Introduction to Algebraic Topology (Part-l)
Professor Anant R. Shastri
Lecture 53
G-coverings

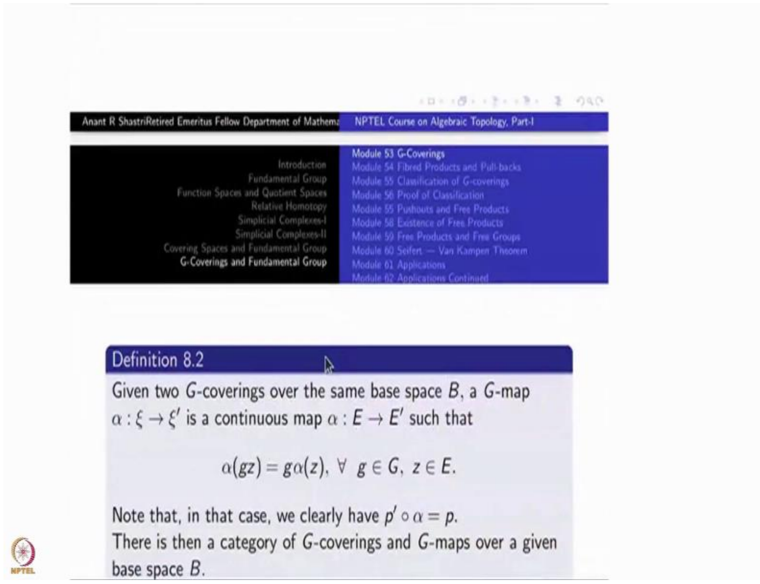
(Refer Slide Time: 0:20)

Today, we start a new topic G-coverings. As we have told earlier the covering space theory has 3 main points of view. Among them the viewpoint of group action is the most ancient. Due to people like Grothendieck this ancient point of view has becoming the forefront now. In this last chapter for this course, we should exploit this viewpoint and reap a wonderful harvest. Among these, proofs of various forms of Seifert-Van Kampen theorem is the foremost. Let us introduce some convenient terminology this time bringing out the group actions in fore-front.

(Refer Slide Time: 1:23)

By a G-covering we mean an ordered triple consisitng of total space E, a projection map p, a bottom space B, wherein p is a covering projection, and this map p is a quotient map of an even action of the group G on E. We have already seen that whenever a group G acts evenly on a topological space through diffeomorphisms the quotient map is a covering projection. So far, we have been studying covering projections without much regard to the group action. Now, we want to bring the group action in the forefront. In principle, all the examples that we have discussed are G coverings, they come out of some group actions.

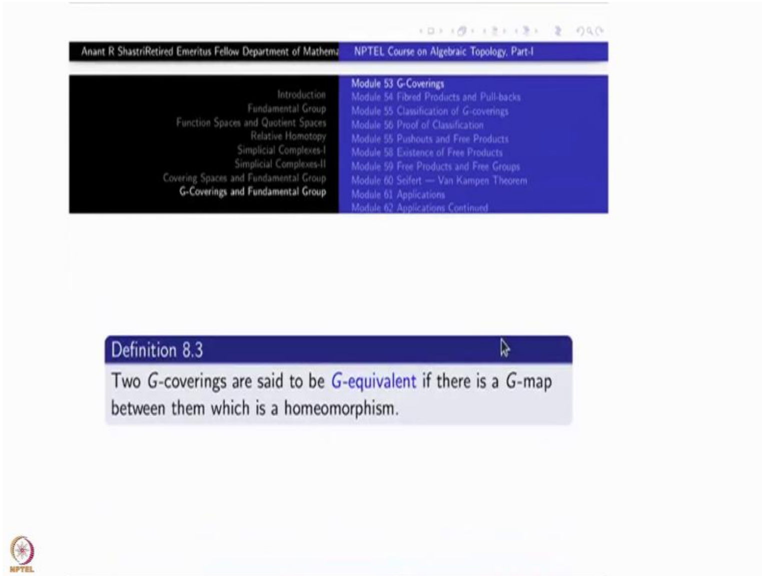
(Refer Slide Time: 2:37)



So, let us just make a few definitions here. Suppose you have two G-coverings first one is (E, p, B) and another one is (E', p', B). So, I am denoting them by ξ, ξ' respectively. Earlier, a map between them was just a map $\alpha : E \to E'$ which `commuted' with the projection maps, viz., $p' \circ \alpha = p$. So, what we are taking here is the base space is the same, the map should be such that it is respecting the G-action on both sides, i.e, it is G-equivariant map; $\alpha(gz) = g\alpha(z)$. This should happen for every $g \in G, z \in E$. Such a thing will be called a G-map. Once it is a G-map, automatically it happens that $p'\alpha = p$. Why?

