Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Indian Institute of Technology, Bombay
Lecture – 51
Properties Shared by Total Space and Base

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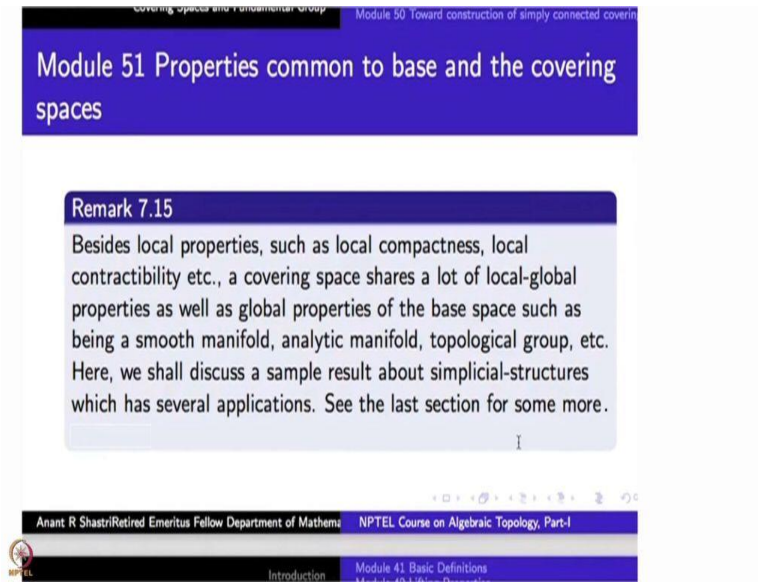

So, today's topic is some kind of a general discussion about covering spaces; namely what are the properties shared by both the base space and covering space. By properties, I mean topological properties. Since, a covering projection is a local homeomorphism, automatically all local properties are shared; such as local compactness, local contractibility, local connectedness, local path connectedness being a first countable space and so on. But it also shares a lot of what are some kind of loca-global properties; and some global properties also; being a manifold, being an analytic manifold, c-infinity manifold or being a topological group.

So, many of these properties are shared by both X and $\bar{X}$; so, it is not possible to discuss all of them. So, here I will give you sample of that one simple thing, because we are just doing a lot of simplicial complexes and so on. So, let me just discuss that being a simplicial complexes being triangulable; that properties shared by base space and the top space of a covering projection.

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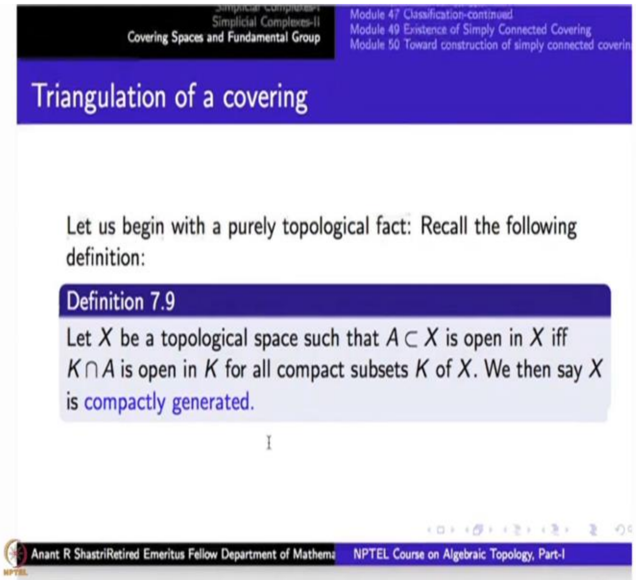

For that I need another topological concept here. Recall that a space is compactly generated, if a set is open, if and only if it is open in its intersection with a compact set is open, inside compact set, for every compact set. So, the compact sets of a topological space determine all other open sets; as well as close sets that is the same thing. K intersection A is closed in K, for every K; then and then only $A \subset X$ will be closed in X. Either open or closed, then it is these two can be one one from the other, you can go back to and go back and forth.

Simplicial complexes, the weak topology is compactly generated; why? Because we have proved that any compact set is first of all covered by compactly many many; sorry finitely many closed simplexes. And each closed simplex is a compact; and finally weak topology is nothing but the topology controlled by each closed simplexes. A set is closed if and only if intersection with each closed simplex is closed in the simplex. That is the relevance of compactly generatedness when you are dealing with simplicial complexes.

