Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Lecture 52
Examples

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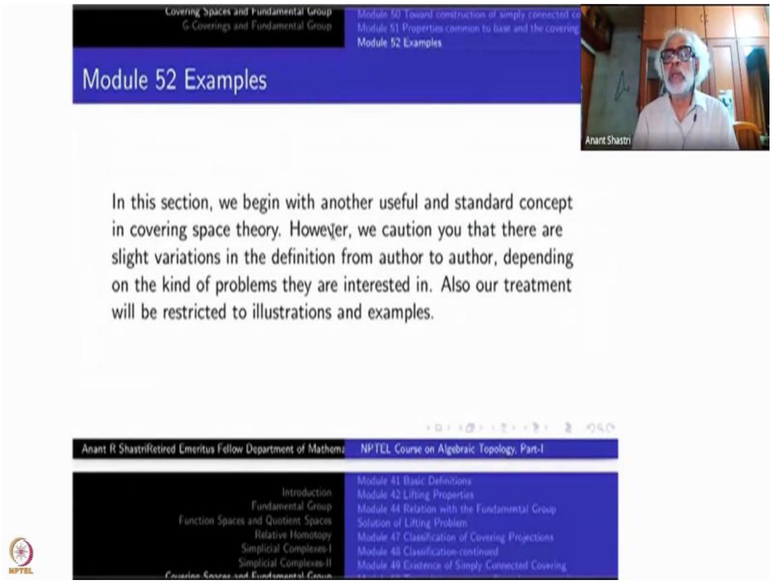

Today's topic is one special concept in covering space theory which is again very classical, we are only touching just the definition and a few simple examples, even the definition may vary from author to author. So, this part we are doing just by examples not much deeper study of this. The simplest model is when you are studying subgroup and its action of the group. The coset representation helps you a lot, this is the kind of thing that we want to do in topology also when a group is acting on a topological space. However, the analogy stops there, we have to bring in more topology than choosing and arbitrarily picking up coset representatives. So, let me stop making comments. Let us first go through the definition and then see that there are a few things which make sense.

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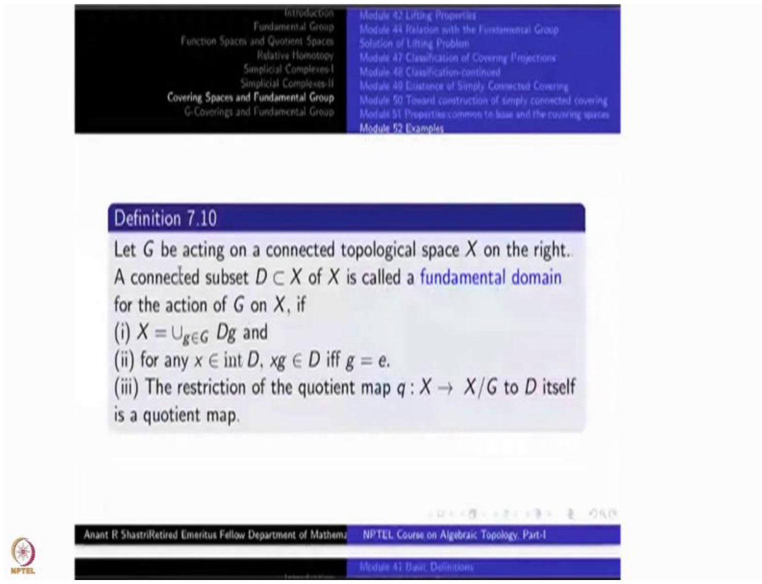

Start with a group G acting on a connected topological space X. Let me just take for the definite sake, that the action is on the right. Then a connected subset $D \subset X$ is called a fundamental domain, if the following happens, namely, (I) X is the union of all the translates of D, (D is a subset of X, you take all right translates Dg, $g \in G$; they must be covering the whole of X. This is similar to the choice of right cosset, but there is no disjointsness here. (ii) The second part brings a little bit of disjointness, namely for any x in the interior of D, $xg \in D$ implies $g = e$must be identity. In other words, the translates of the interior of D are disjoint. (iii) The third point is that if you restrict the entire map the quotient map $q : X \to X/G$, restricted to the domain D that itself must be a quotient map.

So, now we can see what is the idea, the idea is to cut down the top space X to something manageable, something smaller. If we insist on coset representations like they are all disjoint that is not possible because X itself is connected and that is not desirable either. So, we allow minimal overlapping namely in the interior, there should not be any overlap. On the boundary there can be overlap and that will actually happen.

The important thing is that since X is connected, we insist that D is connected. So, now I want to tell you that the definitions may slightly vary from author to author and situation to situation. For example, if X/G is compact then you may want to choose D to be compact. Secondly, there is no uniqueness in the choice of D, each person, depending upon the problem at hand whatever he/she has to study, may choose the fundamental domain differently. So, let us just study a few examples how it helps to understand the quotient space and the action of X on, action of G on X.