Because both p and p' are quotients, they are taking the equivalence classes by the G- action to the same element, they are quotients given by the G-actions. So, automatically onece α is respecting the G-action, it follows that $p' \circ \alpha = p$. You can talk about another map say $\beta : \xi' \to \xi'' = (E'', p'', B)$, then the composite $\beta \circ \alpha$ will be also G-map. This makes it into a category whatever it is. So, this you have to just remember that you can take composites and identity map is there; and the composition is associative. These are the basic things that make up a category.

(Refer Slide Time: 4:50)



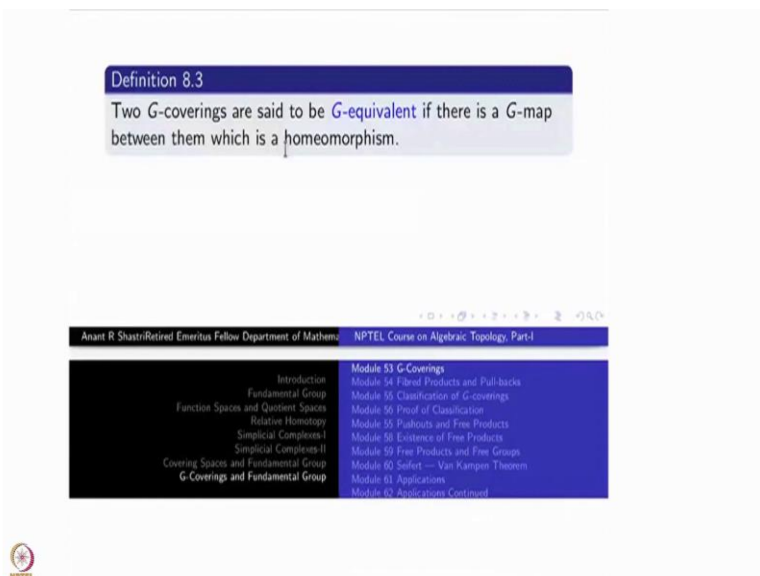
Now, two G-coverings will be called G-equivalent if there is a G-map between them, which is a homeomorphism. If it were not a G-map, just a homeomorphism then remember that was the meaning of covering equivalence, equivalence classes of coverings have been studied thoroughly and we have even classified them earlier, classification covering projections was the topic.

Now, we are putting extra condition namely the covering transformation that we are taking must be respecting the action of G. So, it must be G-map then we call them G-equivalent. Clearly, two G-maps are G-equivalent then as covering maps also they are equivalent. G-equivalence relation is a stronger equivalence relation.

(Refer Slide Time: 5:52)

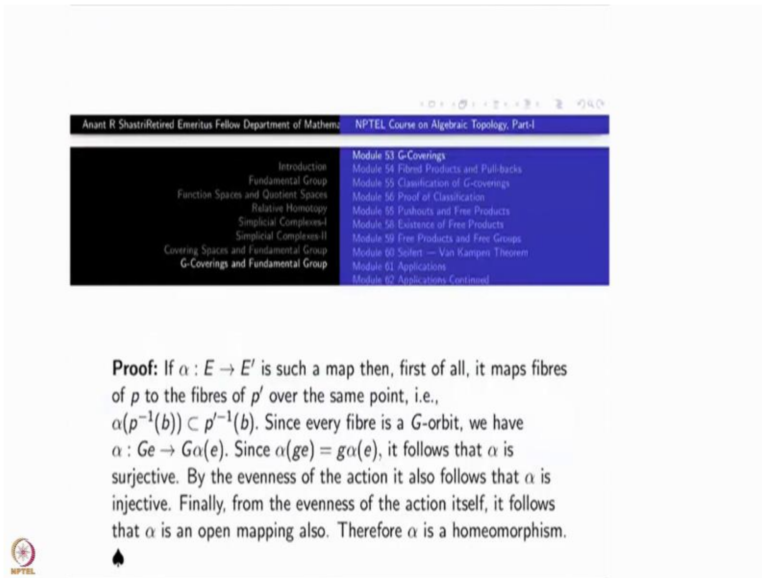
But here is a somewhat unexpected gift you can say unexpected and very important. Namely, every G-map of G-coverings (the base space is the same remember all the time over a single base space) is automatically a G-isomorphism.

(Refer Slide Time: 6:19)



You do not have to assume, in the definition, that it is a homeomorphism, this is not necessary at all, automatically, it is homeomorphism that is the meaning of this. Well not only that, once it is a homeomorphism, the inverse is there, inverse is also a G-map. That is very easy, because that is algebra. If a group homomorphism is invertible automatically the inverse is a group of homeomorphism. It is just like that. But why it is a homeomorphism? that is the beauty here, it is not very surprising, but it is a mild surprise. So, better to go through this proof.

