**Introduction to Algebraic Topology (Part-I)**
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**Indian Institute of Technology Bombay**
**Lecture 54**
**Pull-backs**

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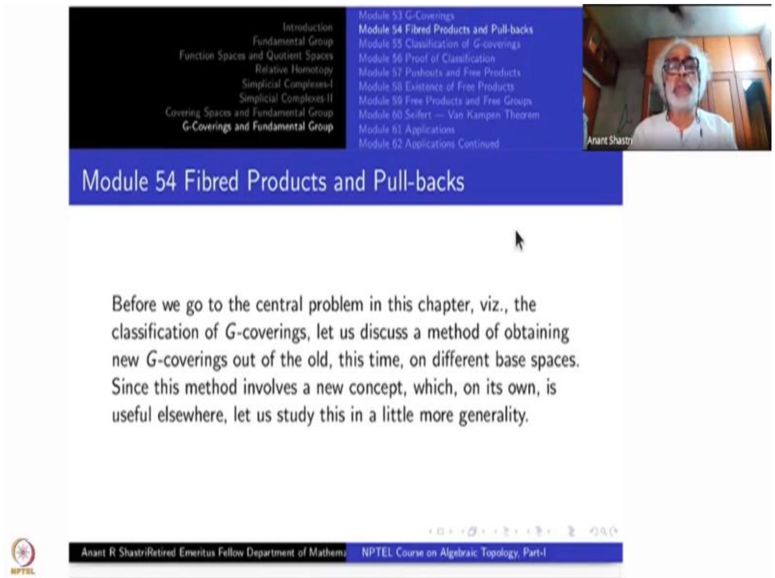


Before we take up the central problem in this chapter of classification of G-coverings, I would like to present a method of obtaining new G-coverings out of the old ones, this time, by a change in the base itself. Since this method involves a new concept which is important on its own, not only in algebraic topology but elsewhere also, let us study this one a little more carefully and in a little more generality, not full generality.

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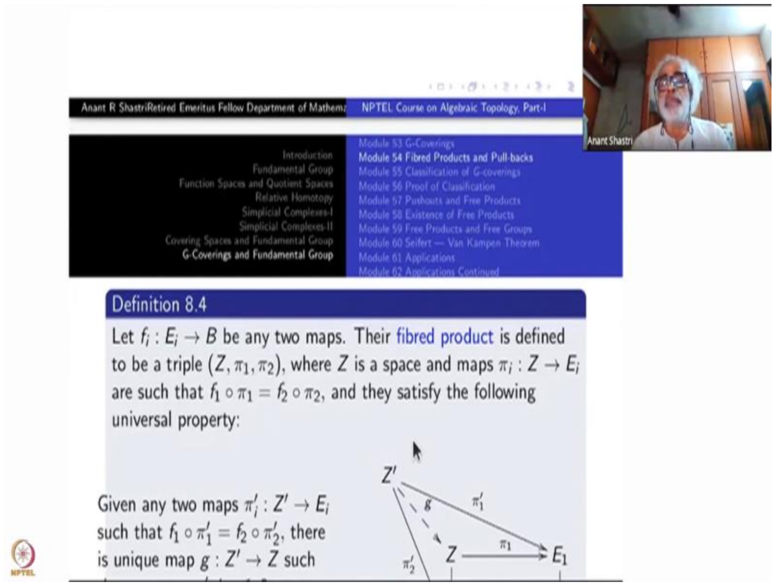


We start with a base space B and take 2 functions $f_i : E_i \to B$. Their fibred product is defined to be a triple consisting of a topological space and two functions $\pi_i : Z \to E_i, i = 1, 2,$ with the condition that $f_1 \circ \pi_1 = f_2 \circ \pi_2$. So, the composite map from Z to E1 to B or Z to E2 to B, they are the same and this entire thing satisfying the so-called universal property.

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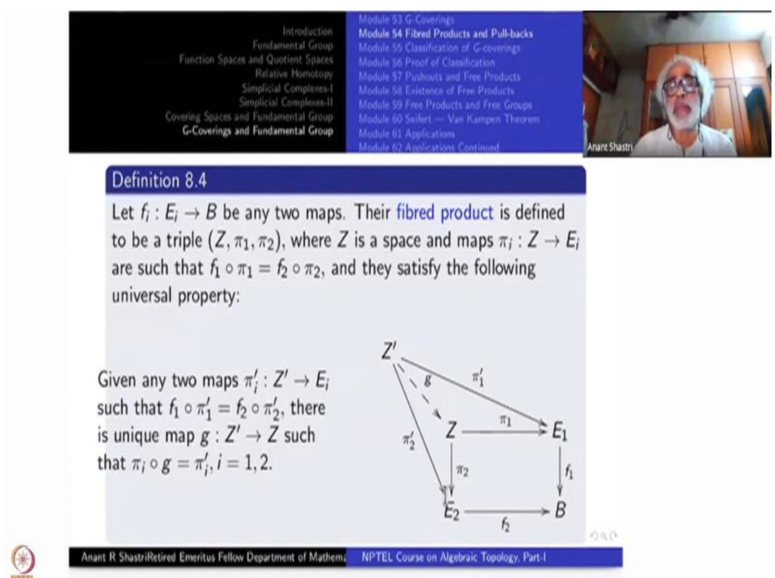


So, this is the picture, $Z$ is here, E1 E2 are here, B is here ok. So, f1 and f2 are functions into B. Then pi 1 and pi 2 respectively are functions into E1 and E2 respectively. If you have another commutative diagram like this, Z prime, pi 1 prime, pi 2 prime respectively maps into E1 to E2 such that when you compose f2 here and f1 there they are the same $f_1 \circ \pi_1' = f_2 \circ \pi_2'$ then we have a map here such and two commutative triangles.

This is the conclusion of the definition, there exist a map $g : Z' \to Z$, a unique map $g$, such that the whole diagram is commutative, namely $\pi_1 \circ g = \pi_1', \pi_2 \circ g = \pi_2'$. For every $Z'$ and pairs of maps which satisfies this property, there must be a unique map like this. But this is a universal property of this commutative square here. Then this Z together with these two maps is called the fibred product of $f_1$ and $f_2$.