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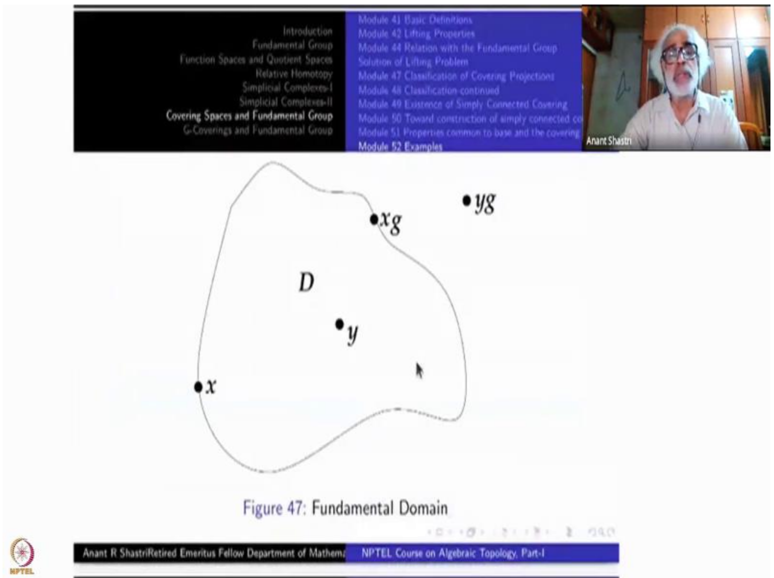

So, here is the picture which shows that if x is in the boundary of D, its translate maybe also in the boundary. But if y is in the interior then its translates will not be in the interior. So, that is the this is a picture that is all.

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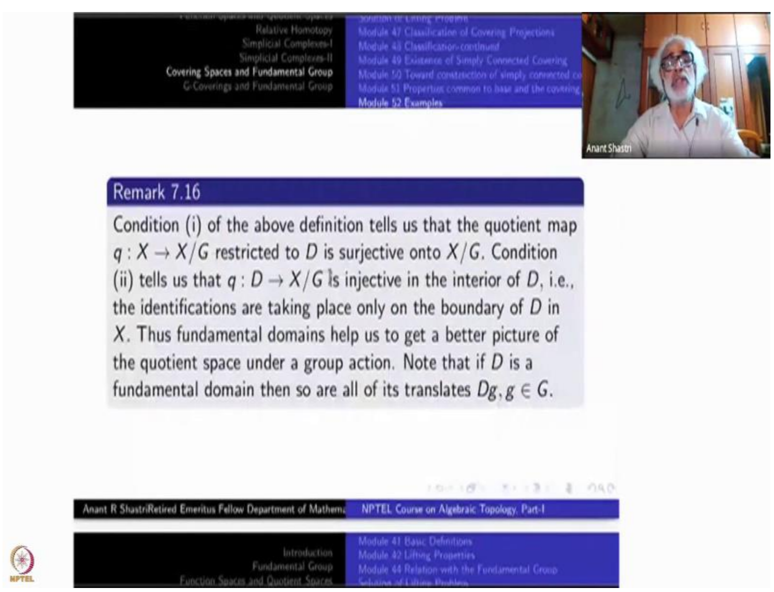

So, I have already told you condition one about, tells us that the quotient map is surjective because translates of this D cover the whole of X. So, D is a set of representatives which is like a coset representatives. Condition (ii) tells you that $q : D \to X/G$, D is injective in the interior of D. And the condition (iii) says that it is actually a quotient map if you take the whole thing.

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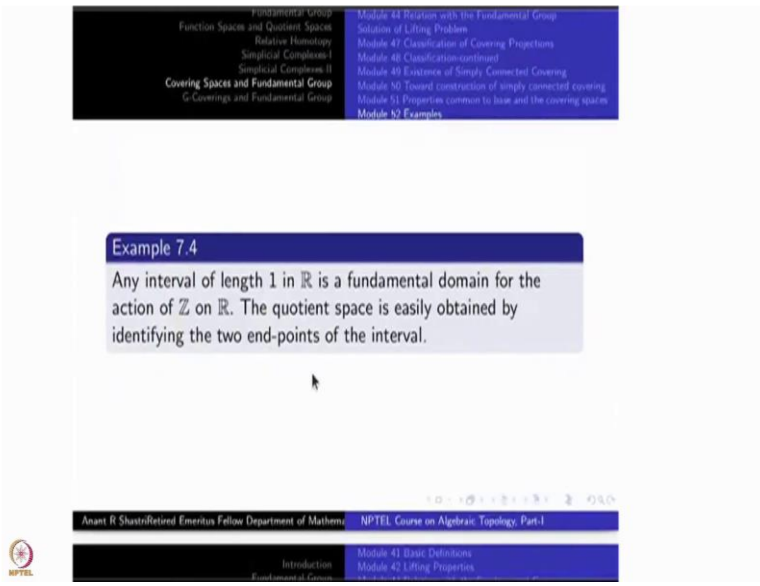

So, let us take an example, the simplest example, all these examples are more or less familiar to you. The first example is $exp : \mathbb{R} \to \mathbb{S}^1$ given by the action of the integers on $\mathbb{R}$ by translation. $\mathbb{R}$ is the group, and $\mathbb{Z}$ a subgroup of $\mathbb{R}$. So it is like a coset representatives.