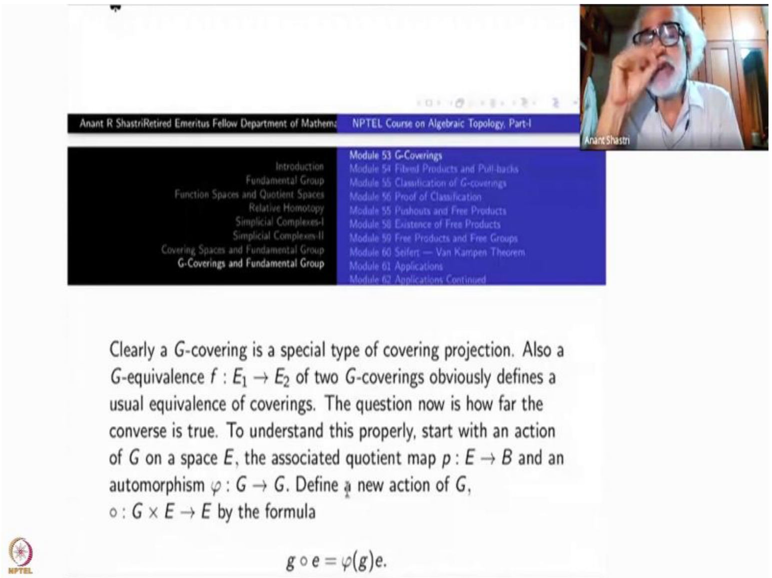
(Refer Slide Time: 7:06)



Let $\alpha : \xi \to \xi'$ be such a map. This is a G-map. As we have seen, once it is a G-map, it takes fibers to fibers, because the condition $p' \circ \alpha = p$ is the same thing as saying that $\alpha(p^{-1}(b)) \subset (p')^{-1}(b), b \in B.$ Since every fiber is an orbit, for both p and p', this is the same as saying $\alpha : Ge \to G\alpha(e).$ Since $\alpha(ge) = g\alpha(e)$, it follows that α is surjective.

But now, action is even also. That will tell you alpha is injective also. Because, suppose $\alpha(g_1 e) = \alpha(g_2 e)$. That means $g_1\alpha(e) = g_2\alpha(e).$ Since the action is fixed-point-free, $g_1 = g_2.$ This means that what we have got here is a bijection. Finally, the evenness of the action also tells you that α an open mapping. Since both $p : E \to B, p' : E' \to B$ are ccovering projections, given any point $e \in E$ you can choose an open set $U \subset B$ around $p(e)$ such that U is evenly covered by both p and $p'.$ It then follows that there are open sets V in E around e_1 and $W \subset E'$ around $\alpha(e)$ such that $p : V \to U$ and $p' : W \to U$ are homeomorphisms. Since $p' \circ \alpha = p$, it follows that $\alpha : V \to W$ is a homeomorphism. Since this is true for all $e \in E$, openness of α follows.

(Refer Slide Time: 9:22)

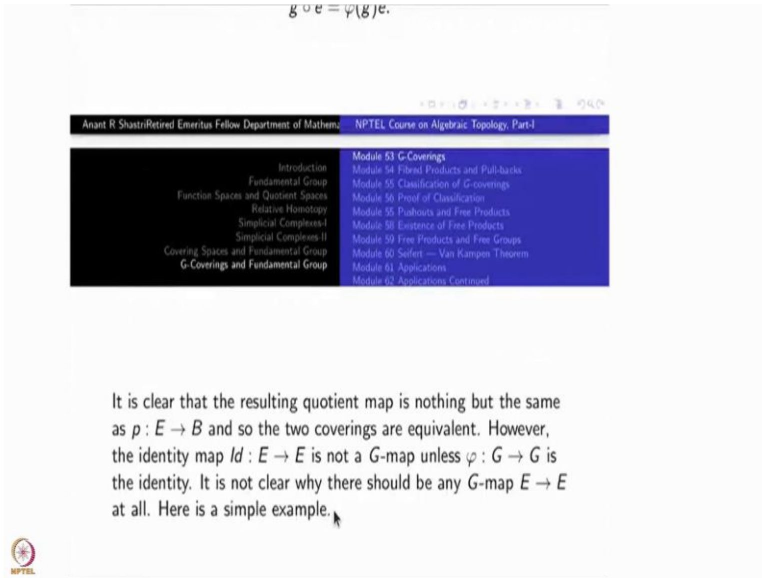


Clearly a G-covering is a special type of covering projection. What I mean to say is it is a covering projection but with extra structure, that is all. So, also a G-equivalence from E 1 to E 2 obviously defines usual equivalence relation. The two coverings are equivalent if they are G-equivalent already, but the other way around may not be true.

The question is now how far the converse is true. To understand this properly, let us do some artificially looking construction here, but that seems to be the final answer. So, let us see. Start with an action of G on a space E, associate the quotient map $p : E \to B.$ Now, you take an automorphism of G, a self-automorphism.

