2 is a covering projection. If f_1 is a G-covering, π_2 will be G-covering. So, many topological properties of f_1 will be reflected here. Of course, there are some which do not. For example, if f_1 is a cofibration then π_2 may not be cofibration.

If f_1 is a cofibration there is no guarantee that π_2 will be cofibration. Cofibration will happen if we reverse all these arrows and then take the co-fibral product. Let me say something like co dual tallies when arrows are all reverse. So cofibration would not fit here. Fibrations is vibration this will be work of fibration. So let us verify a few of them, which I am going to use them also. Meanwhile, we will get familiar with this definition and the construction also.

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Injectivity

Suppose now that f_1 is injective. Given $(e_1, e_2), (e'_1, e'_2) \in Z$ such that $\pi_2(e_1, e_2) = \pi_2(e'_1, e'_2)$, we get, $e_2 = e'_2$. We also have $f_1(e_1) = f_2(e_2)$ and $f_1(e'_1) = f_2(e'_2)$. Therefore,

$$f_1(e_1) = f_2(e_2) = f_2(e'_2) = f_1(e'_1).$$

Therefore, by injectivity of f_1 we get $e_1 = e'_1$. Therefore $(e_1, e_2) = (e'_1, e'_2)$.

Let us verify surjectivity. f_1 is surjective I am assuming. I want to show that π_2 is surjective, what do I do? Take a point $e_2 \in E_2$. Pick up the point $e_1 \in E_1$ which sits over $f_2(e_2)$, i.e., $f_1(e_1) = f_2(e_2)$. $f_2(e_2) \in B$ and f_1 is surjective. Therefore, there is such a $e_1 \in E_1$. Now, by the very construction, $(e_1, e_2) \in Z$. And the second projection element is e_2 . Starting with any $e_2 \in E_2$, I produce an element in Z such that π_2 of that element is e_2 . So, π_2 is surjective all right.

Let us verify the injectivity which is slightly little lengthier, one line lengthier. Suppose f_1 is injective, I want to show that π_2 is injective. So, pick up two points in Z . How will they look like? $(e_1, e_2), (e'_1, e'_2)$. Suppose π_2 of them are equal. What is the meaning of that? The second coordinate e_2 must be equal to e'_2 . We have to then conclude that first coordinates are also equal.

But what is the meaning of these points are inside Z ? They are not arbitrarily elements of $E_1 \times E_2$ right? That they are inside Z means that $f_1(e_1) = f_2(e_2)$; $f_1(e'_1) = f_2(e'_2)$. Therefore, if you start with $f_1(e_1)$ which is equal to $f_2(e_2)$ but $e_2 = e'_2$ and it is equal to $f_2(e'_2)$ which is turn is equal to $f_1(e'_1)$. But f_1 is injective is our assumption. Therefore, $e_1 = e'_1$ from which we conclude that $(e_1, e_2) = (e'_1, e'_2)$. So, we are done.

In particular, if f_1 is a bijection then π_2 will be a bijection.

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Local Homeomorphism

Let f_1 be a local homeomorphism. Given $(e_1, e_2) \in Z$, let U_1 be an open neighbourhood of e_1 in E_1 such that $f_1|_{U_1} : U_1 \rightarrow f_1(U_1) =: V$ is homeomorphism onto an open set V in B . Take $U_2 = f_2^{-1}(V)$. Then $W := (U_1 \times U_2) \cap Z$ is an open neighbourhood of (e_1, e_2) . We claim that $\pi_2|_W : W \rightarrow U_2$ is a local homeomorphism. Bijectivity of π_2 is proved just as before. If $s : V \rightarrow U_1$ is the inverse of f_1 , define $h : U_2 \rightarrow W$ by the formula

$$h(e_2') = (s \circ f_1(e_2'), e_2').$$

So, let us go little further now. Suppose f_1 is a local homeomorphism. Then I have to show π_2 is a local homeomorphism. So, start with a point (e_1, e_2) in Z . I must produce an open subset of Z restrict π_2 on it, that must be a homeomorphism on to its image in B . For this I have to find an open subset of $E_1 \times E_2$ and intersect it with Z . For that, first of all I should use the fact that f_1 is a local homeomorphism.

