Introduction to Algebraic Topology (Part-I)
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Lecture – 50
Construction of Simply Connected Covering

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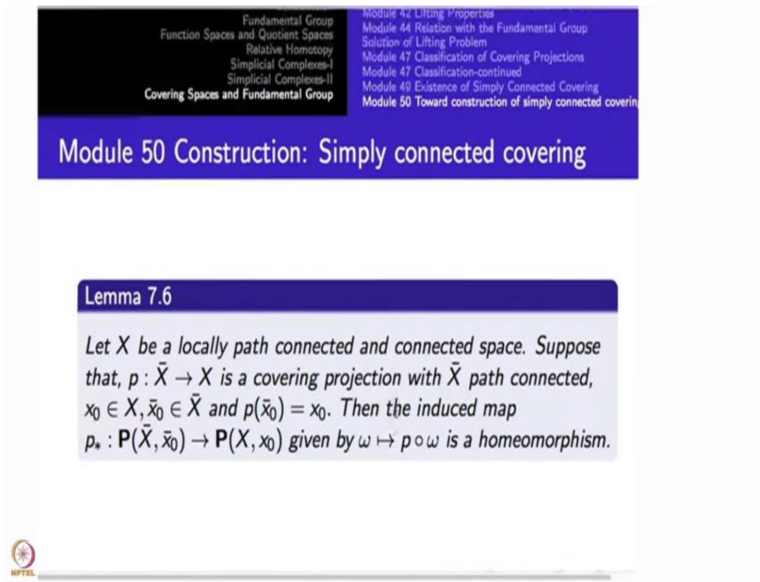

Continuing with the construction of simply connected covering spaces; recall that we made a definition of semi-locally path connected, semi-locally simply connected spaces. Then we studied the path space over such a space and showed that evaluation map is continuous there and it is an open mapping also. So, this lemma today is a key to what we should expect, what we should expect? So, where to look for the simply connected covering space for a given X? So, X is locally path connected and connected space. Suppose $p : \bar{X} \to X$ is a covering projection with $\bar{X}$ path connected. Take $x_0 \in X, \bar{x}_0 \in \bar{X}$ such that $p(\bar{x}_0) = x_0$.

Then the induced map $p^* : P(\bar{X}, \bar{x}_0) \to P(X, x_0)$ namely, a path ω going to $p \circ \omega$. So, that is the map induced by p on the path spaces; this itself is a homeomorphism. Now, this lemma is not essential for the proof or for the construction of simply connected covering space. But it tells you where to look for the simply connected covering space. It tells you that it does not matter , whichever covering space you take, its path space is the `same' as the path space of X. So, you do not have to construct the covering space afresh; it is going to be a quotient to the path space of X.

Therefore, the simply connected covering space if it exist, must be a quotient of the path space itself; path space of X itself. So, this is the lemma which directs you to for that one. So, the proof of this lemma that `homeomorphism' is essentially to proving that this p star is also an open map; which is somewhat similar to that evaluation map is an open mapping that we have done. So, I will presently skip the proof of this one and go to the construction of a simply connected covering. If time permits we can comeback to the proof of this one at some other stage. So, here is a proof but I will skip this proof now; the proof has a nice diagram also here and so on.