Define a new action of G on the same E by this formula, namely, g of surc e, I am using a different notation here: $g \circ e = \phi(g)(e).$ The action on the right hand side here is the given action, but after taking $\phi.$

(Refer Slide Time: 11:07)

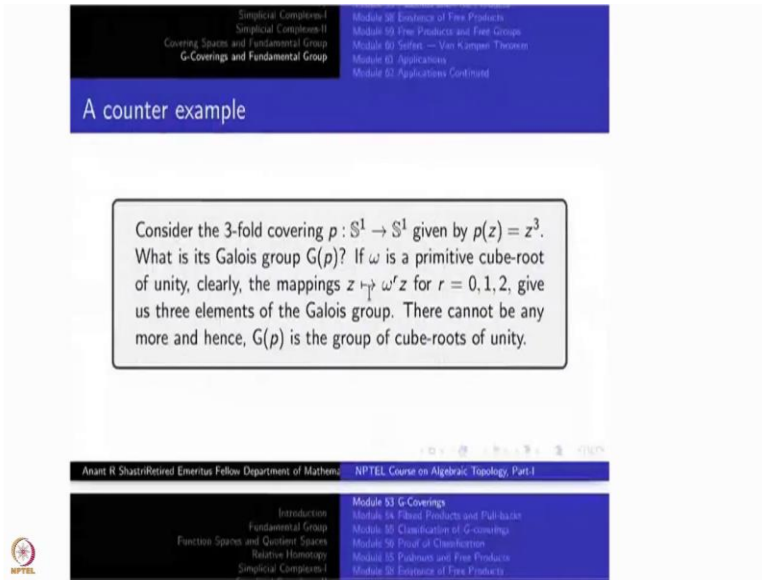


Now, the quotient map is the same quotient map p, because the orbits of the two actions are identical, for $e_2 = ge_1$ iff $e_2 = \phi(\phi^{-1}(g))e_1 = \phi^{-1}(g) \circ e_1$. Therefore, the orbits under the two actions are the same. Therefore, the quotients are the same, the topological space E was the same, quotient is the same, orbit is same, so B is the same, the map is the same.

But I am thinking of G acting on once this way and once that way, are they are different. Are they really different? I should check whether there is a map from this G covering to that G covering which is in G-map. If I find one such, then the two coverings are G-equivalent; they are the same upto a G-equivalence.

The strange thing is there may not be any such equivalence. For example, you think everything is the same. So, maybe identity map itself will be G map. Check that identity map from $E \to E$ is a G-map iff $ge = g \circ e = \phi(g)e$. By the fixed point freeness of the action, it follows that $\phi(g) = g$. Thus the the automorphism ϕ must be identity. So, identity map is not a G-map whenever ϕ is different from identity but there may be some other. So, the answer is not clear; that maybe some other map. So, finally, I wuld like to give a very simple example, wherein no covering transformation will be there which is a G-map, so let us see that example. So, let us see that example.

(Refer Slide Time: 13:47)

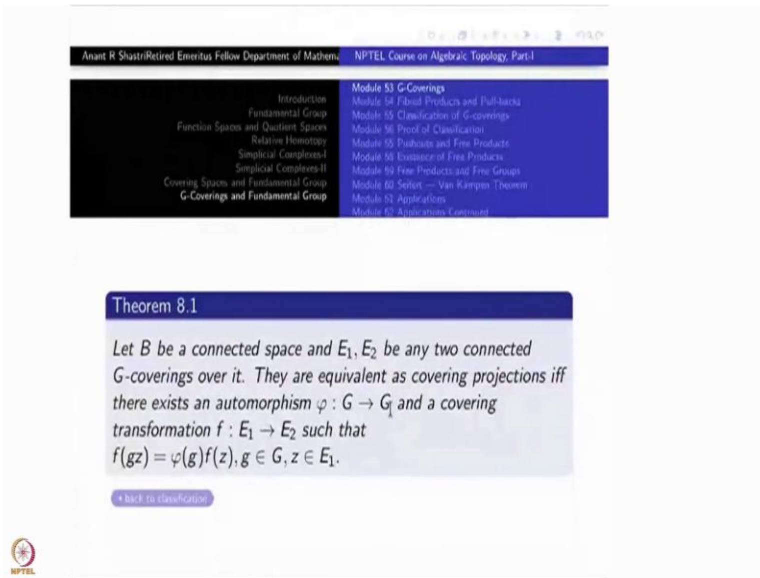


Take the 3-fold covering $p : \mathbb{S}^1 \to \mathbb{S}^1$, given by $p(z) = z^3$. What is the action here? Action is by the cube roots of unity, $G = \{1, \omega, \omega^2\}$ which is a group of order 3; the actionis defined by $(\omega, z) \mapsto \omega z$. What is the Galois group p? What are all the covering transformations? Obviously, they are nothing but $z \mapsto z; \quad z \mapsto \omega z; \quad \& \ z \mapsto \omega^2 z;$ So, these are the three different covering transformations. There cannot be anything more because the order of the covering itself is 3. The fibre of p has only 3 elements. We have proved once that the group of covering transformations injects into the fibre. Therefore, there can be at most 3 such covering transformations and we have already produced 3. So, it must the full group, the Galois group is exactly equal to the group G. So, we know all the covering transformations. Now, you just check that none of them is a G-map and are what I am going to produce that namely, I have to take a automorphism of the few groups of unity, group of 3 elements.