So, I look at $e_1 \in E_1$. e_1 has a neighborhood U_1 in E_1 such that $f_1 : U_1 \rightarrow f_1(U_1) = V \subset B$ is a homeomorphism and V is open in B . That is the definition of local homeomorphism. Then put $U_2 = f_2^{-1}(V) \subset E_2$. So, that will open in E_2 . Now, I have two open subsets, one U_1 , another U_2 open respectively, in E_1 and E_2 . Therefore $U_1 \times U_2$ will be open in $E_1 \times E_2$. Intersection with Z put $W = Z \cap (U_1 \times U_2)$. That will be an open subset of Z , and it contains the point (e_1, e_2) why? Because first of all $e_1 \in U_1$ and you look at $f_1(e_1)$ is in V and is equal to $f_2(e_2)$. That implies that $e_2 \in f_2^{-1}(V) = U_2$.

Claim is that this W has this required property, namely, $p_2 : W \rightarrow U_2$ is a homeomorphism. Now, first thing to observe is that the bijectivity of this map follows from the bijectivity of $f_1 : U_1 \rightarrow V$, exactly as in the previous two steps. Earlier f_1 was assumed to be bijective on the whole space E_1 there. Here you restrict f_1 to U_1 . That instead of E_1, E_2 , use U_1 and U_2 respectively. Carry out the same construction, the new Z will now become the old Z intersected with $U_1 \times U_2$. That is why the step 1 and step 2 are valid here also. So, bijectivity of $p_2 : W \rightarrow U_2$

follows by bijectivity of $f_1 : U_1 \rightarrow V$. Now, we have to show that it is a homeomorphism inverse is continuous that is what we have to show. So, look at $s : V \rightarrow U_1$ which is the inverse for this f_1 . f_1 is a homeomorphism U_1 to V , then this is the inverse of f_1 and therefore, continuous. Using that, we define $h : U_2 \rightarrow W$ by the formula $h(e_2) = (sf_2(e_2), e_2)$ Automatically, when you apply π_2 to this you will get back e_2 . So, this h will be inverse of π_2 . Why it is continuous? Because this second coordinative of h is just the identity map and the first coordinate is $s \circ f_2$, which is continuous. Therefore, h is a continuous inverse.


So, we have produced a neighbourhood of each $(e_1, e_2) \in Z$ on which π_2 is a homeomorphism all right. So, it is a local homeomorphism. Actually, this also proves that if f_1 itself is a homeomorphism, then π_2 will be a homeomorphism. So, that proof is also here. More generally, local homeomorphism implies local homeomorphism is what we have seen now.

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homeomorphism. Bijectivity of π_2 is proved just as before. If $s : V \rightarrow U_1$ is the inverse of f_1 , define $h : U_2 \rightarrow W$ by the formula

$$h(e'_2) = (s \circ f_2(e'_2), e'_2).$$

Then h is the inverse of π_2 on U_2 .



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Covering projection

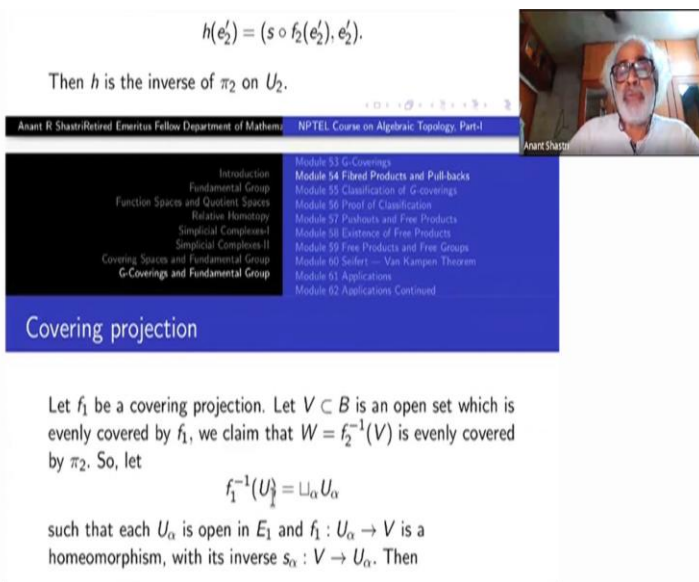
Let f_1 be a covering projection. Let $V \subset B$ is an open set which is evenly covered by f_1 , we claim that $W = f_2^{-1}(V)$ is evenly covered by π_2 . So, let