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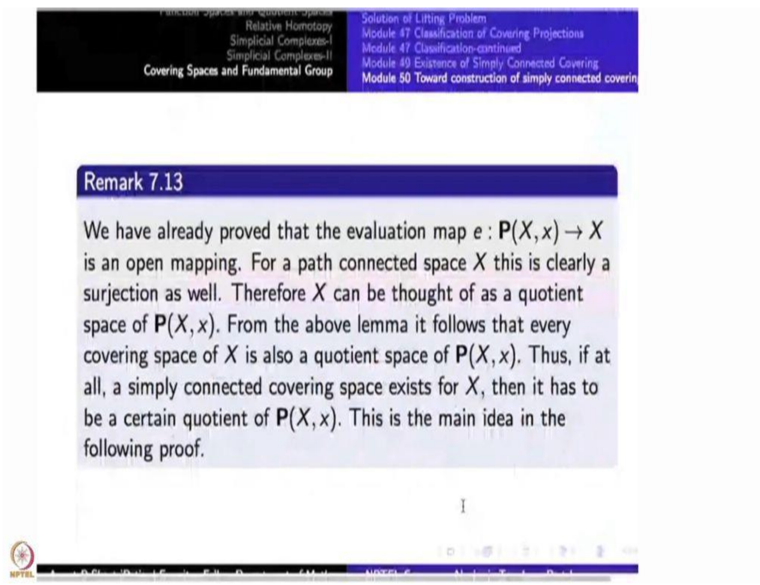
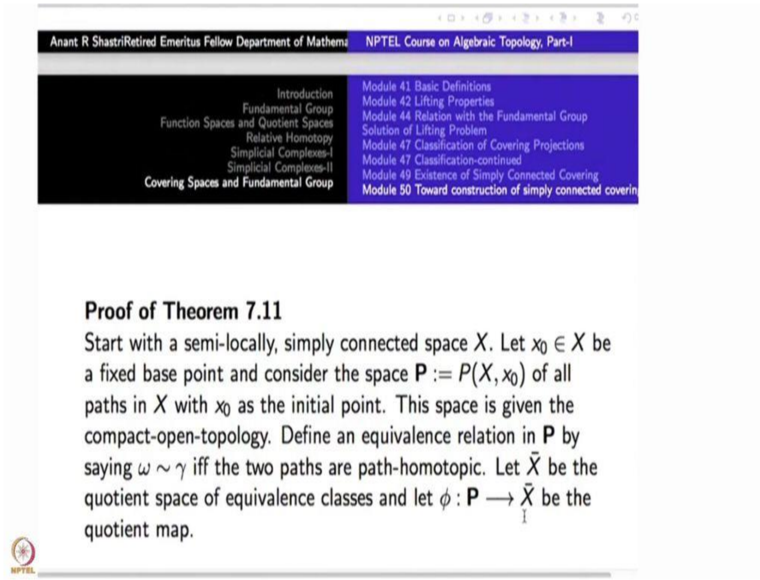


So, as I told you, we have already proved that evaluation map from the path space to X is an open map. Open surjective, because X is path connected; therefore, X is itself can be thought of as a quotient space of $P(X, x_0)$. From the above lemma that just now which we skip, it follows that every covering space of X is also a quotient of $P(X, x_0)$. Why? $P(X, x_0)$ is homeomorphic to $P(\bar{X}, \bar{x}_0)$ and $\bar{X}$ is a quotient of $P(\bar{X}, \bar{x}_0)$. Therefore, $\bar{X}$ is also quotient of $P(X, x_0)$. Therefore, let us work out, whatever we want to do with $P(X, x_0)$ itself. That is the idea.(Refer Slide Time: 05:03)



So, here is the final proof, so we start proof of theorem. Start with a semi-locally, simply connected space X; I am just repeating this thing. Take a point $x_0 \in X$, that is fixed point, base point of X. And this capital P ,bold P, let it denote $P(X, x_0)$. Again and again we do not have to write as a big symbol that is all. This space is given the compact-open-topology, remember that also. We now define an equivalence relation in P, by saying that omega is equivalent to gamma, (remember these are paths starting at x naught), if and only if these two paths are path homotopic.

In the other words, omega 0 is already gamma 0; but end-points must be also the same. And I the two must be path homotopic, namely end-points must be fixed. Path homotopy is an equivalene relation, remember that. Let $\bar{X}$ be a quotient space of all equivalence classes and let$\phi : P \to \bar{X}$ be the quotient map.

So, what is the difference? The difference is in the definition of fundamental group, we took the loops at a single point. Now, we are taking all paths; the end-points could be, the other end-point cold be anything in the space of in inside X, that is a difference. So, this is something much useful than just the set of pi1 of X, x naught; it is not a group either. This is given a topology now; what is the topology? Topology coming from the compact-open-topology of the path space, as a quotient of that; that is, $\bar{X}$ is a quotient space of P.