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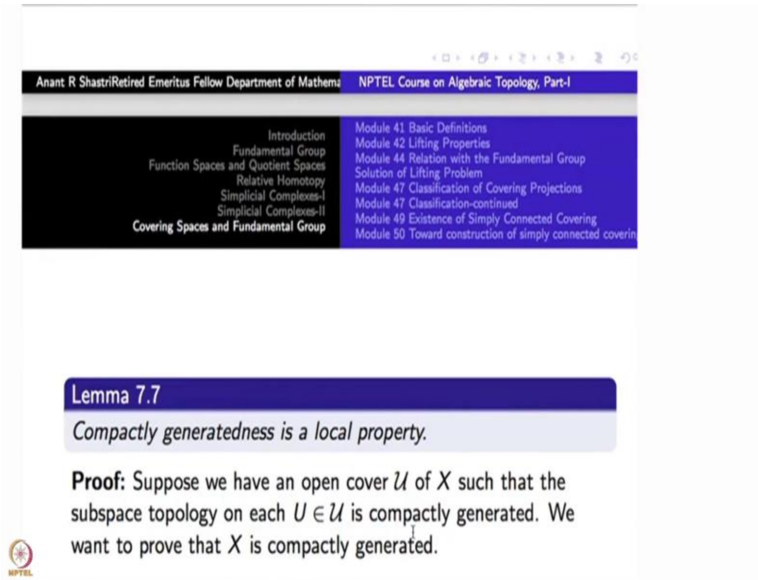

So, the first thing that is being proved here is compactly generatedness is a local property. What is the meaning of local property? I have been using this word, but we may not know what is the meaning I have in mind of this word. What I mean is each point has a neighborhood such that the given property holds in that neighbourhood. Like local compactness and locally connectedness. Is same thing as, we have an open cover, such that on each member of this open cover, the property is true.

So, this can be taken as definition of local property. Right now, suppose we have an open cover $\mathcal{U}$ such that, the subspace topology on each $U \in \mathcal{U}$ is compactly generated. Then I want to say that X is compactly generated. In fact, if X is compactly generated, every subspace $U \subset X$ is compactly generated; that is easy to see. So, now assume that each $U \in \mathcal{U}$ is compactly generated, and $\mathcal{U}$ an open cover; then I want to show that you can comeback to X So, this is a proof.

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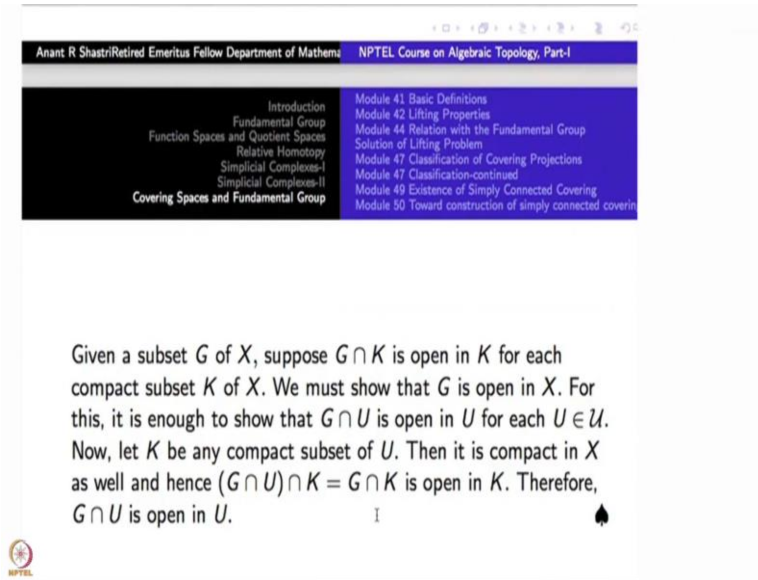