$$f_1^{-1}(U) = \sqcup_{\alpha} U_{\alpha}$$

The next thing is if f_1 is a covering projection then so is π_2 . In fact we claim that if you take an open subset of B evenly covered by f_1 and take its inverse image by f_2 inside E_2 , that open set is evenly covered by p^{i_2} . So, let $V \subset B$ be an open set, which is evenly covered by f_1 . Claim is that $W = f_2^{-1}(V)$ evenly covered by π_2 . Suppose I prove this statement. Then by the very definition of covering projection, you have an open covering of B , consisting evenly covered open subsets. Inverse image of those things will form an open cover for E_2 , each of them being evenly covered π_2 . That will complete the proof that π_2 is a covering projection.

So, it is enough to prove this statement, namely, W is evenly covered. That means I have to look at $\pi_2^{-1}(W)$, write it as a disjoint union of open sets such that restricted to any one of them p^{i_2} is a homeomorphism. What is my source for this? Source is similar statement for f_1 .

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$h(e'_2) = (s \circ f_2(e'_2), e'_2).$

Then h is the inverse of π_2 on U_2 .

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Covering projection

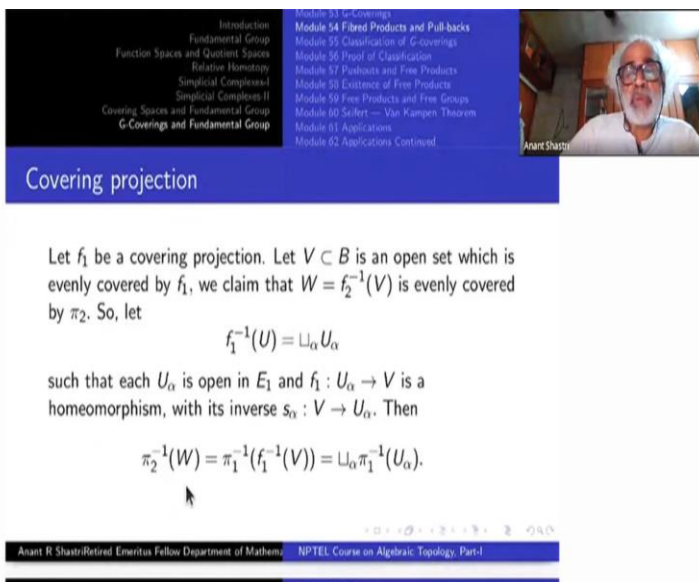
Let f_1 be a covering projection. Let $V \subset B$ is an open set which is evenly covered by f_1 , we claim that $W = f_2^{-1}(V)$ is evenly covered by π_2 . So, let

$$f_1^{-1}(U) = \sqcup_{\alpha} U_{\alpha}$$

such that each U_{α} is open in E_1 and $f_1 : U_{\alpha} \rightarrow V$ is a homeomorphism, with its inverse $s_{\alpha} : V \rightarrow U_{\alpha}$. Then

Namely, write $f_1^{-1}(V)$ as disjoint union of U_{α} 's, where each U_{α} is open in E_1 , and $f_1 : U_{\alpha} \rightarrow V$ is a homeomorphism. Let us put s_{α} to be the inverse of that. That would depend upon alpha right. So, $s_{\alpha} : V \rightarrow U_{\alpha}$ is the inverse of $f_1 : U_{\alpha} \rightarrow V$.

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Covering projection

Let f_1 be a covering projection. Let $V \subset B$ is an open set which is evenly covered by f_1 , we claim that $W = f_2^{-1}(V)$ is evenly covered by π_2 . So, let

$$f_1^{-1}(U) = \sqcup_{\alpha} U_{\alpha}$$

such that each U_{α} is open in E_1 and $f_1 : U_{\alpha} \rightarrow V$ is a homeomorphism, with its inverse $s_{\alpha} : V \rightarrow U_{\alpha}$. Then

$$\pi_2^{-1}(W) = \pi_1^{-1}(f_1^{-1}(V)) = \sqcup_{\alpha} \pi_1^{-1}(U_{\alpha}).$$

Then $\pi_2^{-1}(W) = \pi_2^{-1}(f_2^{-1}(V)) = \pi_1^{-1}(f_1^{-1}(V))$ because we have $f_2 \circ \pi_2 = f_1 \circ \pi_1$. Therefore $\pi_2^{-1}(W) = \sqcup_{\alpha} \pi_1^{-1}(U_{\alpha})$. π_1 being continuous, these are open subsets of $E_1 \times E_2$. But I am restricting it to Z , taking intersection with Z on both sides. So they are open subsets of Z

Disjointedness is still there because U_α are disjoint open subsets here. So I have got a disjoint family. I want to show that each one of them projects homeomorphically on to this W under π_2 . This part already follows from what we have seen in the local homeomorphism picture. What if the inverse corresponding inverse use this s_α to get the corresponding inverse, just the way I have done it here.