But this is the definition of the fibred product of two functions taking values in the same space B. The uniqueness of such a triple $(Z, \pi_1, \pi_2)$, $\pi_1 : Z \to E_1, \pi_2 : Z \to E_2$, up to a homeomorphism, follows easily by this universal property. Namely, suppose $(Z'; \pi_1', \pi_2')$ is another such triple which also has the same universal property, then it will admit a map from $Z \to Z'$ in the reverse direction here, again giving a similar diagram, all these diagrams are commutative. Then what you have is g prime let us say, $g'$ going from $Z$ to $Z'$, and coming back by $g$ that will be another map from Z to Z itself fitting this diagram. Identity map fits this diagram from Z to Z also, So, there would be two of them. By the uniqueness property, this map composite map must be identity, $Id = g \circ g' : Z \to Z$. That means what? By the same argument, we get $Id_{Z'} = g' \circ g : Z' \to Z'$, that is $g : (Z', \pi_1', \pi_2') \to (Z, \pi_1, \pi_2)$ is a homemorphism. This is typical of all universal properties, definition given by universal properties. Existence of such objects is not guaranteed but uniqueness is guaranteed. So, let us look at the existence, the existence here is very easy compared to many such existence theorems.

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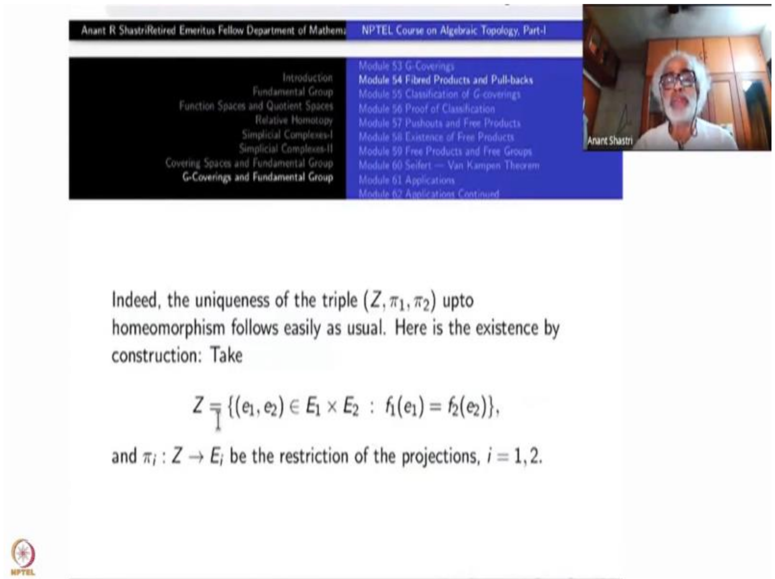


So, for the existence, what I do is I define Z to be the subspace of $E_1 \times E_2$ consisting of pairs $(e_1, e_2)$ such that under $f_i' s$, we have $f_1(e_1) = f_2(e_2)$. Take the product and take the, this subspace. On this subspace we already have the restriction of the projection maps given by this product, $\pi_i : E_1 \times E_2 \to E_i, i = 1, 2$, just take restrictions on $Z$. So, these new maps, I am using the same notation but I am thinking them as maps from $Z \to E_i$. Obviously, $f_1 \circ \pi_1(e_1, e_2) = f_1(e_1) = f_2(e_2) = f_2 \circ \pi_2(e_1, e_2)$.

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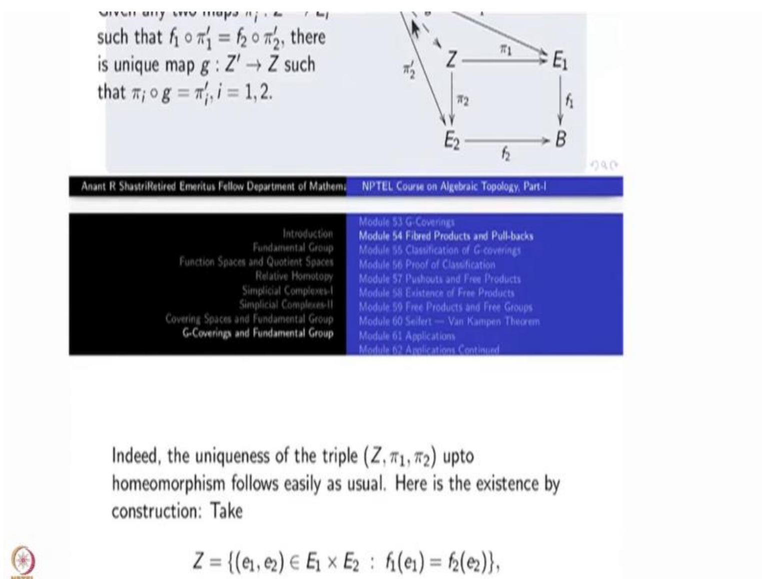

So, the commutativity of this diagram is obvious. You have got, you have got a commutative diagram like this, all right? That we have constructed inside the product space.

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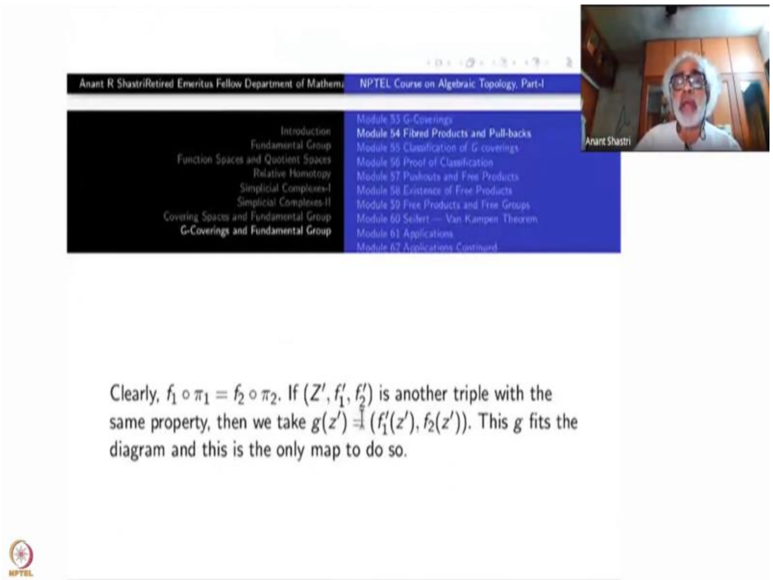