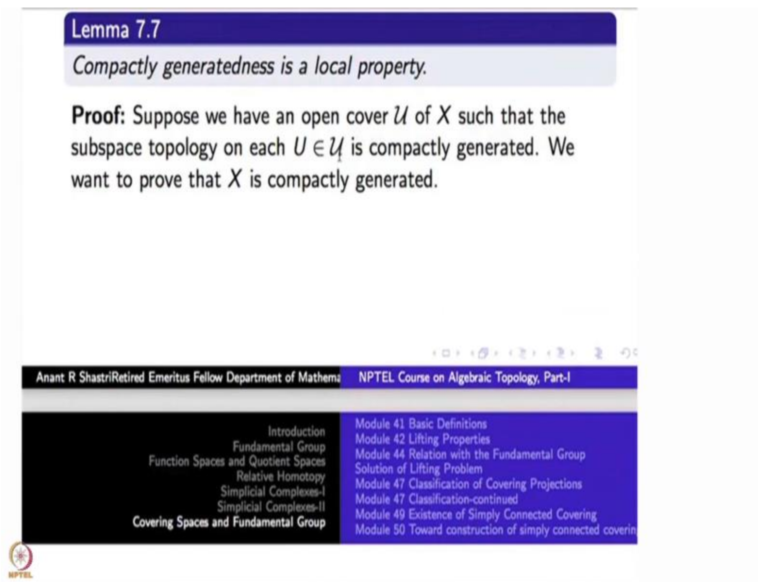

Take any subset $G \subset X$ such that $G \cap K$ is open in K for each compact subset K of X. We have to show that G is open in X. For this, because $\mathcal{U}$ is an open cover for X, it is enough to show that $G \cap U$ is open in U for every $U \in \mathcal{U}$. Because then G will be union $\bigcup_{U \in \mathcal{U}} G \cap U$, abd each $G \cap U$ is open in X. So, now let K be any compact subset of U, then it is compact in X as well as. (This compactness is not a relative notion; it is independent of where the subspace is contained in.) Therefore, by the hypothesis, we have, $G \cap K$ is open in K. But $G \cap U \cap K = G \cap K$; because K is subset of U already. Sine this is true for every compact subset K of U, therefore, $G \cap U$ is open in U. So, we have done both `if' and `only if'; Thus, compactly generatedness is a local property-- if it true for an open cover, then it is true for the whole space.

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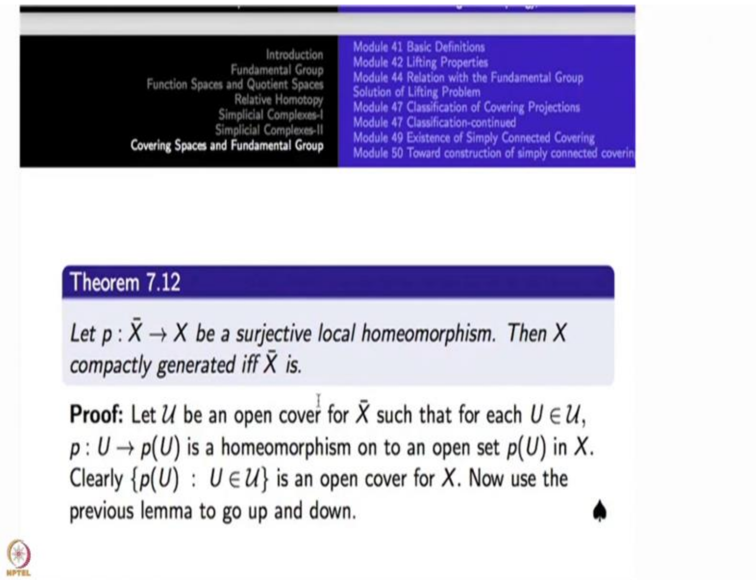

Now, take any covering $p : \bar{X} \to X$. Actually, instead of covering, I am proving a more general result; namely, for any surjective local homeomorphism, then apply it for covering also. Let p is surjective local homeomorphism; then X is compactly generated, if and only if $\bar{X}$ is compactly generated. To see that what we have to take? start with an open covering $\mathcal{U}$ for $\bar{X}$ such that on each $U \in \mathcal{U}$, $p : U \to p(U)$ is a homeomorphism. This we can be done, because p is a local homeomorphism. And of course, being a local homeomorphism, it follows that $p(U)$ is an open set in X. Then $p(\mathcal{U}) := \{p(U) : U \in \mathcal{U}\}$ is an open cover for X; $\mathcal{U}$ is an open cover for $\bar{X}$ implies $p(\mathcal{U})$ is an open cover for X. And $p : U \to p(U)$ is homeomorphism for each $U \in \mathcal{U}$.