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$f_1|_{U_1} : U_1 \rightarrow f_1(U_1) =: V$ is homeomorphism onto an open set V in B . Take $U_2 = f_2^{-1}(V)$. Then $W := (U_1 \times U_2) \cap Z$ is an open neighbourhood of (e_1, e_2) . We claim that $\pi_2 : W \rightarrow U_2$ is a local homeomorphism. Bijectivity of π_2 is proved just as before. If $s : V \rightarrow U_1$ is the inverse of f_1 , define $h : U_2 \rightarrow W$ by the formula

$$h(e_2) = (s \circ f_2(e_2), e_2).$$

Then h is the inverse of π_2 on U_2 .

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Covering projection

Let f_1 be a covering projection. Let $V \subset B$ is an open set which is evenly covered by f_1 . we claim that $W = f_1^{-1}(V)$ is evenly covered


So, h here, we would define as s composite to f_2 identity. So, that is what you have to do here.

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$\pi_2 \circ \nu = \pi_1 \circ \tau_1 \circ \nu = \omega_\alpha \pi_1 \circ \nu$

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From what we have seen in the previous paragraph, it follows that for each α ,

$$\pi_2 \circ \pi_1^{-1}(U_\alpha) \rightarrow W$$

is a homeomorphism with its inverse given by $h_\alpha(e_2) = (s_\alpha \circ f_2(e_2), e_2)$.




So, just like what we have seen in the previous paragraph, it follows that $\pi_2 \circ \pi_1^{-1}(U_\alpha) \rightarrow W$ is a homeomorphism.

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G-coverings

Finally, suppose a group G acts on E_1 evenly and $f_1 : E_1 \rightarrow B$ is the resulting quotient map, which is a covering. We claim that there is an even action of G on Z such that $\pi_2 : Z \rightarrow E_2$ is the corresponding quotient map and $\pi_1 : Z \rightarrow E_1$ is a G -map.



Now, we come to one of the most interesting part, namely, if f_1 is a G -covering then π_2 is a G -covering OK? If it is a covering, this is a covering it is already done. So, only thing is now we have to see that they are given actually by an even action of the group G . Suppose G acts on E_1

E_1 evenly and f_1 is the corresponding projection map from E_1 to B , which is automatically a covering projection.

We claim that there is an even action of G on Z such that this π_2 becomes the corresponding quotient map. You cannot change π_2 by the way. On Z , the maps π_1, π_2 etc. are already defined. Once on E_1 and E_2, f_1, f_2 are given respectively, Z is already defined. So, I want to claim that π_2 itself becomes a quotient map corresponding to the action of G and therefore it is a covering projection.

Not only that, ok the π_1 from Z to E_1 , there are G action on both sides. This π_1 becomes a G map. Indeed, this extra observation will tell you what you have to do. So, this is more or less like a last part is more or less like a hint here for the construction of G action on Z .

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The action of G on Z is defined in an obvious way through the first factor:

$$g(e_1, e_2) = (ge_1, e_2).$$

Since $f_1(ge_1) = f_1(e_1) = f_2(e_2)$ this action is well defined. Clearly each orbit is mapped to the same point e_2 by π_2 . Moreover, if $(e'_1, e_2) \in Z$ is any other element, then $e'_1 \in f_1^{-1}\{f_2(e_2)\}$ and hence there must exist $g \in G$ such that $ge_1 = e'_1$ and therefore $g(e_1, e_2) = (e'_1, e_2)$. This shows that the orbits of the G -action on

So, we define the action of G on Z , obvious action. There is an action of G on E_1 and we do not know anything about E_2 . E_2 is just an arbitrary topological space. Therefore, take the action of this G on $E_1 \times E_2$ via the first factor, $Gg(e_1, e_2) = (ge_1, e_2)$. Now, suppose (e_1, e_2) is already inside Z . Then when you act by this namely $g(e_1, e_2)$ this will be also inside Z . Why? Why it is inside Z ? I have to verify that. For that we have to check that $f_1(ge_1) = f_2(e_2)$.