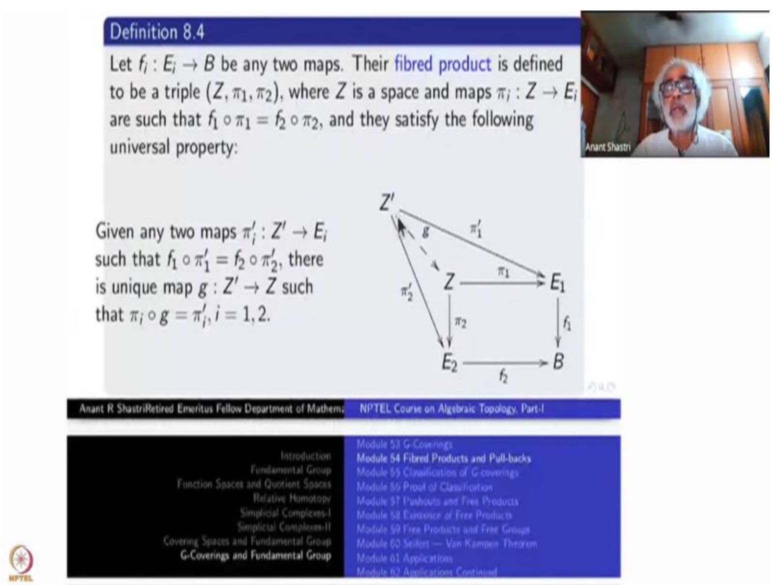


But now I have to show this $Z$ has the universal property right? That is also easy, for if $(Z', \pi_1', \pi_2')$ is another triple with the same property, then I am taking $g : Z' \to Z$ to be $g(z') = (\pi_1'(g'), \pi_2'(g'))$ which is an element in $E_1 \times E_2$ as such. But when you go here, they agree that is why, by the very definition of $Z$ the pair $(\pi_1'(z'), \pi_2'(z'))$ is acutally inside $Z$.

So that gives you a map into Z. So, that is what I am doing here. F1 prime Z1 and f2 prime Z2. It is not, this definition in this notational output pi 1 prime here. So, I should take pi 1 prime, pi 2 prime ok not f1 prime f2 prime. So, that g fits the diagram in the only way a map can fit there.

Because any map into a product, first of all, is determined by the two projection maps pi 2 and pi 1 the two coordinates. So therefore, if there is a map it has to be this map. The first projection first coordinate has to be this one, this map. The second coordinate as this map and that defines an element of Z because that is taken as those points were in when you go to f1 they agree and f1 and f2 they agree. So, that is the construction and the construction and the uniqueness are over, all right. So, this is a very simple minded thing to begin with but it has wonderful properties.

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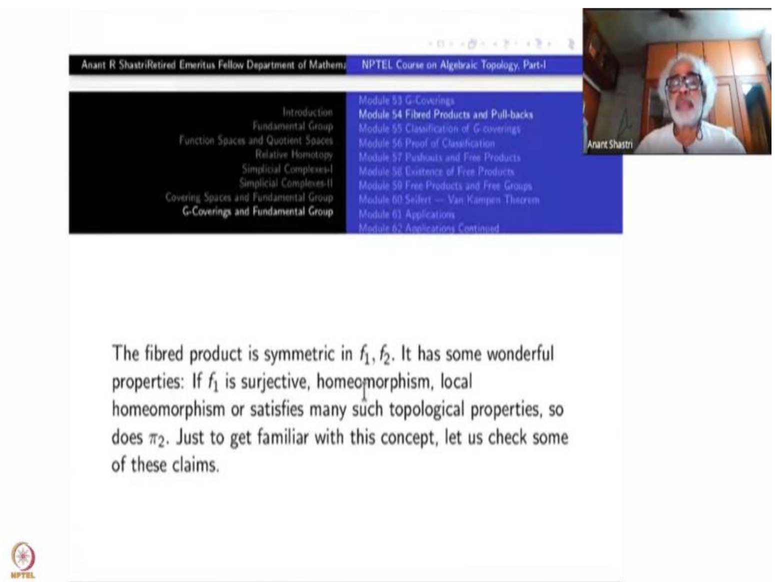


The fibred product by the very definition is symmetric in $(f_1, f_2)$. You can say, you can write here $f_1 \times_B f_2$ or $f_2 \times_B f_1$. So I just omit having a notation for this one. It has some wonderful properties---I will use this diagram.

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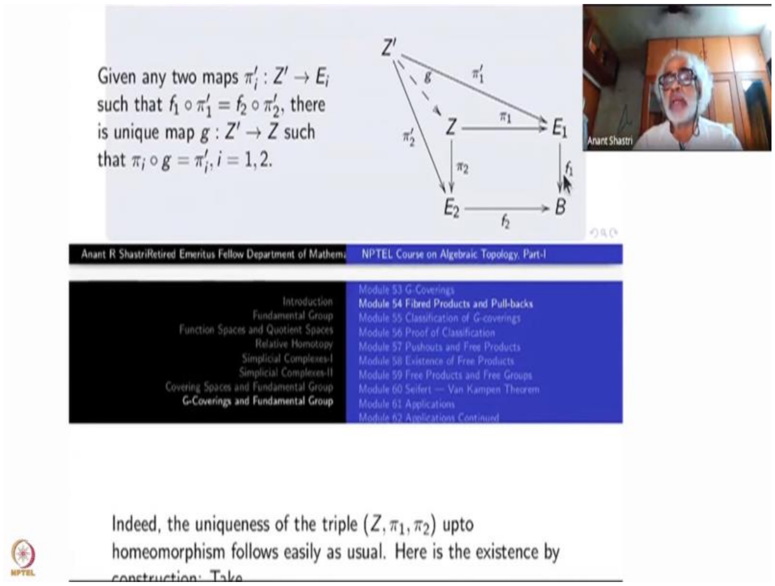


If $f_1$ is surjective, then $\pi_2$ is surjective. If $f_1$ is injective and $\pi_2$ is injective. If $f_1$ is a homeomorphism then $\pi_2$ is a homeomorphism. If $f_1$ is a covering projection then $\pi_2$ is a covering projection. If $f_1$ is a G-covering, $\pi_2$ will be G-covering. So, many topological properties of f1 will be reflected here. Of course, there are some which do not. For example, if $f_1$ is a cofibration then $\pi_2$ may not be cofibration.

If $f_1$ is a cofibration there is no guarantee that $\pi_2$ will be cofibration. Cofibration will happen if we reverse all these arrows and then take the co-fibral product. Let me say something like co ok dual tallies when arrows are all reverse. So cofibration would not fit here. Fibrations is vibration this will be work of fibration. So let us verify a few of them, which I am going to use them also. Meanwhile, we will get familiar with this definition and the construction also.