Now, you can use the previous lemma to go up and down. Suppose, X is compactly generated; what does that mean that the topology on U for each u is compactly generated. Therefore, the topology on U instead of p U, because p is a homeomorphism; topology on U is compactly generated. But U is an open cover; therefore X bar is compactly generated. So, we have to coming back, if x bar is compactly generated; each U on each open subset is compactly generated. Therefore, p U is compactly generated; so, p u is an open cover for X is compact.

So, we just need a local homeomorphism, which is surjective; of course it should cover the whole thing that is in X, p U should be cover. So, it is a surjective homeomorphism, local homeomorphism is good enough.

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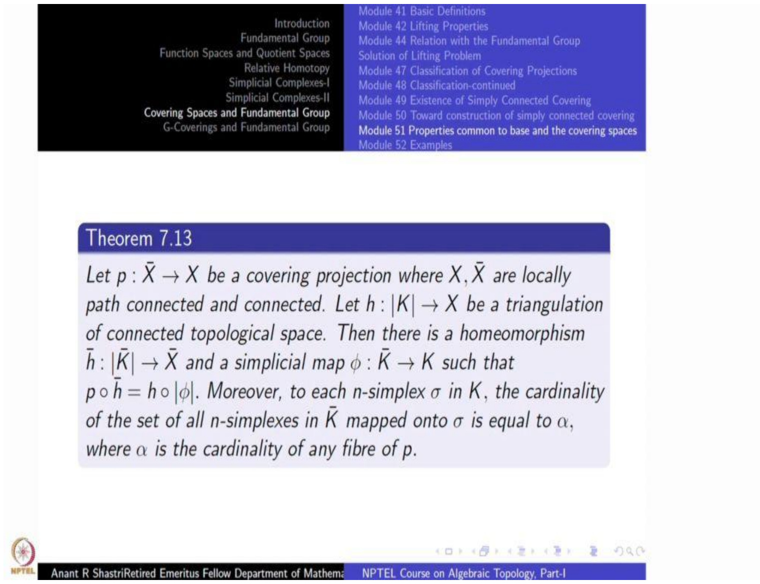

So, now we come to the main topic of today's agenda. Let $p : \bar{X} \to X$, be a covering projection; X and $\bar{X}$ are locally path connected and connected. (This connectivity condition is not exactly needed all the time anyway; and I have told this several times.) Let $h : |K| \to X$ be a triangulation of a connected topological space X. Then for every covering space $\bar{X}$, like this, there is a triangulation of $\bar{X}$, namely $\bar{h} : |\bar{K}| \to \bar{X}$. And a simplicial map $\phi : \bar{K} \to K$ such that this diagram is commutative. So, even the h bar here, which is it is a triangulating map is not arbitrary.

In other words, you can forget about $\bar{X}$ and X and work with the simplicial complexes $\bar{K}$ and K. What we get is a simplicial complex $\bar{K}$ and simplical map $\phi : \bar{K} \to K$ such that $|\phi| : |\bar{K}| \to |K|$ is a covering projection. That map phi bar itself would be a covering projection. In other words, start with mod K, take a covering; then there will be a simplicial structure on this covering, such that the map p the original covering map becomes a simplicial map. So, this is just of first part here. Moreover, for each n-simplex in K, cardinality of the set of n-simplexes in $\bar{K}$ mapped onto this simplex, is equal to alpha where alpha is cardinality of the fibre any fibre.

Remember, this is where connectivity is necessary that is all. If X is connected, the cardinality of the fibre is a constant. So, this gives us for every n-simplex of K, there will be alpha n-simplexes in X bar, exactly same. So, look at the vertices, how many vertices are there for each vertex here? There will be alpha many vertices above. So, I will make it clear in the proof of this one.