So, what you want to do is you can take D to be any closed interval which we have been doing, any closed interval of length 1. Then in the interior, there will not be any identification, when you translate, any interior point will go outside the interval of length 1, but one boundary point may go the boundary point; for 0 will go to 1 when you add 1 . So, that is the only point of intersection between the translates of the interval, close interval and itself, either way, either add or subtract only one of them that intersect.

After that there are no intersection at all, this interval closed interval, this happens to be connected that justifies the domain, the word domain here is called a fundamental domain. But in this particular case, it is also compact, we did not bargain for that compactness, but because the quotient space which is the circle is compact, this was possible obviously, it is a fundamental domain itself is compact, the quotient which is a image of that will have to be compact, this is actually was this remark was used in proving that the projective space is compact, let us come to that example then I will explain it again.

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So, here is a comment that says that most of the very interesting examples come similar to this result contained inside R. Namely, what the, what are called Lie groups, then inside that Lie group you are taking a discrete subgroup. When you take discrete subgroup inside a Lie group the quotient becomes a covering space projection and then we can talk about choosing a fundamental domain there. To study what is happening to the action as well as the quotient space and so on.

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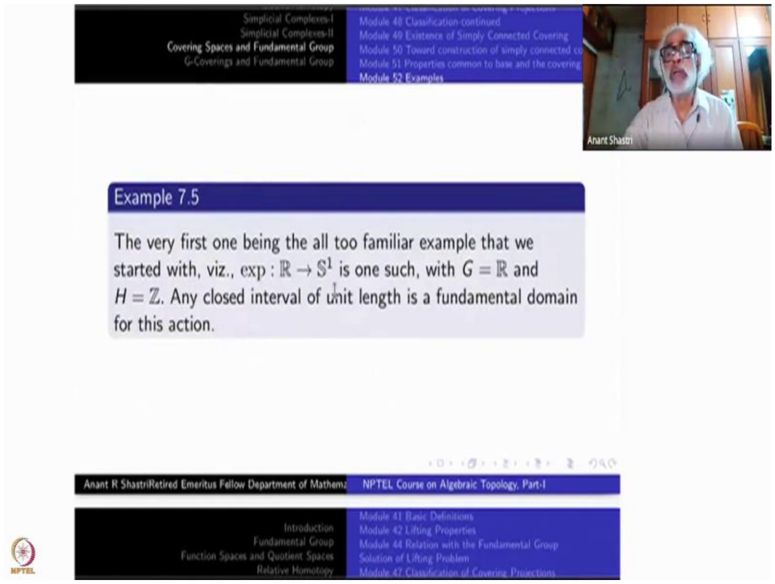

So, this is the prototype of that example, since we have not studied or we are not assuming any knowledge of the Lie groups and so on, we cannot pursue that angle more than that.

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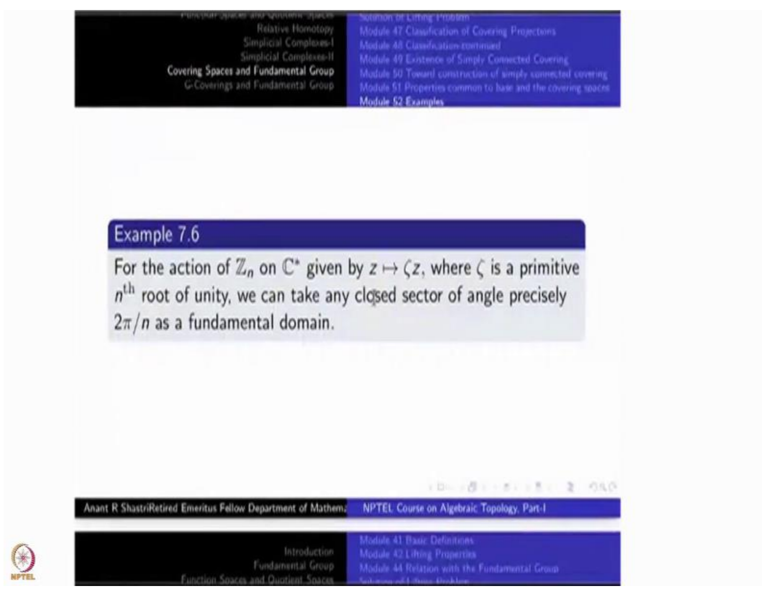

But we can take another simple example, simpler than the exponential function namely, $\mathbb{S}^1 \to \mathbb{S}^1$ or $\mathbb{C}^* \to \mathbb{C}^*$, namely $z \mapsto \zeta z$, where ζ is a primitive nth root of unity, which defines an action of the group $\mathbb{Z}/n\mathbb{Z}$. If $n = 2$, this action is nothing but $z \mapsto -z$.

If $n = 3$ you are multiplying by ω or ω^2 and so on. So, this is a group of order n, we have studied its action on $\mathbb{S}^1$. The quotient space is again $\mathbb{S}^1$. Same thing happens in $\mathbb{C}^*$ also--- the polar coordinate representative, the norm of the vector does not get affected because zeta is of unit vector, is of unit length.