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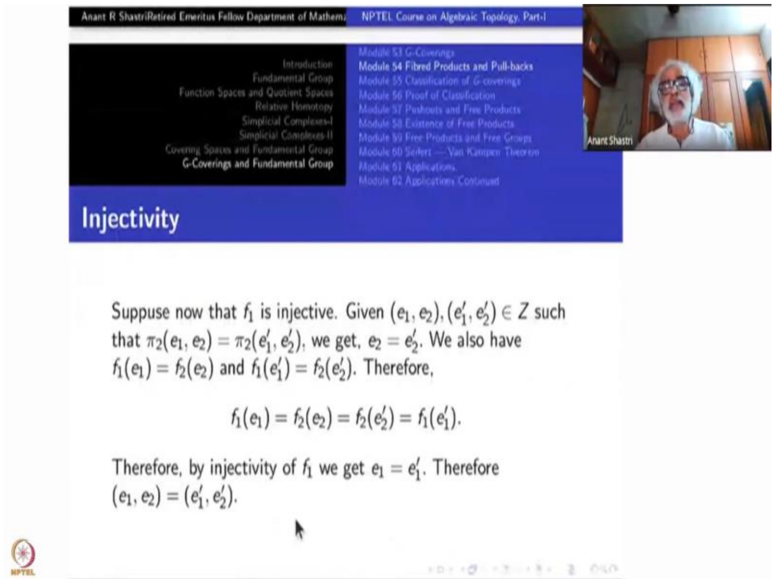


Let us verify surjectivity. $f_1$ is surjective I am assuming. I want to show that $\pi_2$ is surjective, what do I do? Take a point $e_2 \in E_2$. Pick up the point $e_1 \in E_1$ which sits over $f_2(e_2)$, i.e., $f_1(e_1) = f_2(e_2)$. $f_2(e_2) \in B$ and $f_1$ is surjective. Therefore, there is such a $e_1 \in E_1$. Now, by the very construction, $(e_1, e_2) \in Z$. And the second projection element is $e_2$. Starting with any $e_2 \in E_2$, I produce an element in $Z$ such that $\pi_2$ of that element is $e_2$. So, $\pi_2$ is surjective all right.

Let us verify the injectivity which is slightly little lengthier, one line lengthier. Suppose $f_1$ is injective, I want to show that $\pi_2$ is injective. So, pick up two points in $Z$. How will they look like? $(e_1, e_2), (e_1', e_2')$. Suppose $\pi_2$ of them are equal. What is the meaning of that? The second coordinate $e_2$ must be equal to $e_2'$. We have to then conclude that first coordinates are also equal.

But what is the meaning of these points are inside $Z$? They are not arbitrarily elements of $E_1 \times E_2$ right? That they are inside $Z$ means that $f_1(e_1) = f_2(e_2); f_1(e_1') = f_2(e_2')$. Therefore, if you start with $f_1(e_1)$ which is equal to $f_2(e_2)$ but $e_2 = e_2'$ and it is equal to $f_2(e_2')$ which is turn is equal to $f_1(e_{\prime 1})$. But $f_1$ is injective is our assumption. Therefore, $e_1 = e_1'$ from which we conclude that $(e_1, e_2) = (e_1', e_2')$. So, we are done.

In particular, if $f_1$ is a bijection then $\pi_2$ will be a bijection.

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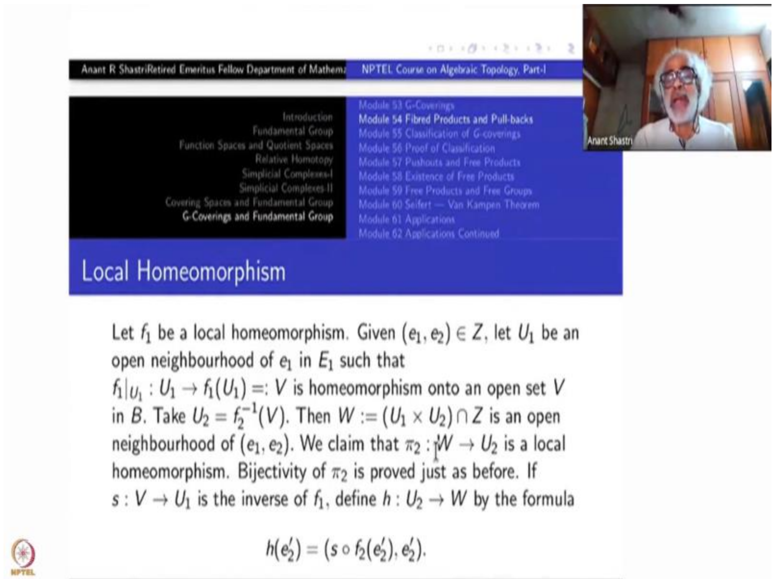


So, let us go little further now. Suppose $f_1$ is a local homeomorphism. Then I have to show $\pi_2$ is a local homeomorphism. So, start with a point $(e_1, e_2)$ in $Z$. I must produce an open subset of $Z$ restrict $\pi_2$ on it, that must be a homeomorphism on to its image in $B$. For this I have to find an open subset of $E_1 \times E_2$ and intersect it with $Z$. For that, first of all I should use the fact that $f_1$ is a local homeomorphism.

So, I look at $e_1 \in E_1$. $e_1$ has a neighborhood $U_1$ in $E_1$ such that $f_1 : U_1 \to f_1(U_1) = V \subset B$ is a homeomorphism and $V$ is open in $B$. That is the definition of local homeomorphism. Then put $U_2 = f_2^{-1}(V) \subset E_2$. So, that will open in $E_2$. Now, I have two open subsets, one $U_1$, another $U_2$ open respectively, in $E_1$ and $E_2$. Therefore $U_1 \times U_2$ will be open in $E_1 \times E_2$. Intersection with $Z$ put $W = Z \cap (U_1 \times U_2)$. That will be an open subset of $Z$, and it contains the point $(e_1, e_2)$ why? Because first of all $e_1 \in U_1$ and you look at $f_1(e_1)$ is in V and is equal to $f_2(e_2)$. That imples that $e_2 \in f_2^{-1}(V) = U_2$.

Claim is that this $W$ has this required property, namely, $p_2 : W \to U_2$ is a homeomorphism. Now, first thing to observe is that the bijectivity of this map follows from the bijectivity of $f_1 : U_1 \to V$, exactly as in the previous two steps. Earlier $f_1$ was assumed to be bijective on the whole space $E_1$ there. Here you restrict $f_1$ to $U_1$ . That instead of $E_1, E_2$, use $U_1$ and $U_2$ respectively. Carry out the same construction, the new $Z$ will now become the old $Z$ intersected with $U_1 \times U_2$. That is why the step 1 and step 2 are valid here also. So, bijectivity of $p_2 : W \to U_2$

follows by bijectivity of $f_1 : U_1 \to V$. Now, we have to show that it is a homeomorphism inverse is continuous that is what we have to show. So, look at $s : V \to U_1$ which is the inverse for this $f_1$. $f_1$ is a homeomorphism $U_1$ to $V$, then this is the inverse of $f_1$ and therefore, continuous. Using that, we define $h : U_2 \to W$ by the formula $h(e_2) = (sf_2(e_2), e_2)$ Automatically, when you apply $\pi_2$ to this you will get back $e_2$. So, this $h$ will be inverse of $\pi_2$. Why it is continuous? Because this second coordinative of $h$ is just the identity map and the first coordinate is $s \circ f_2$ , which is continuous. Therefore, $h$ is a continuous inverse.