Now the group G has precisely one nontrivial automorphism, viz., $\phi(1) = 1; \ \phi(\omega) = \omega^2; \ \phi(\omega^2) = \omega.$ Let E, E' be the two G-coverings with Id, ϕ as G-actions respectively. It is easily checked that none of the three covering transformations we have above will be a G-map from E to E'.

Therefore, the same covering transformation can be thought of as a G-covering in more than 1 way. This was totally ignored in the usual study of covering transformations that we have done so far. So, why this is so, important is precisely the question here that we are going to study. Namely, homomorphisms of one group to another group, will be taken care here. The defintions takes care of automorphisms of the same group.

(Refer Slide Time: 17:27)



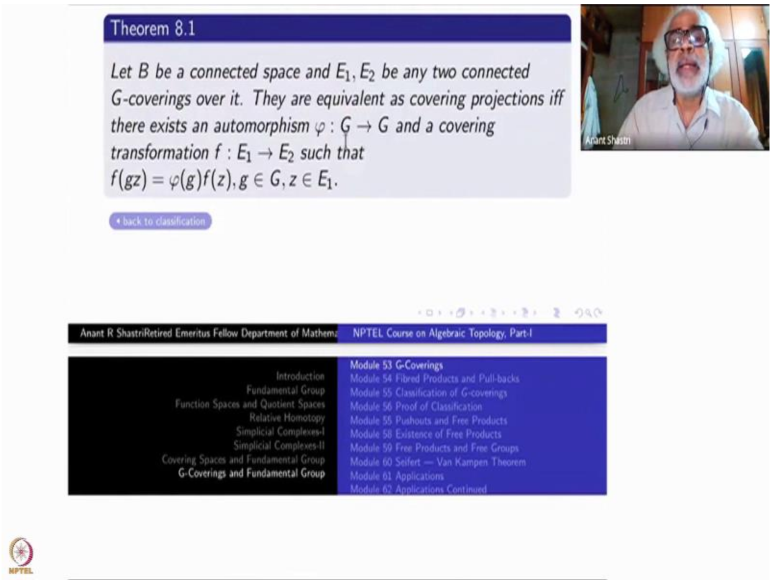
So, the first simple example, counter example, you have given this is the key, namely the next theorem says so, that is what I have completely given here. This theorem says that nothing else will be wrong this is all that is going to happen in the case of the same covering transformation, same covering transformation here, same covering projection you have taken and only action could be different.

How they are related in what way they are related is precisely stated here. Namely, start with a connected space B and connected coverings E_1 and E_2 which are G-coverings. They are equivalent as covering transformations if and only if you have an automorphism ϕ of G and a covering transformation $f : E_1 \to E_2$ such that this f becomes a G-map after you take the action on E_2 with a twist by ϕ.

On the right side, you have to treat E_2, you have to take a different action, what is that action, it corresponds to an automorphism. Both E_1 and E_2 are given a G action. So, if we take f of gz here equal to g of fz, that then that would have be a G-map of that but what we get is phi g of fz and this phi g is an automorphism. So, this is the theorem. This is the, this is the difference between covering transformations which are both G-coverings on a connected space and that is all. E 1 E

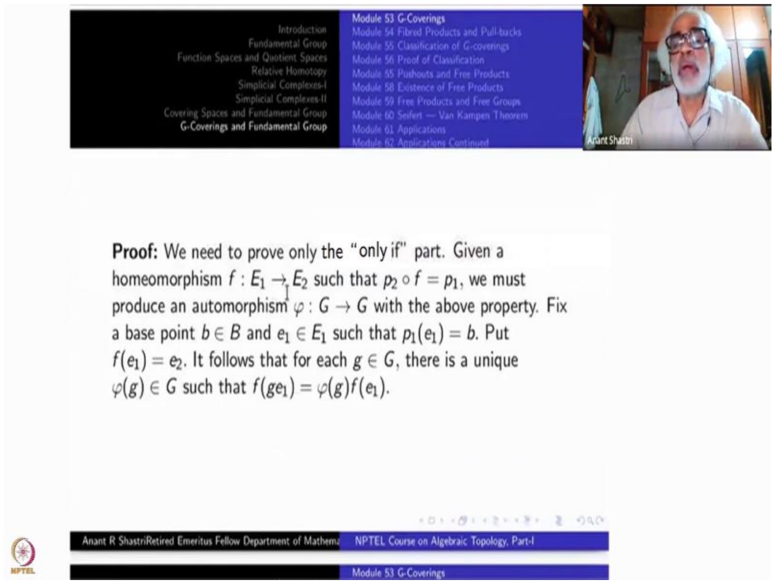
2 connected, B is connected, let us prove this one. And that gives some kind of satisfaction for introducing G-coverings.