This is the condition. But what is $f_1(ge_1)$? Because f_1 is a quotient map under G action, $f_1(ge_1) = f_1(e_1)$. Therefore, $f_1(ge_1) = f_2(e_2)$ and So, therefore the right hand side is inside Z if (e_1, e_2) is inside Z . So, this action of G on $E_1 \times E_2$ actually gives you action of G on Z .

That is the meaning of that this is well defined. Now, clearly under G action, the second coordinate of a point does not change. Therefore, the fibers of π_2 are G -orbits. And they are precisely the orbits. Because $(e_1, e_2), (e'_1, e_2) \in Z$ implies $f_1(e) = f_2(e_2) = f_1(e'_1)$. That means e_1, e'_1 are in the same orbit. And hence there must be a $g \in G$ such that $ge_1 = e'_1$ because they are in the same orbit.

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The action of G on Z is defined in an obvious way through the first factor:

$$g(e_1, e_2) = (ge_1, e_2).$$

Since $f_1(ge_1) = f_1(e_1) = f_2(e_2)$ this action is well defined. Clearly each orbit is mapped to the same point e_2 by π_2 . Moreover, if $(e'_1, e_2) \in Z$ is any other element, then $e'_1 \in f_1^{-1}\{f_2(e_2)\}$ and hence there must exist $g \in G$ such that $ge_1 = e'_1$ and therefore $g(e_1, e_2) = (e'_1, e_2)$. This shows that the orbits of the G -action on Z are precisely the fibres of π_2 .

That just means that $g(e_1, e_2) = (e'_1, e_2)$. So, this shows that the fibers of π_2 inside Z are precisely equal to the orbits G action on Z . This just means that this map π_2 is the quotient map. Well tell me that the whole thing is quotient map but is the covering projection already that we have already seen. A covering projection is a open quotient therefore it is a open surjective that holds a quotient map. The fibers are correct but why it is a quotient map requires some explanation. But we have already seen that it is a covering projection.

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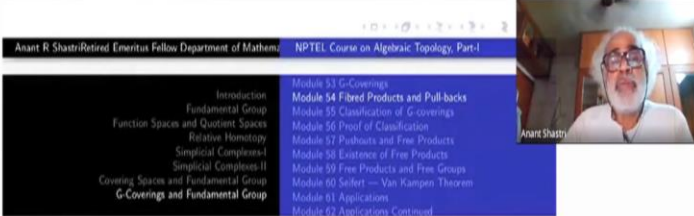
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We have already seen that f_1 is a covering projection implies so is π_2 . In particular, it also follows that $\pi_2 : Z \rightarrow E_2$ is a quotient map.

There are slightly different ways of seeing that quotient that does not matter. So, what we have seen is that π_2 from Z to E_2 is actually corresponds to the covering projection by G action.

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The action of G on Z is defined in an obvious way through the first factor:

$$g(e_1, e_2) = (ge_1, e_2).$$

Since $f_1(g e_1) = f_1(e_1) = f_2(e_2)$ this action is well defined. Clearly each orbit is mapped to the same point e_2 by π_2 . Moreover, if $(e'_1, e_2) \in Z$ is any other element, then $e'_1 \in f_1^{-1}\{f_2(e_2)\}$ and hence there must exist $g \in G$ such that $g e_1 = e'_1$ and therefore $g(e_1, e_2) = (e'_1, e_2)$. This shows that the orbits of the G -action on Z are precisely the fibres of π_2 .

Look at the way we have defined, defined this action here, G of $E_1 \times E_2$ is $G \times E_1 \times E_2$. What happens to when you come to π_1 , π_1 of this is π_1 of $G \times E_1$ ok which is G of just $G \times E_1$. This π_1 we see are the just $G \times E_1$ which G of E_1 . On this side it is E_1 . So, π_1 of $E_1 \times E_2$ operated by G is the same thing as $G \times E_1$. So, this very definition shows that the projection map to π_1 is a G action, G map.