So, we have produced a neighbourhood of each $(e_1, e_2) \in Z$ on which $\pi_2$ is a homeomorphism all right. So, it is a local homeomorphism. Actually, this also proves that if $f_1$ itself is a homeomorphism, then $\pi_2$ will be a homeomorphism. So, that proof is also here. More generally, local homeomorphism implies local homeomorphism is what we have seen now.

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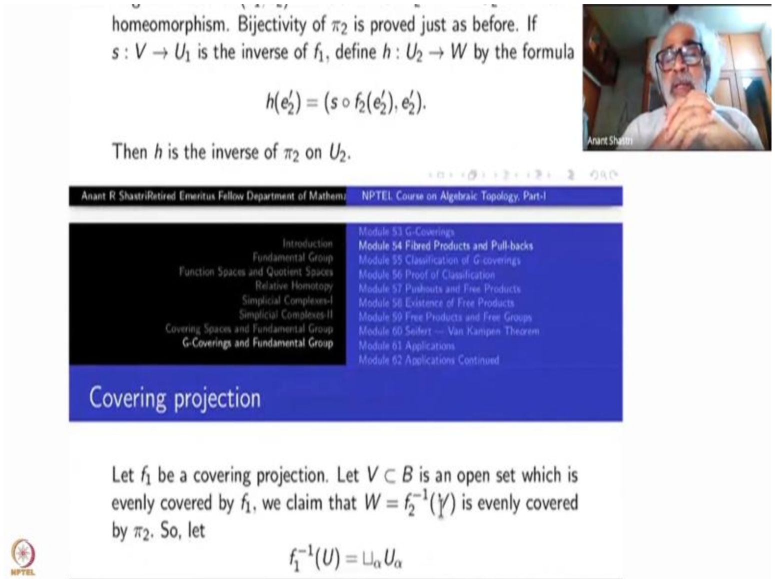


The next thing is if $f_1$ is a covering projection then so is $\pi_2$. In fact we claim that if you take an open subset of $B$ evenly covered by $f_1$ and take its inverse image by $f_2$ inside $E_2$, that open set is evenly covered by $pi_2$. So, let $V \subset B$ be an open set, which is evenly covered by $f_1$. Claim is that $W = f_2^{-1}(V)$ evenly covered by $\pi_2$. Suppose I prove this statement . Then by the very definition of covering projection, you have an open covering of $B$, consisting evenly covered open subsets. Inverse image of those things will form an open cover for $E_2$, each of them being evenly covered $\pi_2$. That will complete the proof that $\pi_2$ is a covering projection.

So, it is enough to prove this statement, namely, W is evenly covered. That means I have to look at $\pi_2^{-1}(W)$, write it as a disjoint union of open sets such that restricted to any one of them $pi_2$ is a homeomorphism. What is my source for this? Source is similar statement for $f_1$.

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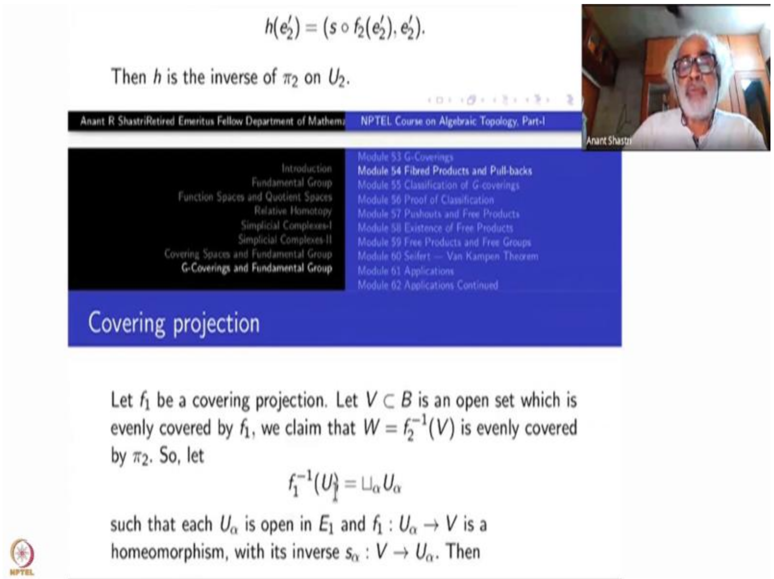


Namely, write $f_1^{-1}(V)$ as disjoint union of $U_\alpha$'s, where each $U_\alpha$ is open in $E_1$, and $f_1 : U_\alpha \to V$ is a homeomorphism. Let us put $s_\alpha$ to be the inverse of that. That would depend upon alpha right. So, $s_\alpha : V \to U_\alpha$ is the inverse of $f_1 : U_\alpha \to V$.

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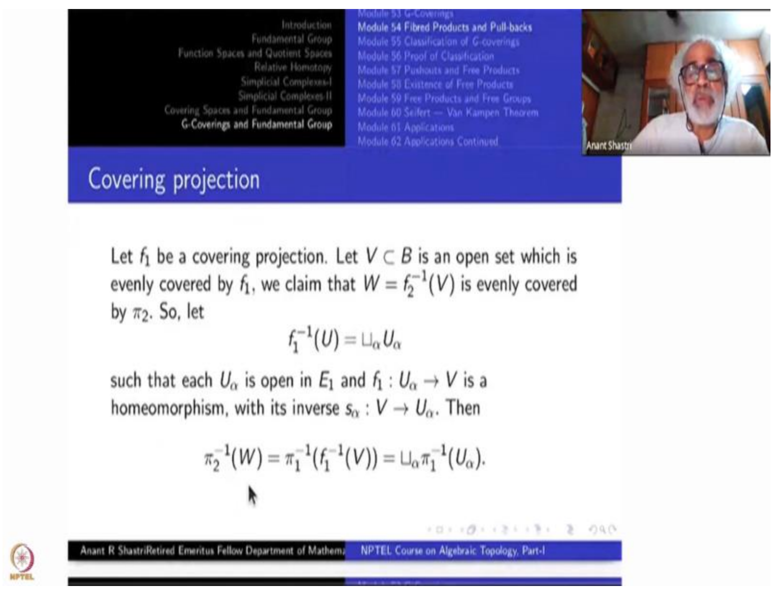