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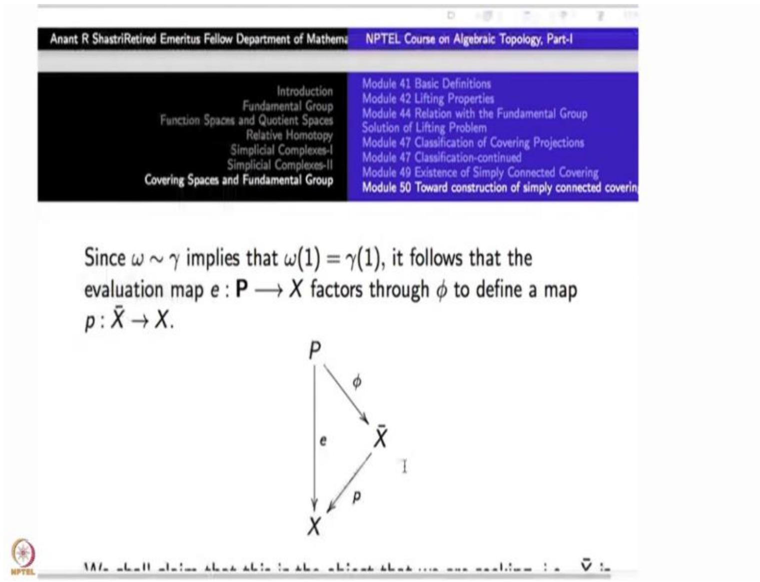

So, by definition, these two are path homotopic means the end-points must be the same. Therefore, it follows that the evaluation map, remember, is just the end-point; so that factors through the equivalence classes. On the entire equivalence class, it takes the same value; therefore, $e : P \to X$ factors through the quotient map ϕ like this and gives $p : \bar{X} \to X$. X bar to Remember we have to construct not only the space, but also the projection map. So, this p is nothing but e factor through the quotient. Take any class here; it is represented by any path; the endpoint is independent of the representaive path, and take the end-point here; so that is p of that. So, p of this is also just the endpoint, but it is independent of the path class. Now, one easy thing here is that p is an open mapping, because we have shown that e is an open mapping. Taking open set by the very definition of quotient space, inverse image is open. The image of that is same thing as the image under p of the original open set here. So, e is open, so p is also open. We have to show that p is a covering projection; we have to show that $\bar{X}$ is simply connected. So, these two are the task now for.

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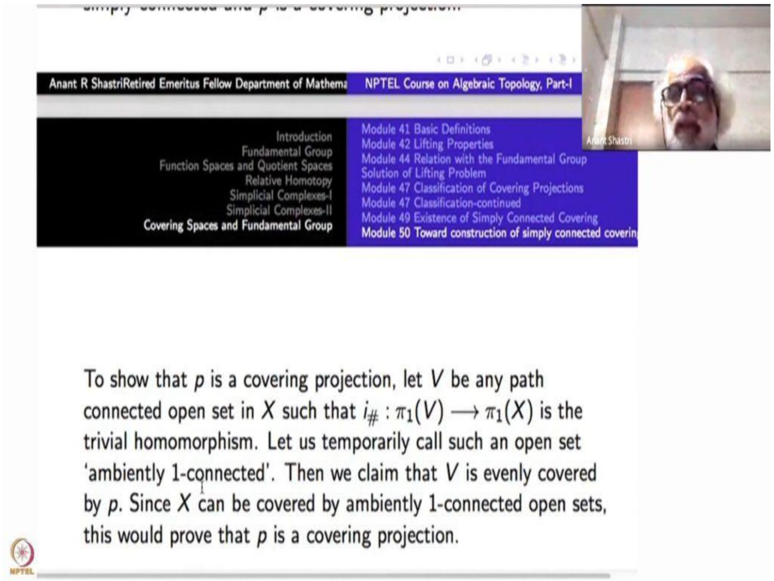

So, let us start with V be any path connected open set in X, such that inclusion induced map on the fundamental group $\pi_1(V) \to \pi_1(X)$ is trivial. Such an open subset around every point is guaranteed by the hypothesis of semi-locally simply connected. Just for being a bit economical in with the words, let us call some such a set ambiently 1-connected. It is not simply connected by itself, it need not be. When you pass to the whole space, its fundamental group becomes trivial that is the meaning ambiently 1-connected. So, what we know is that X is covered by open subsets which satisfy this property. they are ambiently 1-connected. We claim that if V is ambiently 1-connected, then V is evenly covered by p; and that will finish the proof that p is a covering projection. The idea of the claim is clear, so we have to execute this one now.