So, what will the fundamental domain for this action on $\mathbb{C}^*$? You will have to choose a sector of angle $2\pi/n$. For example, take the positive real axis and then take another line passing through origin which makes an angle $2\pi/n$ and everything lying in between that is called a sector, all $re^{\imath\theta}, 0 \le \theta \le 2\pi/n$.

So, that forms a fundamental domain, in the interior will not be any identification, but on the boundary the whole line the X axis, part of the X axis, positive X axis is turned into the next line, you keep turning it n times you will come back to the real axis.

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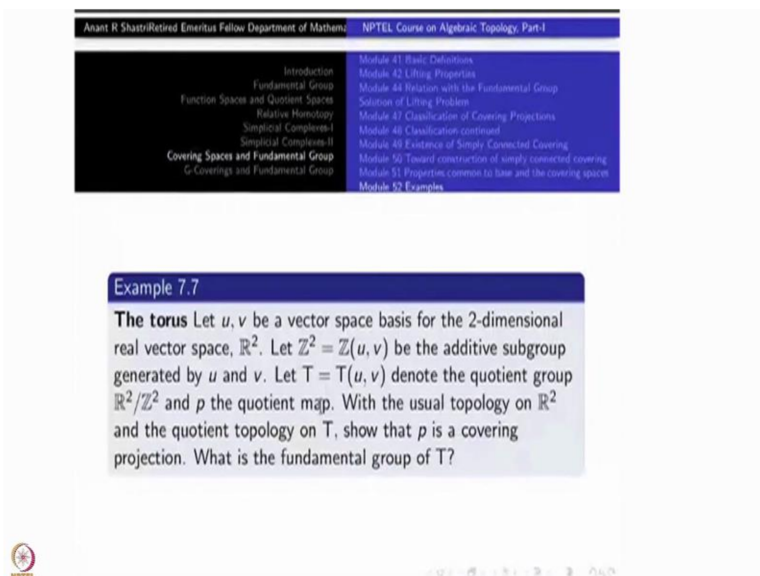

Next example is a little more interesting or a little more complicated. Start with the 2- dimensional vector space $\mathbb{R}^2$, pick up any two vectors u, v as basis for $\mathbb{R}^2$. Now, you take the subgroup generated by these two vectors, the abelian subgroup not the vector subspace. They generate the whole vector space $\mathbb{R}^2$, but we take the subgroup generated by these two elements. That will be a free abelian group of rank two. So, I am writing Z^2 for the set of all elements of the form $mu + nv$ where m, n range over all integers.

So, that happens to be a discrete subgroup, you can mark these things. Starting with u and v, make a parallelogram. So, you get 4 vertices of the parallelogram, keep translating this parallelogram both up and down right and left and so on. So, you get all those lattice points. So, that is the subgroup.

Quotient is just the quotient of one of these parallelograms, any one of the closed parallelogram is good enough to cover the entire thing and the quotient is again a covering projection indeed the quotient space is nothing but homeomorphic $\mathbb{S}^1 \times \mathbb{S}^1$ again. Just like if you have taken the the special case $u = e_1, v = e_2$ or any two perpendicular vectors.

The importance of this one is that no matter what your choice of u and v that they must be independent that is all. The quotient is always homeomorphic $S^1 \times \mathbb{S}^1$, the torus.

But remember that $\mathbb{R}^2$ can be thought of as a complex plane with complex structure. Then, there is a way to put a complex structure to the torus. The structure will depend on what vectors you have chosen, the basis vectors you have taken. Actually, it will depend upon just the angle between these vectors. Depending on the angle, choice of the angle, you will get different complex structures on the torus. They are all called elliptic curves. In fact, they are all smooth elliptic curves, they are all of them smooth elliptic.

This is a very classical subject and extremely important in other areas of mathematics also. Like this was used in solving Femat's last theorem also. And this is classical subject, which goes back to Weierstrass, Abell and so on, the study of the elliptic curves.

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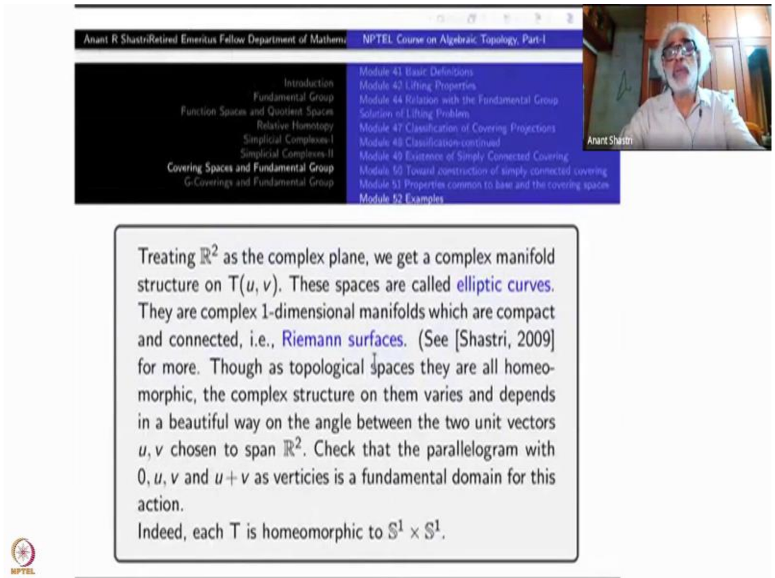

And it goes back to Riemann also. You know that these are examples of Riemann surfaces--not the first one, the first one is a sphere, the next set of examples, the simplest examples of surfaces surfaces with a complex structure.