(Refer Slide Time: 19:48)



Obviously, we have to proof only the if part here, sorry, we need to prove the `only if' part only. Once this is satisfied automatically it is covering transformation. So, start with a covering transformation $f : E_1 \to E_2$ and produce the required automorphism ϕ.

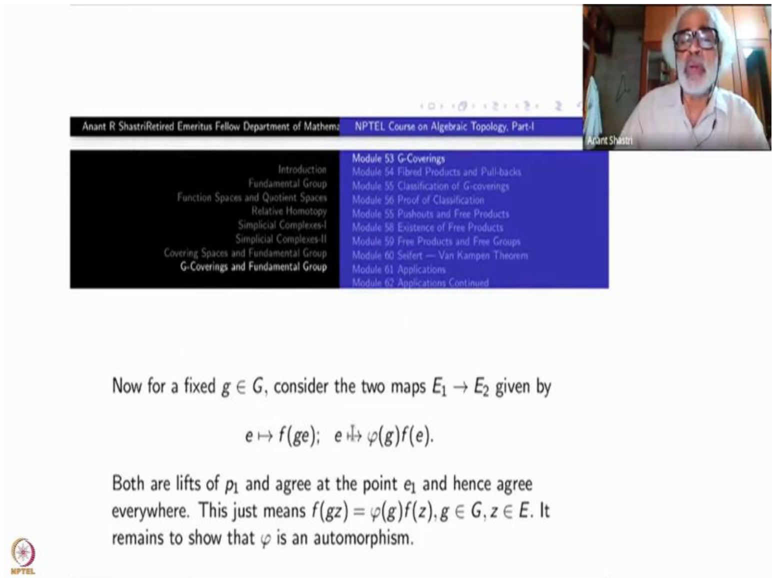
(Refer Slide Time: 20:04)



So, given a homeomorphism $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$, we must produce an automorphisms $\phi : G \to G$ with the property that $f(gz) = \phi(g)f(z)$. So, fix base points $b \in B, e_1 \in E_1, e_2 \in E_2$ such that $p_1(e_1) = b = p_2(e_2)$ and $f(e_1) = e_2$. This much is fine.

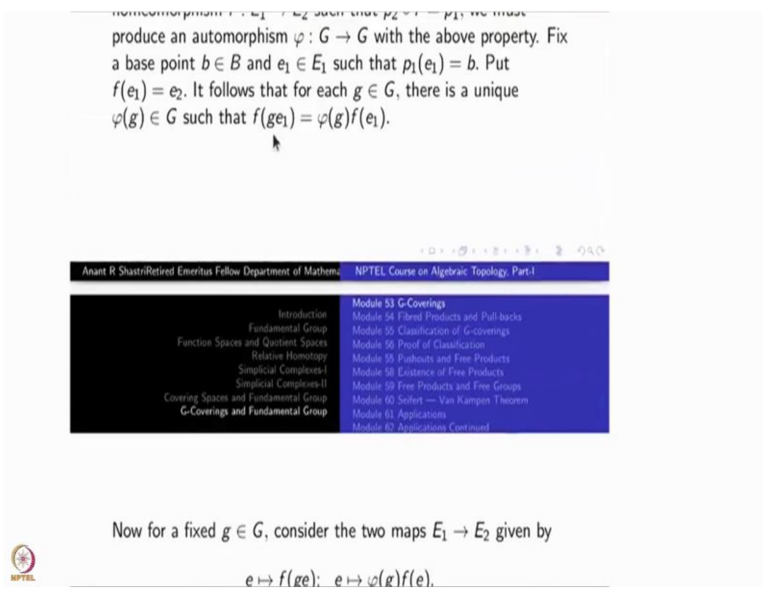
Now, it follows that for each $g \in G$, there is a unique $\phi(g) \in G$ such that $f(ge_1) = \phi(g)e_2$. I am going to define $\phi : G \to G$ by this rule. e_1, ge_1 are in the same fibre and hence $e_2 = f(e_1), f(ge_1)$ are also in the same fibre. Therefore the element $\phi(g)$ exists and is unique bcause of the fexed point free action of G. So, this $\phi : G \to G$ is well-defined as a function. So, $f(ge_1) = \phi(g)e_2 = \phi(g)f(e_1)$. This is happening at one single point, namely, the point that we have chosen as the base point. Just like in all other covering space theory, this will tell you the function will have all the required properties.