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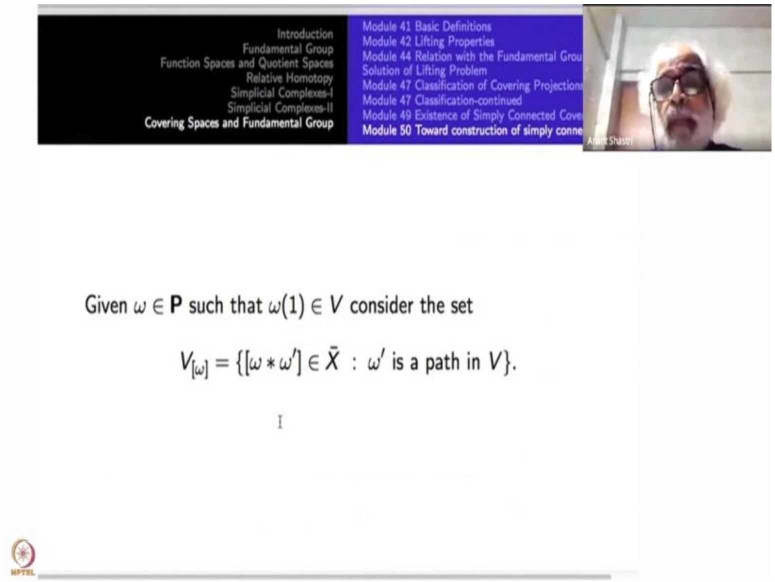

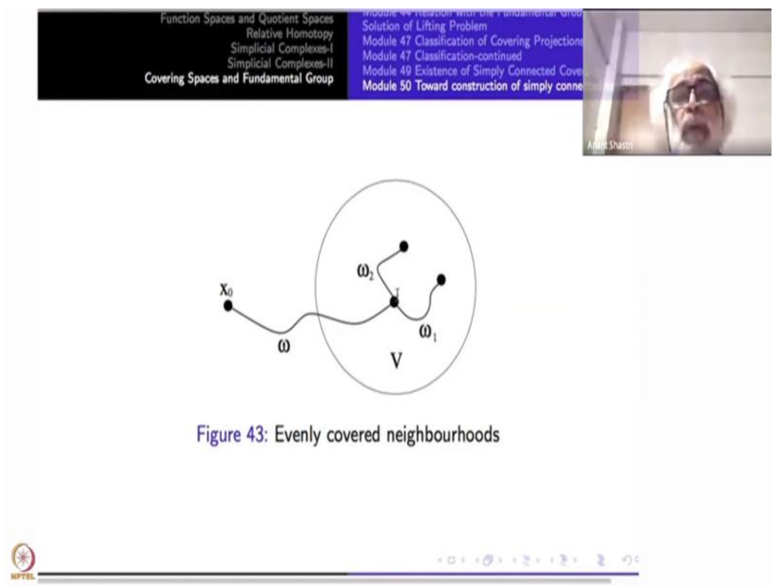

Figure 43: Evenly covered neighbourhoods

Given a path ω starting at x_0 that is an element of P, such that the endpoint is inside V, (V is chosen to be some open set, which is ambiently 1-connected), we are going to define a set of subsets of $\bar{X}$; what are they? Consider the set $V_{[\omega]} = \{[\omega * \omega' : \omega' \text{ is a path in } V\}$. This notation is because this is going to depend upon V as well as the class of omega. The class of omega is an element of what? is an element of $\bar{X}$ remember that. So, all such classes they are also elements of X bar, which are omega star omega prime; path is which look like omega star omega prime. Where, in this omega prime is a small loop, small a path completely contained inside V.

Remember omega to begin with is a path from x_0; x_0 to $\omega(1)$. $\omega(1)$ is what? $\omega(1)$ is some point of $V \subset X$. Then I am taking this class; this class is an element of $\bar{X}$. I am extending it to $\omega * \omega'$ by some path ω' within V, which starts at $\omega(1)$. Look at all those that collection is going to be $V_{[\omega]}$ So, the claim is so I have here picture here; start with omega like this, which ends inside this open subset V. Then I can extend it by some omega1 or omega2 and so on; these extensions are completely inside V. So, this picture I keep referring to again and again; so we have defined V omega like this.