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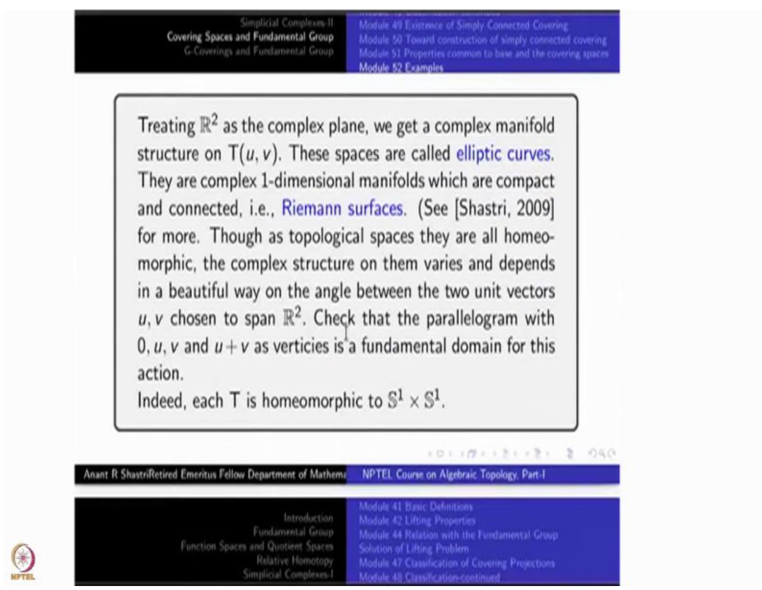

So, beyond that, I cannot touch this one, this maybe just a motivation to study these things because the study of elliptic curves itself is a very very deep subject one can study the whole thing for the entire of ones life.

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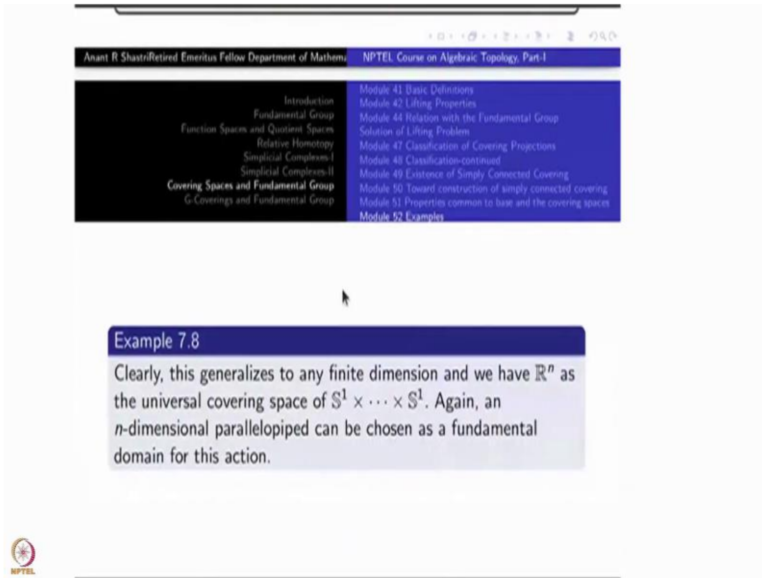

Now we have gone from $\mathbb{S}^1$ to $\mathbb{S}^1 \times \mathbb{S}^1$. We generalize this further to all dimensions $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. All that you have to do is to take n independent vectors $\mathbb{R}^n$, any basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$; that will give you a subgroup of $\mathbb{R}^n$, an free abelian subgroup of rank n, that will be a discrete subgroup. The quotient will be again a compact space and that is nothing but homeomorphic to $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Now, what will be the fundamental domain here? Once again you look at the box, the box given by all these vertices $0, v_1, \ldots, v_n$ etc, your n vectors and the origin. Just like a parallelogram in $\mathbb{R}^2$, you generate the box, parallelopiped, that will become a fundamental domain for this action. In dimension 3 it would be a, this is like a cube; if you take vectors to be perpendicular each other and of same length, then it will be actually a cube.