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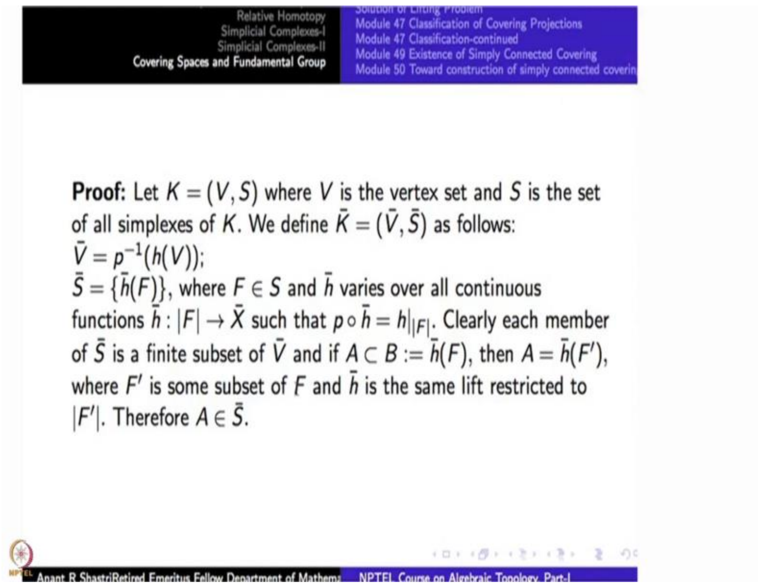

So, what I am going to do is let V be the set of vertices and S be the set of simplexes for K. Then, we are going to define $\bar{K} = (\bar{V}, \bar{S})$, as follows. $\bar{V} = p^{-1}(h(V))$. Remember h is a homeomorphism from $|K|$ to X. So, I am picking up the vertices in $\bar{X}$, to be p inverse of the vertices in X; that is the meaning of this; h of V are images of those points which are vertices. Vertices are just p inverse of that; that is a vertex set. What are the simplexes? This $\bar{S}$ is collection of all $\bar{h}(F)$. What is h bar and what is F bar I am going to tell you; where F is a simplex of S; like vertices, triangles edges all those things have to be coming from S.

Then, look at this h bar; h bar is a function from mod of F to X bar; that means a lift h restricted to $|F|$F is a subspace of mod K, h is a homeomorphism mod K to X; restricted to the mod F, then the you must get a lift of this one. $\bar{h} : |F| \to \bar{X}$ is a lift through p, the covering projection. Take all possible lifts; all of them; look at these all these h bars; then h bar of F. F is the finite set remember that; so h bar of F will be a finite set. And where are these points? they will all be inside $p^{-1}(h(V)) = \bar{V}$. So, $\bar{h}(F)$ will be the vertices of $\bar{F}$. Declare them as simplexes in $\bar{K}$.

If we have a subset $A \subset B = \bar{h}(F)$, then A is nothing but $\bar{h}(F')$ for some $F' \subset F$. Therefore, you can restrict the same $\bar{h}$ to this $|A|$. So every subset of B in $\bar{S}$ is also in $\bar{S}$. So, what we have seen is that this K bar is a simplicial complex; and that is easily verified. But why $|K|$ is homeomorphic to $\bar{X}$, in such a way that when you take composite with p; it will be mod of a simplicial map $\bar{K} \to K$? So, that is the harder part to show; already definition is over.

Before proceeding further, I want to tell you about a simple situation. Look at the circle, easiest way to triangulate it is to take $\{1, \omega, \omega^2\}$ the three points as vertices. Then join 1 to ω, ω to ω^2, ω^2 to 1, by line segments. Map them stereographically on the corresponding arcs. That is a triangulation. It is actually a triangle. Now, take a covering, let us take simply the covering $exp : \mathbb{R} \to \mathbb{S}^1$. So, what is the corresponding triangulation on the real line, which will make this exponential map itself, look like a linear map. So, what is that? Declare all the inverse image of these three points as vertices.

The inverse image of 1 we are familiar already, we have done several times; all the integers come. Inverse image of omega-- all one third integers will come; $1/3, 4/3, 7/3 \cdots$ all of them will come. Then the inverse of omega square … 2/3, 5/3/ 8/3, ….. will come. So, each unit interval $[n, n+1]$ will divided into 3 parts. So, now exponential map from each interval to another interval can be thought of as a linear map. Of course, it is an arc there. So, when you take actual triangle to that one that homeomorphism is precisely the linear map.