Then $\pi_2^{-1}(W) = \pi_2^{-1}(f_2^{-1}(V)) = \pi_1^{-1}(f_1^{-1}(V))$ because we have $f_2 \circ \pi_2 = f_1 \circ \pi_1$. Therefore $\pi_2^{-1}(W) = \sqcup_\alpha \pi_1^{-1}(U_\alpha)$. $\pi_1$ being continuous, these are open subsets of $E_1 \times E_2$. But I am restricting it to $Z$, taking intersection with $Z$ on both sides. So they are open subsets $of Z$

Disjointedness is still there because $U_\alpha$ are disjoint open subsets here. So I have got a disjoint family. I want to show that each one of them projects homeomorphically on to this $W$ under $\pi_2$. This part already follows from what we have seen in the local homeomorphism picture. What if the inverse corresponding inverse use this $s_\alpha$ to get the corresponding inverse, just the way I have done it here.

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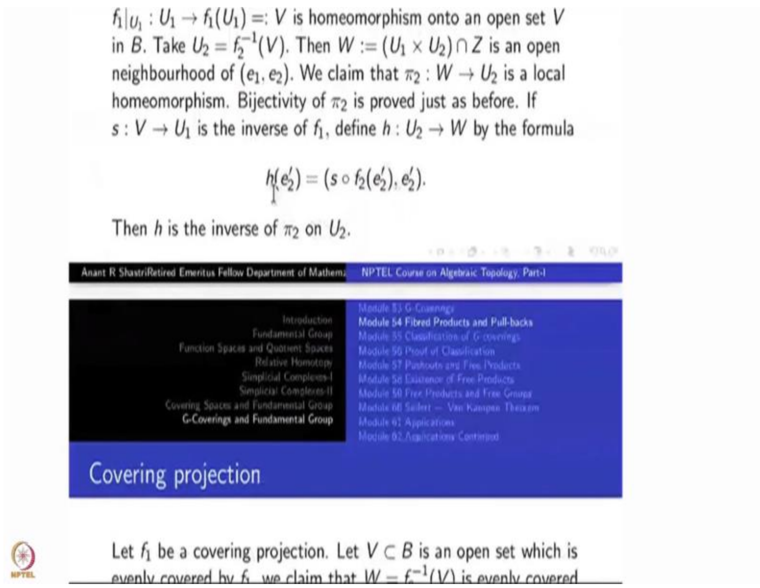


So, H alpha here, we would define as S alpha composite to f2 comma identity. So, that is what you have to do here.

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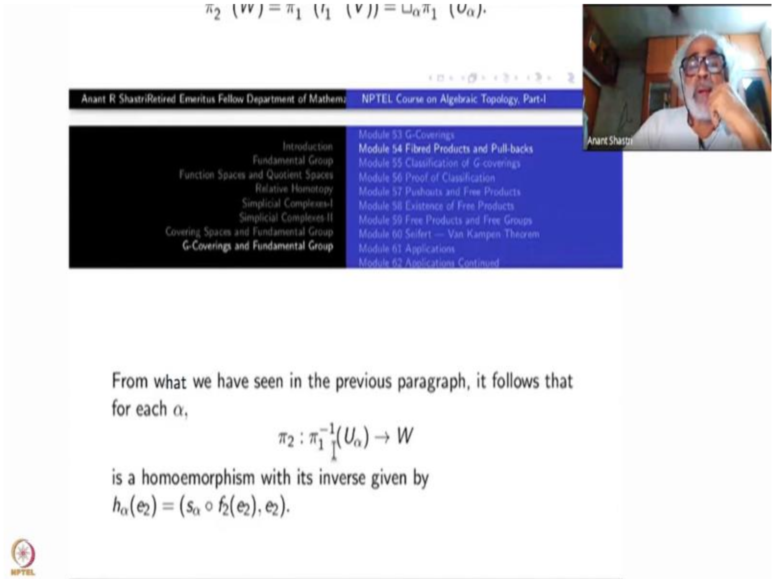


So, just like what we have seen in the previous paragraph, it follows that $\pi_2 : \pi^{-1}(U_\alpha) \to W$ is a homeomorphism.

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Now, we come to one of the most interesting part, namely, if $f_1$ is a G-covering then $\pi_2$ is a G-covering OK? If it is a covering, this is a covering it is already done. So, only thing is now we have to see that they are given actually by an even action of the group $G$. Suppose $G$ acts on E_

$E_1$ evenly and $f_1$ is the corresponding projection map from $E_1$ to $B$, which is automatically a covering projection.

We claim that there is an even action of $G$ on $Z$ such that this $\pi_2$ becomes the corresponding quotient map. You cannot change $\pi_2$ by the way. On $Z$, the maps $\pi_1, \pi_2$ etc. are already defined. Once on $E_1$ and $E_2$, $f_1, f_2$ are given respectively, $Z$ is already defined. So, I want to claim that $\pi_2$ itself becomes a quotient map corresponding to the action of G and therefore it is a covering projection.

Not only that, ok the pi 1 from Z to E1, there are G action on both sides. This pi 1 becomes a G map. Indeed, this extra observation will tell you what you have to do. So, this is more or less like a last part is more or less like a hint here for the construction of $G$ action on $Z$.

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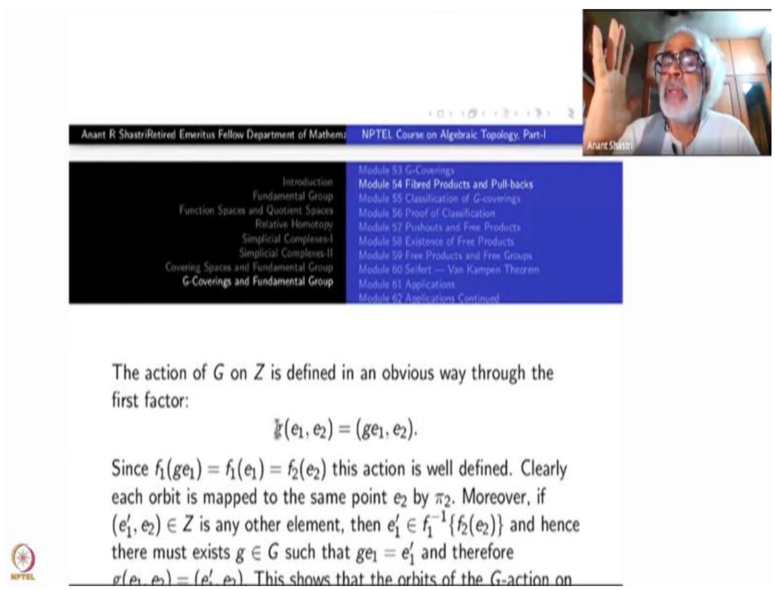


So, we define the action of $G$ on Z, obvious action. There is an action of $G$ on $E_1$ and we do not know anything about $E_2$. $E_2$ is just an arbitrary topological space. Therefore, take the action of this $G$ on $E_1 \times E_2$ via the first factor, G$g(e_1, e_2) = (ge_1, e_2)$. Now, suppose $(e_1, e_2)$ is already inside $Z$. Then when you act by this namely $g(e_1, e_2)$ this will be also inside $Z$. Why? Why it is inside $Z$? I have to verify that. For that we have to check that $f_1(ge_1) = f_2(e_2)$.