(Refer Slide Time: 22:18)



Now, for a fixed g, consider the two maps $E_1 \to E_2$ given by the two rules: $(i) e \mapsto f(ge); (ii) e \mapsto \phi(g)f(e)$. What I am doing here?

(Refer Slide Time: 22:33)

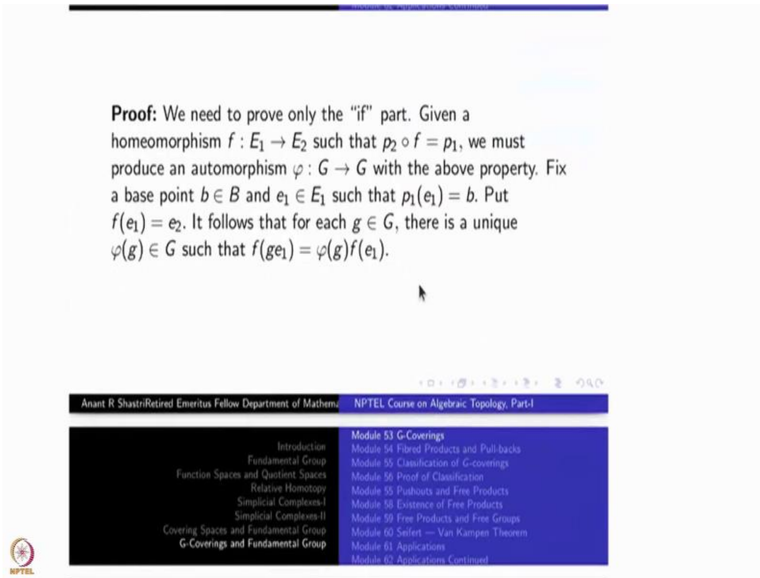


First, I fixed e 1 and got a $\phi(g)$ for each $g \in G$. Now, I keep this $g \in G$ and hence $\phi(g) \in G$ fixed and vary the point $e \in E_1$. I get two maps, let us call them $\alpha : E_1 \to E_2; \beta : E_1 \to E_2$.

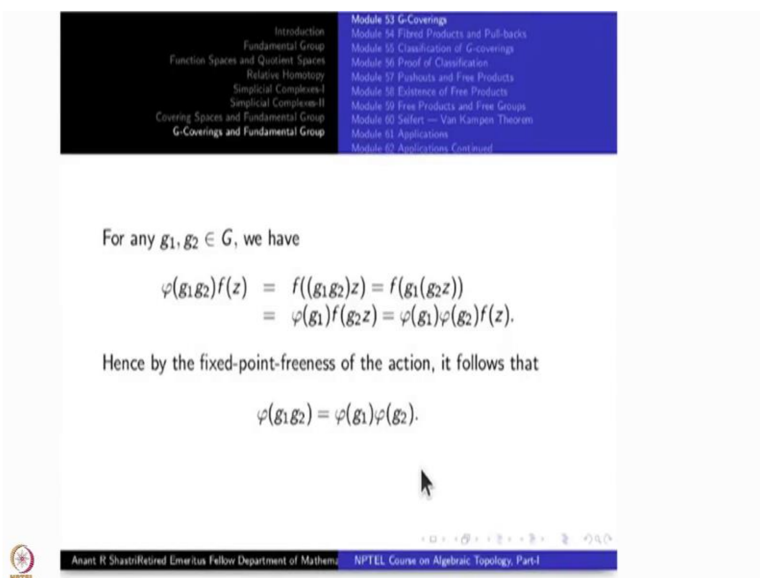
Both are the lifts of p_1 and agreeing at a point. There is a p 1 here, there is a p 2 here to B, you check that $p_2 \circ \alpha = p_1; p_2 \circ \beta = p_1$. They are lefts of the same map p_1. They agree at one point;

$\alpha(e_1) = f(ge_1) = \phi(g)e_2 = \beta(e_1)$. Therefore, they must agree everywhere. Why because we have assumed that E_1 is connected.