What is that homeomorphism? You look at the triangle and push it back to the circle; that that is the homeomorphism, by the stereographic projection. So, what are the triangles? What are what are the edges have declared? By this formula what I have declared is you take an edge here, lift it lift it at various points. If we have lift, then join the two points, the end-points by a simplex. So, that is precisely what we the h bar of S means. I cannot join just one third to some 5 by 3; or some other 25 by 3 that is not one. One third we have to lift, there are two possibilities; so one going backward and coming backward; that is the only two lifts will comeback.

So, what you get is the real line. So, this condition is very important that we cannot take arbitrary subsets of V bar as simplexes of this one; of course, that will be a simplicial complex. But it will not be giving you mod, mod K bar will not be put X bar. For that, we have to use this X bar itself, and it lifts of lifts of this homeomorphism, p lifts of homeomorphisms; restricted to its simplex.

And then, you get a simplicial complex. So, even if you do not understand all the details which are getting lost in notation; you must be able to figuring it out yourself.

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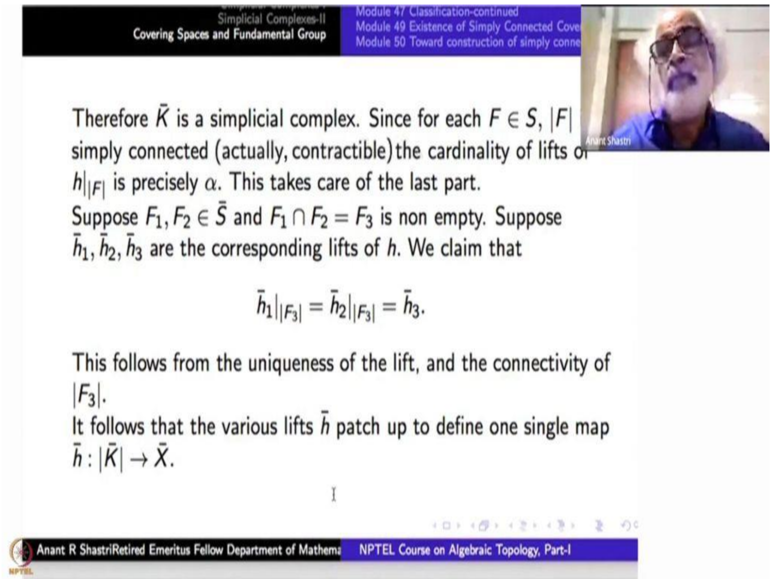

So, let us go through some of the proofs here fairly. $\bar{K}$ is a simplicial complex-- this we have seen. Since for each F in S, $|F|$ is simply connected; $|F|$ is actually contractible, it is homeomorphic to a disc. So, lifts are always possible; the cardinality of lifts is precisely equal to α At each point in the fibre, you lift it; you will get that many disjoint lifts. In fact, under every covering projection, for every simply connected space, we have seen that the inverse image will be disjoint union of copies of this; they are all evenly covered. So, look at this h bar, h bar or whatever you lifted; and look at the end-points, declare those end-points as union of those things as simplexes.

And then you have to fill it. The filling will be done by the homemorphism $\bar{h}$ that is the trick. So, what happens is compatibility; compatibility comes automatically, because of uniqueness of the lifts. So, suppose F1, F2 are inside S bar and intersection is F3 is nonempty. If it is empty, there is no question of compatibility. Suppose $\bar{h}_1, \bar{h}_2,$ are the corresponding lifts of the h restricted $|F_1|, |F_2|,$ into X respectively so that there images intersect. We claim that $\bar{h}_1|_{|F_3|} = \bar{h}_3 = \bar{h}_2|_{|F_3|}$. Therefore compatibility is there. Why this is true?

This follows because as soon as two liftings of a single map coincide at one point then they are equal on any connected set. Since we assume $\bar{h}_1(|F_1|) \cap \bar{h}_2(|F_2|) \neq \emptyset,$ take a point $\bar{x} \in \bar{X}$ which is in this intersection. It follows that $\bar{h}_1(x) = \bar{h}_2(x) = \bar{x}$. Therefore by the uniqueness of the lifts, $\bar{h}_1|_{|F_3|} = \bar{h}_3 = \bar{h}_2|_{|F_3|}$.