This is the condition. But what is $f_1(ge_1)$? Becasue $f_1$ is a quotient map under $G$ action, $f_1(ge_1) = f_1(e_1)$. Therefore, $f_1(ge_1) = f_2(e_2)$ and So, therefore the right hand side is inside Z if $(e_1, e_2)$ is inside Z. So, this action of $G$ on $E_1 \times E_2$ actually gives you action of $G$ on $Z$.

That is the meaning of that this is well defined. Now, clearly under G action, the second coordinate of apoint does not change. Therefore, the fibers of $\pi_2$ are $G-$orbits. And they are precisely the orbits. Because$(e_1, e_2), (e_1', e_2) \in Z$ implies $f_1(e) = f_2(e_2) = f_1(e_1')$. That means $e_1, e_1'$ are in the same orbit. And hence there must be a $g \in G$ such that $ge_1 = e_1'$ G because they are in the same orbit.

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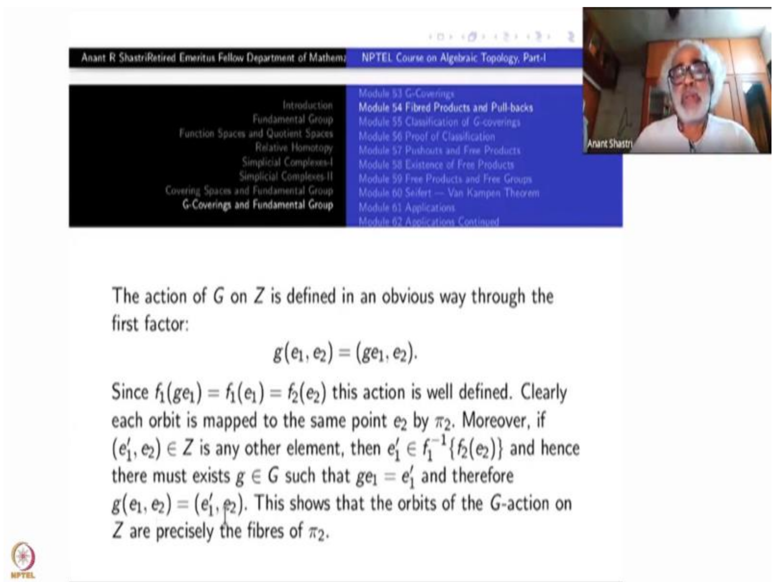


That just means that $g(e_1, e_2) = (e_1', e_2)$. So, this shows that the fibers of $\pi_2$ inside $Z$ are precisely equal to the orbits $G$ action on $Z$. This just means that this map $\pi_2$ is the quotient map. Well tell me that the whole thing is quotient map but is the covering projection already that we have already seen. A covering projection is a open quotient therefore it is a open surjective that holds a quotient map. The fibers are correct but why it is a quotient map requires some explanation. But we have already seen that it is a covering projection.

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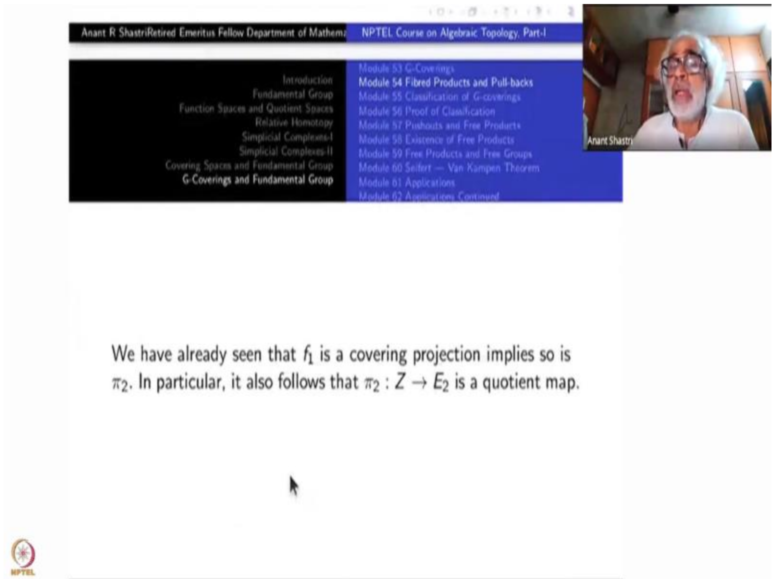


There are slightly different ways of seeing that quotient that does not matter. So, what we have seen is that pi 2 from Z to E2 is actually corresponds to the covering projection by G action.

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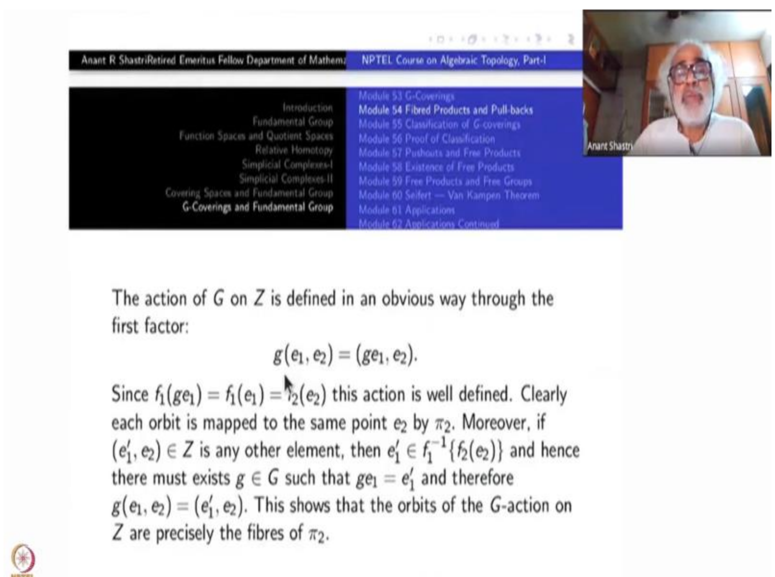


Look at the way we have defined, defined this action here, G of E1 E2 is G E1 E2. What happens to when you come to pi 1, pi 1 of this is pi 1 of G E1 ok which is G of just G E1. This pi 1 we see are the just G E1 which G of E1. On this side it is E1. So, pi 1 of E1 E2 operated by G is the same thing as G E1. So, this very definition shows that the projection map to pi 1 is a G action, G map.