Another more familiar example which we have studied already is projective space, projective space $\mathbb{P}^{n-1}$, we have defined first as the quotient of $\mathbb{R}^n \setminus \{0\}$; but then you can restrict the quotient map to $\mathbb{S}^{n-1}$ to get a quotient map again. $\mathbb{S}^{n-1}$ is a fundamental domain in a trivial way because I has no interior in $\mathbb{R}^n \setminus \{0\}$. But $\mathbb{P}^{n-1}$ is again a quotient of $\mathbb{S}^{n-1}$ by the action of $\mathbb{Z}_2$. So, to get a fundamental domain for this, all that you have to do is cut it down the sphere in half, take only the upper hemisphere then the interior there is no identification, on the boundary X going to minus X is still identified that is allowed.

So, this was used in understanding the projective space inductively, I have discussed this one earlier. For example, in the case of n equal to one this will immediately tell you that the projective space p 1 is again homeomorphic to S 1 because then the fundamental domain is just an arc, the upper hemi circle and 1 and minus 1 are identified. When you identify a string, the endpoints are identified by single point what you get is circle.

And that will give you a picture of $\mathbb{S}^1$ and it will also give you a picture for $\mathbb{P}^2$. $\mathbb{P}^2$ is now nothing but a disk being attached to the circle via the map $z \mapsto z^2$, namely, z and $-z$ on the boundary are identified. So, that is the picture of $\mathbb{P}^2$. Unfortunately, though it is easy to describe, t you cannot construct a model remaining inside $\mathbb{R}^3$, because $\mathbb{P}^2$ is not embeddable in $\mathbb{R}^3$.

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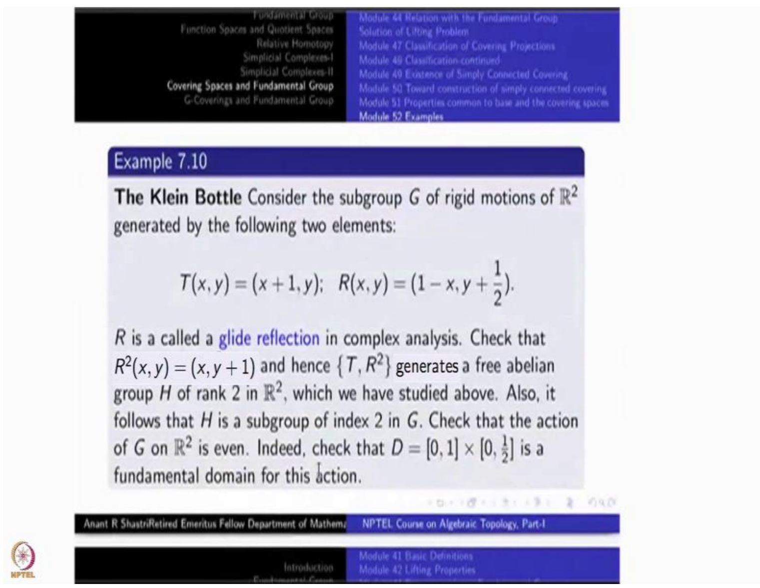

The next example is of more interest to topologists. So, Klein bottle is again a two dimensional thing. So, I will explain this one because it seems that many people have wrong conception of what a Klein bottle is. Consider the subgroup G of rigid motions in $\mathbb{R}^2$. What is a rigid motion? Which preserve the distance in $\mathbb{R}^2$ --- translations, rotations and such things. The group of rigid motions in $\mathbb{R}^2$, generated by following 2 elements. One is just the translation along the x- axis, $x \mapsto x + 1$ and y remains the same. The second one, the first variable is reflected in the point ½. So, it is given by $x \mapsto 1 - x$, and the second coordinate shifts y, ie., $y \mapsto y + 1/2$. So, this is called a glide reflection, gliding along the y-axis reflecting along a line parallel to the y- axis and passing through x=1/2. So, such things are called glide reflections in complex analysis, it is a rigid motion, it can be thought of as composed of two things namely one reflection and then and translation, reflection is also a rigid motion after all, only thing is it is not orientation preserving, so such thing is glide reflection.

Put $T(x, y) = (x+1, y); R(x, y) = (1-x, y+1/2)$. $G = \langle T, R \rangle$ the group generated by T and R.

So, look the rigid motion R^2. $R^2(x, y) = R(1-x, y+1/2) = (x, y+1)$. So R^2 is a translation along the y-axis. Hence, if you take T, which is transitional along x-axis and R^2 which is translation along the y-axis they will generate a subgroup H isomorphic to $\mathbb{Z}^2$. By iidentifying T, R^2 with the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, the group H can be thought of as a subgroup of $\mathbb{R}^2$. The quotient would be $\mathbb{S}^1 \times \mathbb{S}^1$.

But now, I am going to take the subgroup G generated by T and R; $H = \langle T, R^2 \rangle$ will be in a subgroup of that. So, now, I am going to consider the quotient of $\mathbb{R}^2$ by a larger group, of which $\mathbb{Z}^2$=H happens to be a subgroup of index two. So, it follows that H is a subgroup of index 2 in this G. Check that the action of G on $\mathbb{R}^2$ is even.

So, this is fairly easy, all that you have to do is take the neighborhood of any points say $(0, 0)$ for example, and take a small enough neighborhood, very small enough neighborhood just that either reflection or shifting does not intersect with it, that is all. So, details I will leave to you. So, check that the action is indeed I have written already. So, you can check that the rectangle $[0, 1] \times [0, 1/2]$ is a fundamental domain. So, that will give you a fundamental domain for this action.