So, that is one single function h bar such that p composite h bar is h; automatically on each these simplexes is continuous, so it will be continuous also. So, this is where you have to use that mod F mod K has the, what is this? Weak topology. A function is continuous if and only if it is continuous on each simplex. So, now you have got a function h bar from mod K bar to X bar, such that when you composite with p; comeback come here, and restricted to each K here that is h. So, I have to define what is the map from K bar to K? Can you guess now what is the map? It is very easy.

What are simplexes of K bar? They are lifts of some F; so, you map them accordingly. What is p inverse of what is V bar? V bar is what V bar is p inverse of V. So, wherever it goes to you define phi to be that map; so that is the simplicial map that is the vertex map. It will become simplicial map by the very definition. So, vertex map is already there; so that become simplicial map because of the very definition. So, diagram is converted to this is fine; finally what we have to do is that show that h bar is one-one, onto and homeomorphism. Once again on each simplexes is a homeomorphism by very definition; the lifts are automatically homeomorphisms.

And they go into compatible things; therefore, if the bijectivity automatically follows. Ontoness follows because take any point in X bar; come come down to X, it is inside a simplex. So, that simplex will be continuous lifted up; so that is surjectivity. So, I have written the little more details of this one here.

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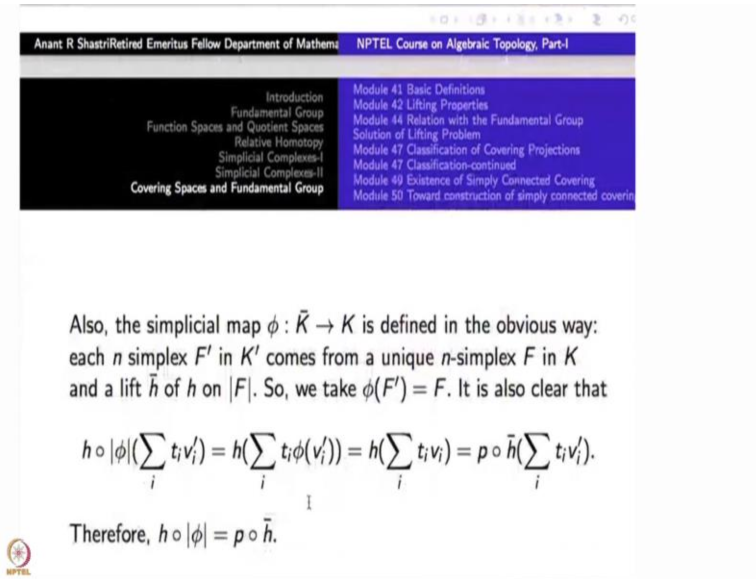

So, h composite phi equal to phi composite h bar; I have written down completely details here.

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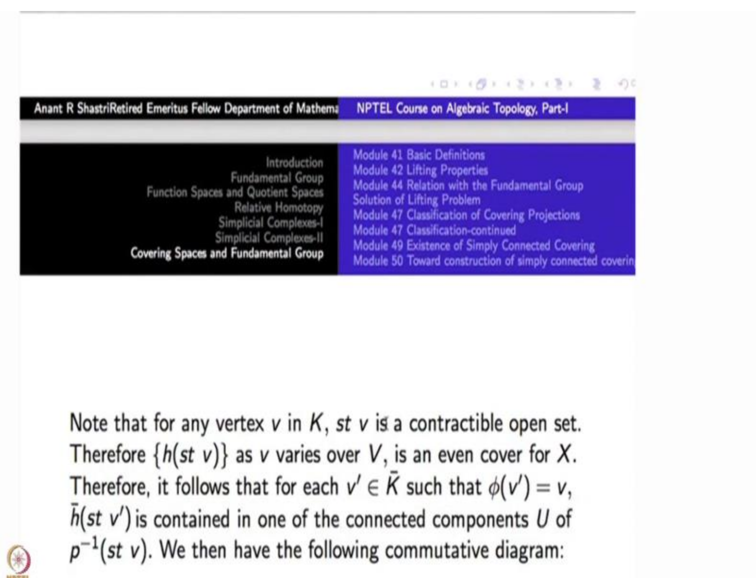

So, what we can do is we look at star of a vertex. The star of a vertex is an open subset of mod K; under h it will be open subset of X. This star of X, star of V are all contractible spaces; so in particular they are simply connected. Therefore, inverse image will be disjoint union of open subsets, which are map down to this star v; this star v is evenly covered.