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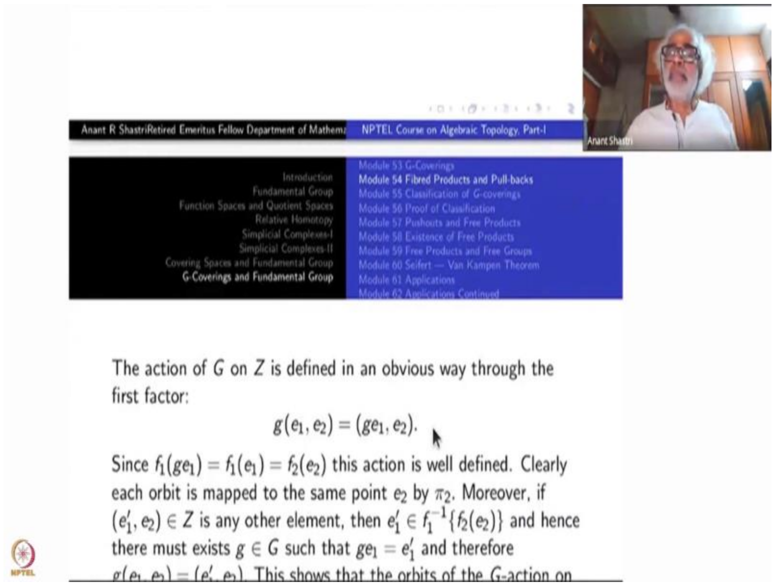


So, this was last part here, projection of G map. So, I have taken that as a key, as a clue to define this action and it works, all right. So, these are the basic things that we need to explain our classification of G coverings. So, this is going to produce large number of G coverings. If we have one G-covering and any map into the base place, you can construct the G-covering on the other space.

So, this procedure is precisely called, has a different name which is non-symmetric in the definition, in the idea. Because the role of f1, all properties of f1 reflected in pi 2, f2 may not have any of these properties maybe just a continuous function. That is why the role is important though in the definition of the fibred product f1 and f2 does not matter, which one does matter, that is the beauty of this.

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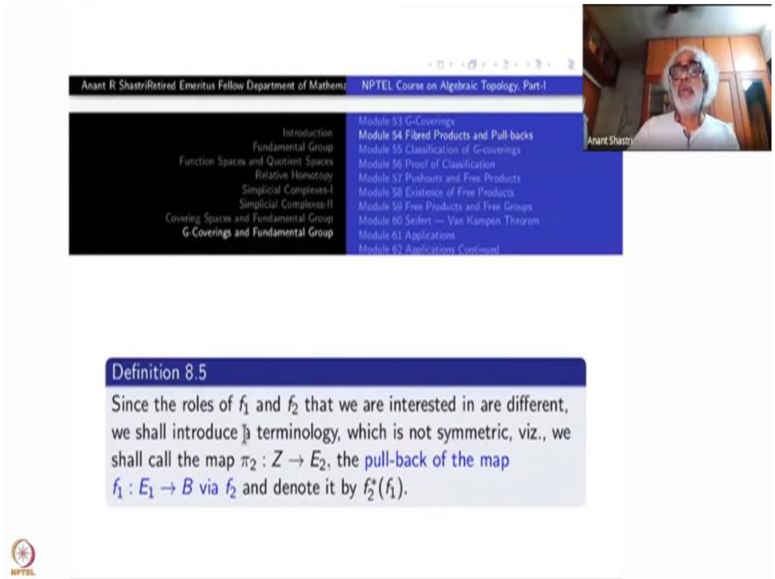


So, we introduce a non-symmetric wording here, namely this is the point f1 f2 though the fibral product is symmetric in that, the role of f1 f2 are different. So, we shall introduce a terminology which is not symmetric in that sense, namely we shall call this map pi 2 from Z to E2 and the pull-back of the map f1 E1 to B via f2 and denoted by f2 star of f1. You see this is a categorical notation which does not depend on the construction.

Here we have heavily used the construction. But all these things can be done in categorical language without appealing to the construction. That will be taking you little more deeper. We do not have any time for that. Here we use the easy set theory which gives you all these answers easily. So, this map will be called the pull-back. The corresponding covering projection will be called pull-back covering projection. So, if you start with the G-covering E to B, I will going to use a different notation now. E to B is a G-covering.

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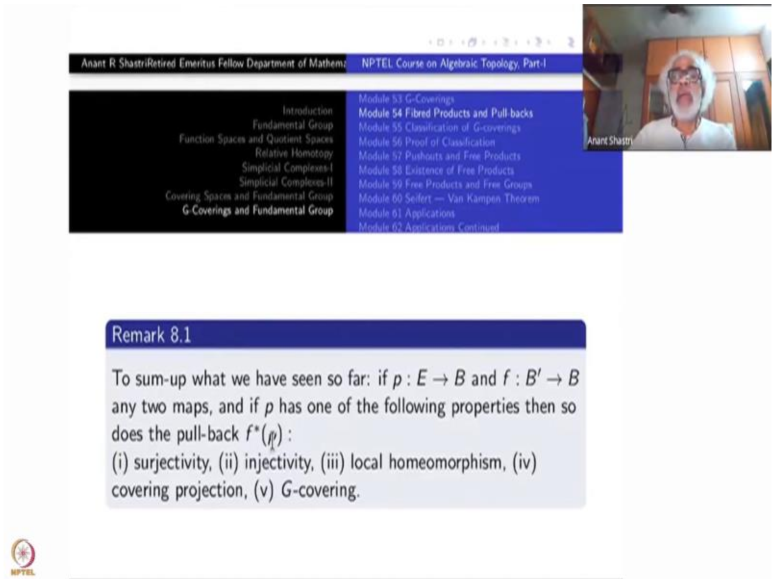


There is a map, there is a map F from B prime to B, then I get a G-covering on B prime which I am going to write as f star of P. P is a G-covering, f star of p will be G-covering. What we have seen is if p is surjective, injective, local homeomorphism, covering projection, G-covering etc. the same is true for f star. Thank you.