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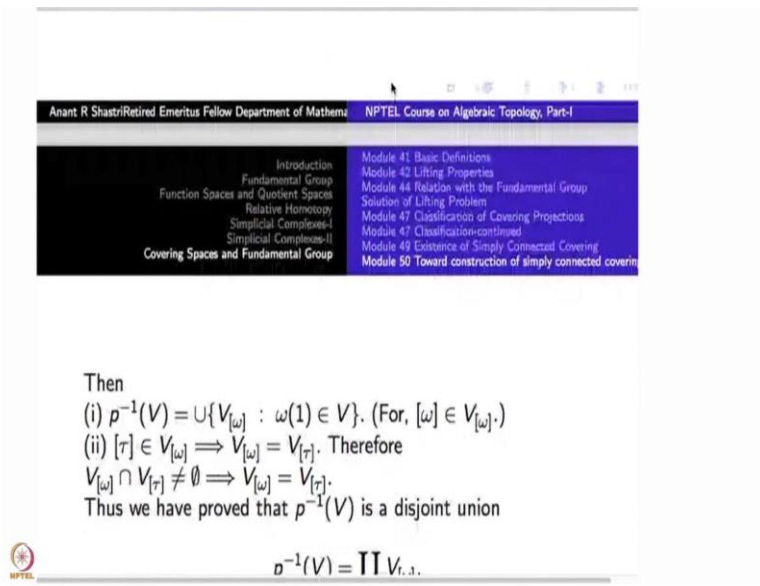

Then $p^{-1}(V)$, V is an open subset in X, $p^{-1}(V)$ is some open subset of $\bar{X}$. And it is a union of all possible $V_{[\omega]}$'s (V is fixed) whereas ω ranges over all possible paths with the condition that $\omega(1) \in V$. Remember p is nothing but the end-point; the end-point must be inside V. So, if you have the point is already inside V, and then you are connecting it with another path within V; the end-point of that will also inside V.

So, this p inverse of V is union of all these $p^{-1}(V) = \{[\omega * \omega']\omega(1) \in V\}$.things so obvious. Second part is: take any class $[\tau] \in V_{[\omega]}$, for some element of $\bar{X}$. That would imply that $V_{[\tau]} = V_{[\omega]}$ So, this is just like in group theory, how group theory wherein H is a subgroup of G then if $x \in H$ then its right-coset Hx will be H itself; it is of that nature. So, let us see what happens.

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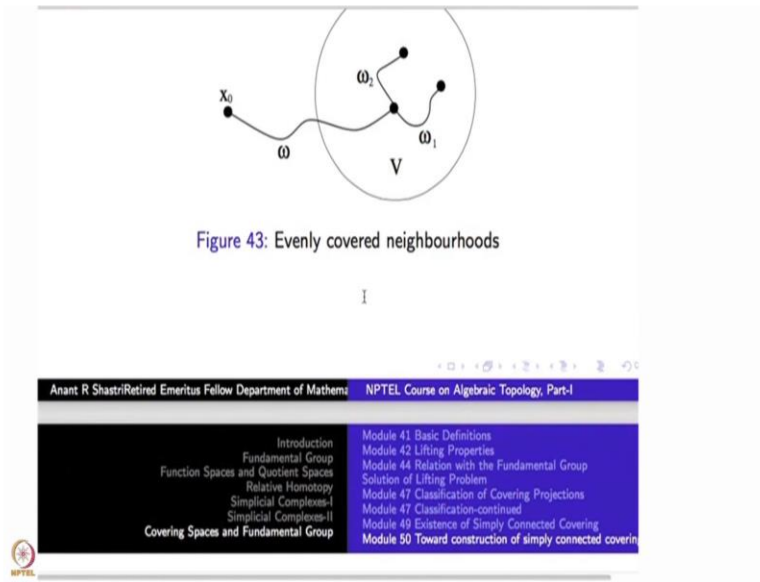