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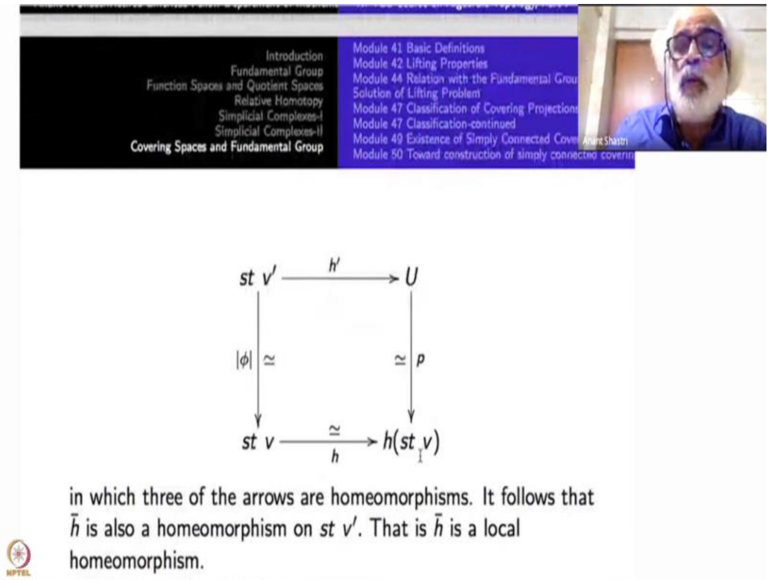

So, we can use that to establish that there is such a homeomorphism, and diagram commutative diagram; h is restricted to star v, this h star of star v. p is a homeomorphism on each connected path portion of the inverse image; and that will become star of this v prime. So, this will automatically show that this h prime which is restriction of h bar, is both open as well as continuous; this another way of doing this one. It is a homeomorphism now at each each each each open covering of this one.

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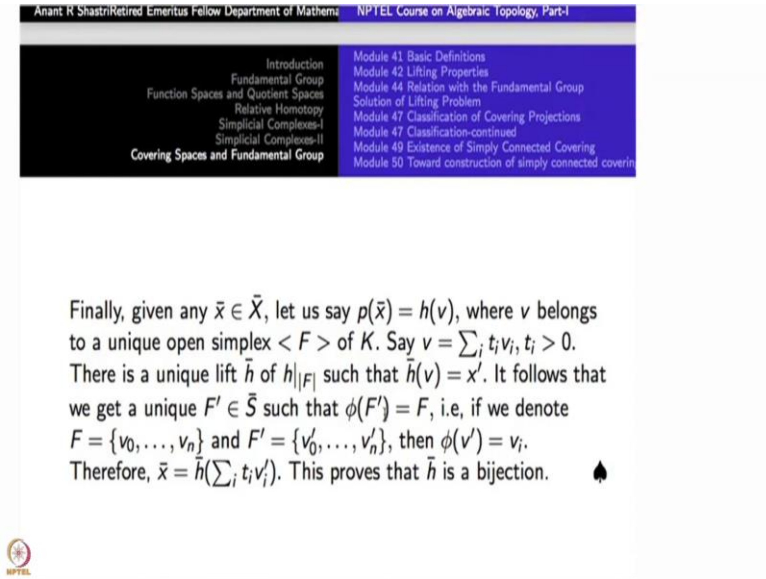

So, I have given you a heuristic argument why it is a bijection. Here is a straightforward proof to show that h bar is a bijection. So, there are many interesting properties shared by covering projection under covering projection, by top space and bottom space. But, in the right in the beginning remember we have we have questioned you that Hausdorff space et-cetera are not shared. Regularity or normality et-cetera are not shared; so we have to be careful for that.

Well, I am giving you some exercises, so there will be chance to look into these exercises; and then discuss them also. Maybe next time we will discuss some more examples; and then go to the next topic. So, we will discuss some more examples of covering projections; then go to the next topic. So, thank you.