(Refer Slide Time: 24:57)

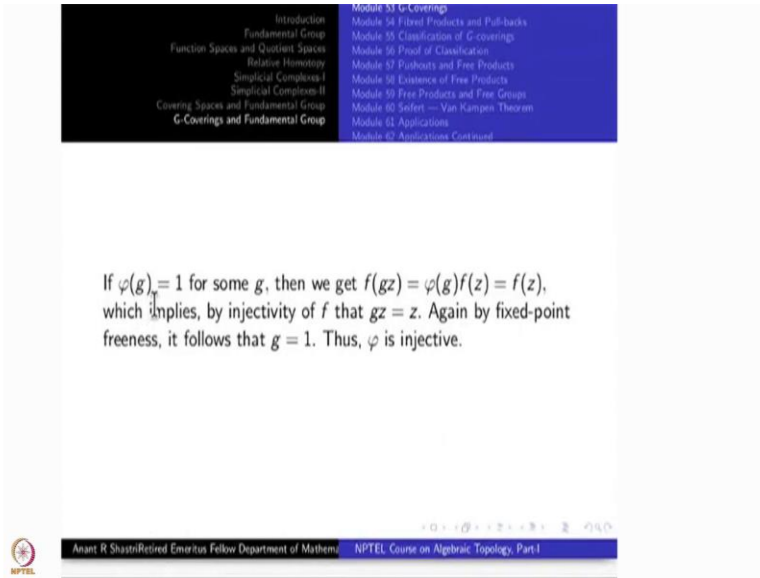


So, they agree everywhere. That means $f(ge) = \phi(g)f(e)$ for all $e \in E_1$. Since $g \in G$ is fixed arbitrarily, this is true for all $g \in G$. So, f becomes a G-map, may be you can call it $\phi(g)-$ map. Except that we have got $\phi : G \to G$ only as a function, and we have yet to to verify that it is an automorphism of G.



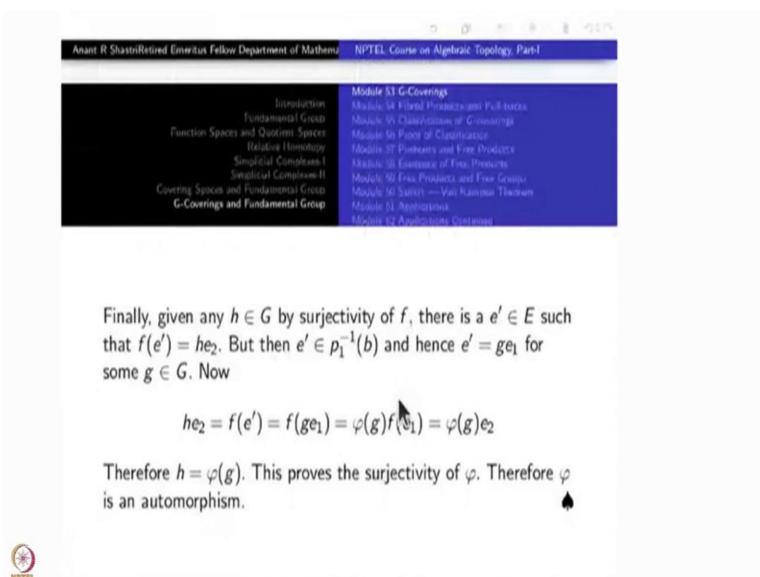
Let us prove that ϕ is a homomorphism first. So, $g_1, g_2 \in G$, we have to show that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. So, start with $\phi(g_1 g_2)$ operating upon $f(z)$; this is equal to $f(g_1 g_2 z)$. But $g_1 g_2 z = g_1(g_2 z)$, by associativity. By the property of f again, $f(g_1(g_2 z)) = \phi(g_1) f(g_2 z)$. Now, again apply the property of f, g_2 will also come out now, so, that will be equal to $\phi(g_1)\phi(g_2)f(z)$. Therefore $\phi(g_1 g_2) f(z) = \phi(g_1)\phi(g_2) f(z)$. Now, use the evenness of the action, in particular what you get is $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. So, that proves that ϕ is a homomorphism. It remains to prove that it is injective and surjective.

(Refer Slide Time: 27:23)



So, let us now prove that ϕ is injective. Suppose $\phi(g) = 1 \in G$ for some $g \in G$. Then by the definition of ϕ, we get $f(gz) = 1.f(z) = f(z)$. This implies, by the injectivity of $f(f$ is a homeomorphism) $gz = z$. But then the evenness of action of G on E_1, $g = 1$. That proves that ϕ is injective.

(Refer Slide Time: 28:04)



Now, finally, to show that ϕ surjective, what you want to do is the following. Given any $h \in G$, by surjective of f, there is an $e' \in E_1$ such that $f(e') = he_2$. But then e' will be also in $p^{-1}(b)$

because $p_1(e') = p_2 \circ f(e') = p_2(he_2) = b.$ Therefore, e' itself will be equal to ge_1 for some $g \in G.$ I started with some e' which goes to he_2 under f, but this e prime must be in the same fibre as e_1, therefore, $e' = ge_1$ for for some $g \in G$. Therefore, $he_2 = f(e') = f(ge_1) = \phi(g)e_2.$Now, again by evenness of the action, we conclude that $h = \phi(g).$ So this proves surjectivity of ϕ and therefore, $\phi : G \to G$ is an isomorphism. Hence this completes the proof of the theorem. Thank you.