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The action of G on Z is defined in an obvious way through the first factor:

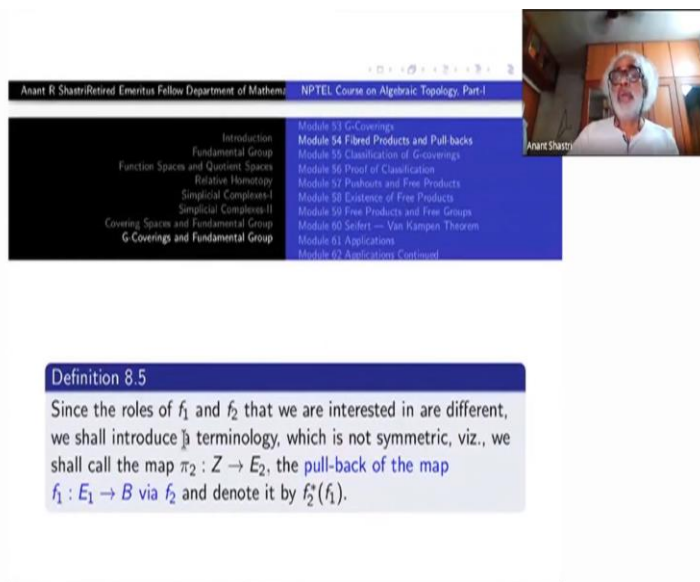
$$g(e_1, e_2) = (ge_1, e_2).$$

Since $f_1(ge_1) = f_1(e_1) = f_2(e_2)$ this action is well defined. Clearly each orbit is mapped to the same point e_2 by π_2 . Moreover, if $(e'_1, e_2) \in Z$ is any other element, then $e'_1 \in f_1^{-1}\{f_2(e_2)\}$ and hence there must exist $g \in G$ such that $ge_1 = e'_1$ and therefore $g(e_1, e_2) = (e'_1, e_2)$. This shows that the orbits of the G -action on

So, this was last part here, projection of G map. So, I have taken that as a key, as a clue to define this action and it works, all right. So, these are the basic things that we need to explain our classification of G coverings. So, this is going to produce large number of G coverings. If we have one G -covering and any map into the base place, you can construct the G -covering on the other space.

So, this procedure is precisely called, has a different name which is non-symmetric in the definition, in the idea. Because the role of f_1 , all properties of f_1 reflected in π_2 , f_2 may not have any of these properties maybe just a continuous function. That is why the role is important though in the definition of the fibred product f_1 and f_2 does not matter, which one does matter, that is the beauty of this.

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Definition 8.5
 Since the roles of f_1 and f_2 that we are interested in are different, we shall introduce a terminology, which is not symmetric, viz., we shall call the map $\pi_2 : Z \rightarrow E_2$, the pull-back of the map $f_1 : E_1 \rightarrow B$ via f_2 and denote it by $f_2^*(f_1)$.

So, we introduce a non-symmetric wording here, namely this is the point f_1 f_2 though the fibral product is symmetric in that, the role of f_1 f_2 are different. So, we shall introduce a terminology which is not symmetric in that sense, namely we shall call this map π_2 from Z to E_2 and the pull-back of the map $f_1 : E_1 \rightarrow B$ via f_2 and denoted by $f_2^*(f_1)$. You see this is a categorical notation which does not depend on the construction.

Here we have heavily used the construction. But all these things can be done in categorical language without appealing to the construction. That will be taking you little more deeper. We do not have any time for that. Here we use the easy set theory which gives you all these answers easily. So, this map will be called the pull-back. The corresponding covering projection will be called pull-back covering projection. So, if you start with the G -covering E to B , I will going to use a different notation now. E to B is a G -covering.

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Remark 8.1

To sum-up what we have seen so far: if $p : E \rightarrow B$ and $f : B' \rightarrow B$ any two maps, and if p has one of the following properties then so does the pull-back $f^*(p)$:

(i) surjectivity, (ii) injectivity, (iii) local homeomorphism, (iv) covering projection, (v) G-covering.

There is a map, there is a map F from B prime to B , then I get a G -covering on B prime which I am going to write as f star of P . P is a G -covering, f star of p will be G -covering. What we have seen is if p is surjective, injective, local homeomorphism, covering projection, G -covering etc. the same is true for f star. Thank you.