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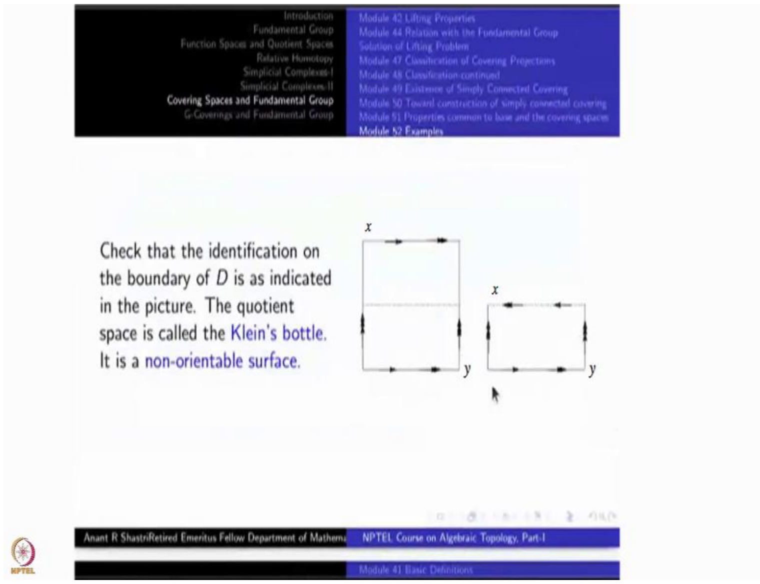

So, here is a fundamental domain for the action of H and the other one is a fundamental domain for the action of G. Only along the border there is some overlap and identifications. Look here, along the x-axis points are shifted by 1 in both the actions. Along the y axis what happens? when you use the torus action, clearly there is shift by 1. In the action of G howevr, shift is by laf and there is a rotation as well. So, what you have to do is when you come here, this will be 1 minus x, this is half, half will go to half, but 0 would have gone to 1, say one third would have gone to two third and so on, this is coming this way. So that is why I am coming this way, and I am going this way and I am coming this way.

Of course, engineering here have been different, but I am shifting also. So, a point here would have gone to point here and point here would have gone to point here on the line y could execute half this will be just shifting from here to here, action is like this, and this, this is shifted like this. So, identification is precisely this line segment is identified this one, after rotating not just like that, this one is identified as it is, you can perform this one first, then you get a cylinder, then you perform and indentifying this one, you can just bend it down and to do it like this, you have to bend it and go inside and glute lactase and that is why you cannot perform this one remaining inside $\mathbb{R}^3$. So, this is a Klein bottle.

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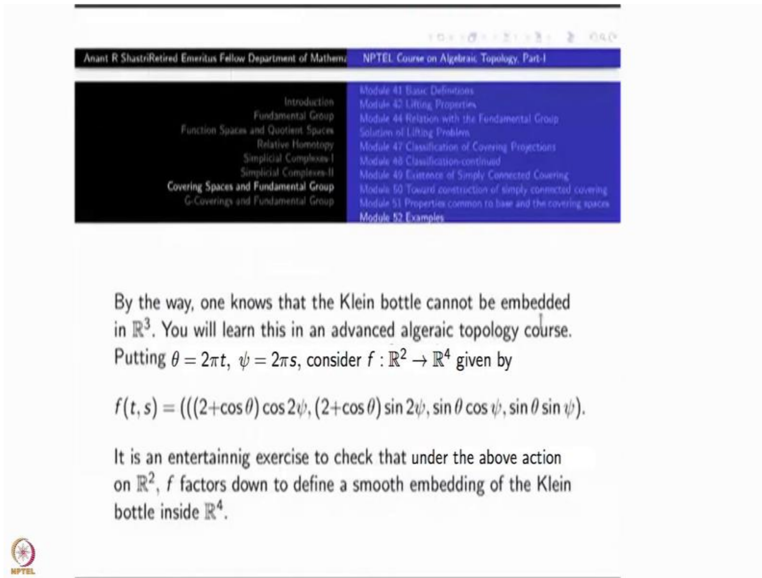

One knows that Klein bottle cannot be embedded in $\mathbb{R}^3$. you will learn this in an advanced algebraic topology course that it cannot be embedded in $\mathbb{R}^3$, it is not a part of this course, you cannot handle that one. Put $\theta = 2\pi t$, $\psi = 2\pi s$. Consider $f : \mathbb{R}^2 \to \mathbb{R}^4$; I am giving you a embeddingof the Klein bottle in $\mathbb{R}^4$~~~~~~; $f(t,s) = ((2+\cos\theta)\cos 2\psi, (2+\cos\theta)\sin 2\psi, \sin\theta\cos\psi, \sin\theta\sin\psi)$.