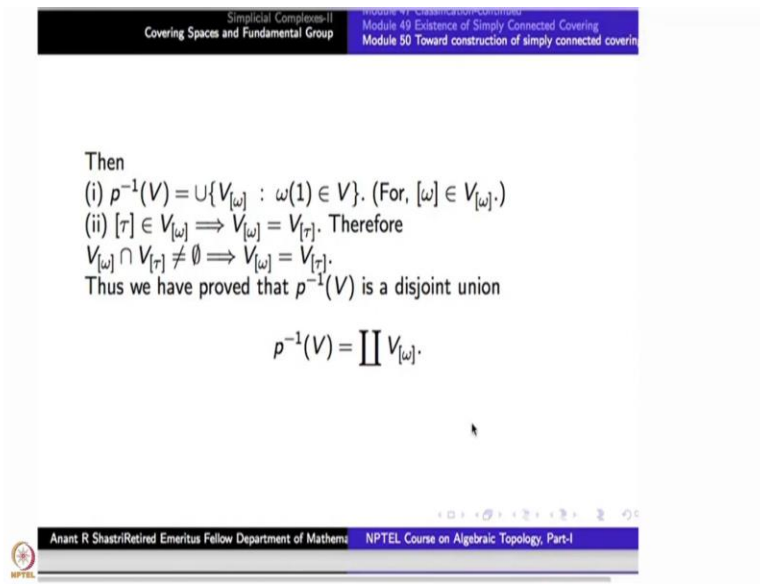

Suppose $[\tau] \in V_{[\omega]}$. Then $\tau \simeq \omega * \omega'$ and hence $\tau * \omega_1 \simeq \omega * (\omega' * \omega_1) \in V_{\omega}$. This means $V_{[\tau]} \subset V_{[\omega]}$. By symmetry $V_{[\omega]} \subset V_{[\tau]}$ and hence equality holds. it is omega star omega1 for some some omega1; suppose, this is tau. Now, look at V tau; V tau is what? All those paths coming up to here; and then paths will go within V from this point. But I am taking the homotopic classes of path; therefore in particular after going here, I can comeback from this path up to this point. That will be homotopic to omega1. Therefore, V tau is contained inside omega1, and V omega1 is V omega is contained inside V tau; so these two are equal. So, this is completely trivial. But if you look at the picture it will be that nature.

So, this will happen, either some elements here when both the classes are same; or it just means that V omega and V tau in general are disjoint. Either they are equal or they are disjoint; if they intersect, they must be same. This is like the cosets, cosets which is inside; cosets inside of a group. V omega intersection V tau is non-empty would imply V omega equal to V tau. Therefore, what we have proved it p inverse of V is a disjoint union of some v omega's. So, you see this is what we wanted out; then I want to say that each of V omega comes to V, which is obviously by a homeomorphism. Namely, p restricted to V omega to V is a homeomorphism; this is what we have to verify.

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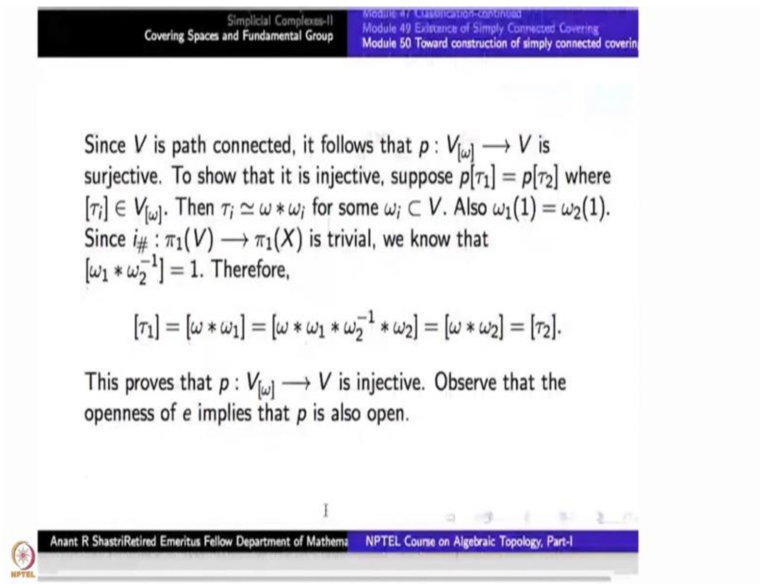