Check that $f(t,0) = f(1-t, 1/2)$ and $f(0,s) = f(1,s)$ for all $0 \le t \le 1, 0 \le s \le 1/2.$

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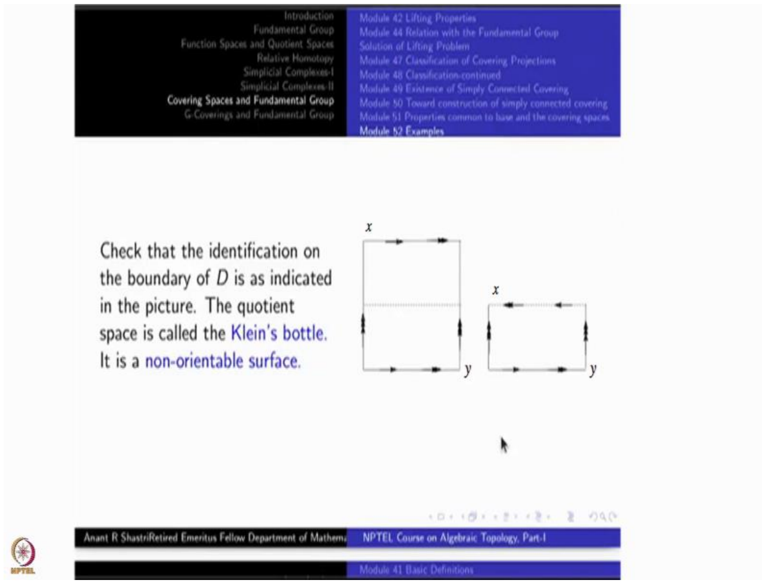

It follows that f will quotient down to define a map $\tilde{f}$ from Klein bottle into $\mathbb{R}^4$. That map, you have to show, is injective. Once you show that it is injective that is enough. because one can show that the quotient is a compact space, and hence $\tilde{f}$ is automatically will be homeomorphism onto the image.

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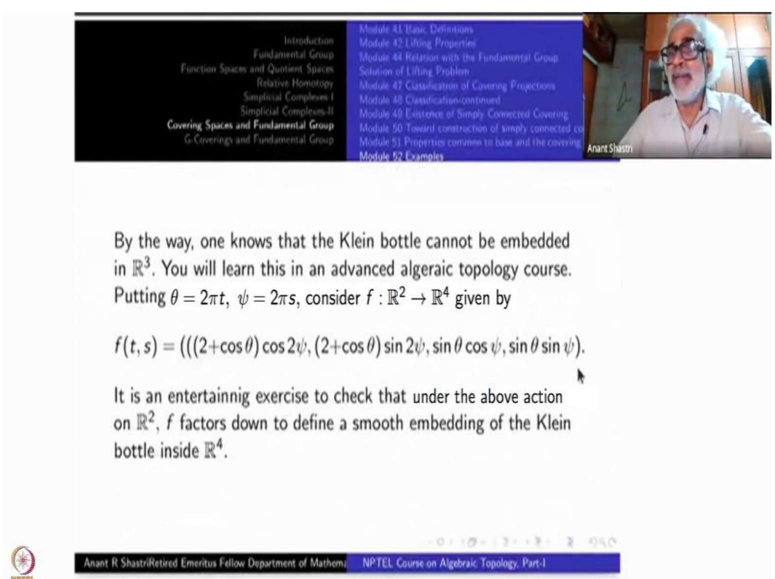

So, this is elementary checking, but many people make mistakes here, elementary mistakes. So, I have written down this one, so please check that I have not made a mistake. That is all.

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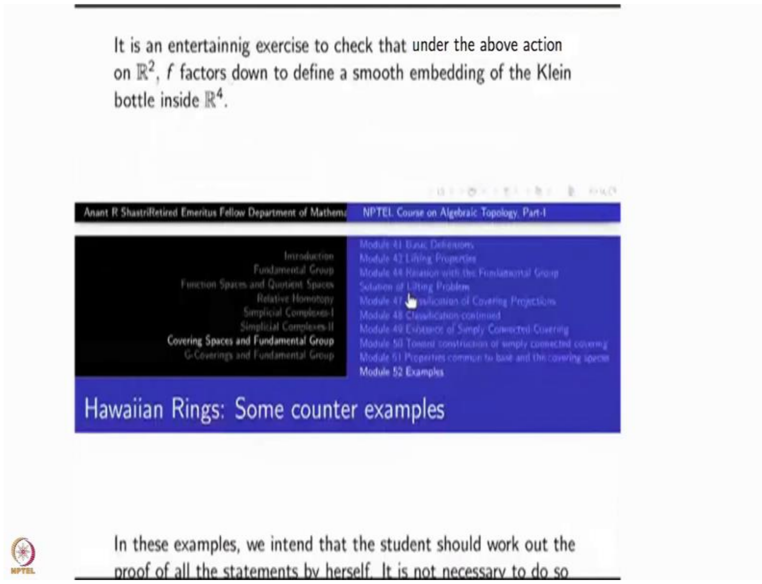

And rest of the thing that I have put here, we will discuss it some other time. They are all concrete samples. So, they do not play much role in the theory that we are developing, but they are good for understanding what is going on. So, we will cover it some other time. Thank you.