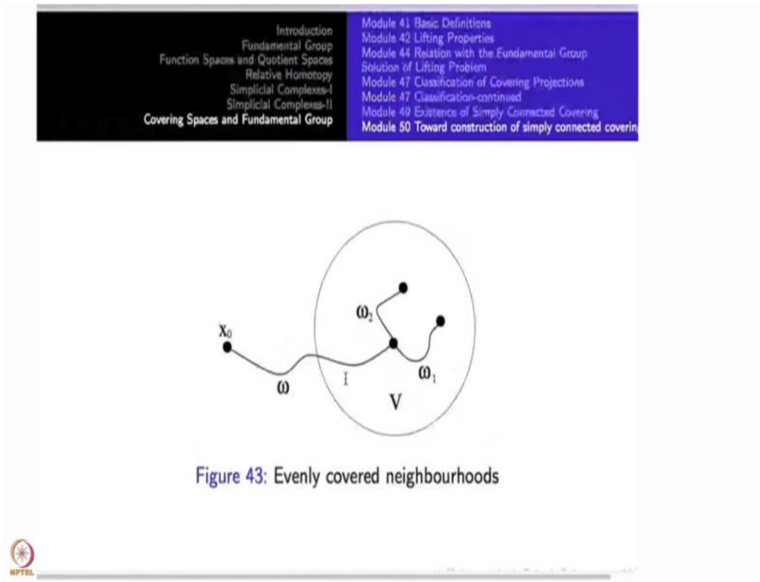

Figure 43: Evenly covered neighbourhoods

So, V is path connected, it follows that $p : V_{[\omega]} \to V$ is surjective. Once a path has come inside omega has come inside V, from there I can join it to every point inside V. So, that gives you that p is surjective, from V omega to V. Let us show that this is injective. And this is where, so far we never used this fact, namely, V is ambiently 1-connected. So, far we not, where the injectivity, we used that one; having said that, I can leave it that as an exercise; but now let us verify this one. Suppose $p([\tau_1]) = p([\tau_2])$, tau1 and tau2 are two elements in X bar.

Actually, I should assume that they are inside V omega; where tau i is inside V omega. That means what? $[\tau_i] = [\omega * \omega_i]$ where $\omega_i, i = 1, 2$ are paths in V. I am assuming that $p([\tau_1]) = p([\tau_2])$ which means that their end-points are the same. We can go back to this picture modulo that these two end-points here of omega1 and omega2 are the same. What does that mean? $\omega_1 * \omega_2^{-1}$ is a loop inside V. Therefore, inside X, it is null homotopic; But then $\tau_2 \simeq \omega * \omega_2 \simeq \omega * (\omega_2 * \omega_2^{-1}) * \omega_1 \simeq \omega * \omega_1 \simeq \tau_1$. Therefore $[\tau_1] = [\tau_2]$.

This omega followed by omega2 is the same thing as omega followed by omega2; all the way from omega1 and come back to omega1. It just means that tau1 is homotopic to tau2. These kinds of things we have seen several times; so, I repeat this. So, omega1 equal to omega2 means they have same end-points. I check from pi1 of V to pi1 of X is trivial; so we know that omega1 composite omega2 inverse is null homotopic. The class is 1; therefore, tau1 which is omega star omega1 is omega star omega1; I can put omega2 inverse star omega2, because this is this is trivial. But, now put the bracket this way, omega1 star omega2 inverse is trivial; so, this cancels out. What is left out is omega star omega2, which is tau2.

So, we have got $p : V_{[\omega]} \to V$ a bijective mapping; it is already continuous. We have shown that it is an open mapping also.

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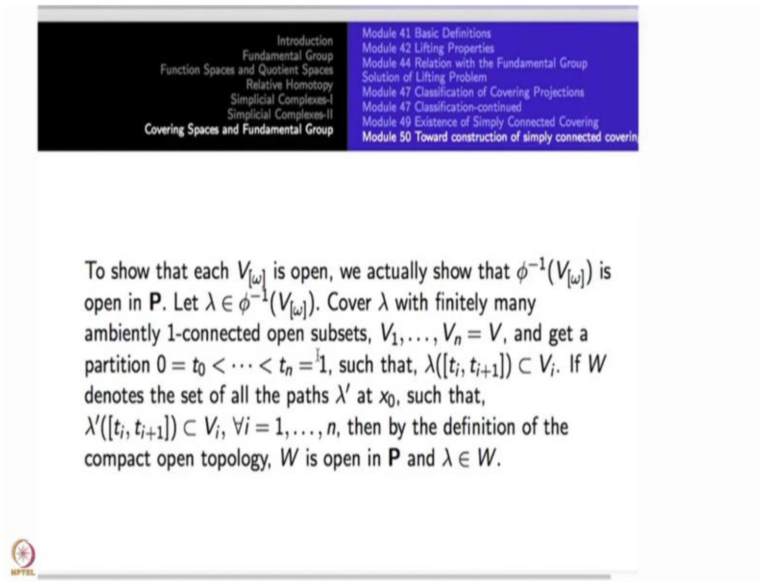

Only thing what remains is to show that each $V_{[\omega]}$ is an open set in $\bar{X}$. So, that that is that is that is remaining; then proof that p is a covering projection will be over. Because we have to show that p inverse of V is disjoint union of open sets; each of them coming homeomorphically onto V. So, everything else is shown except V omega must be an open set. So, V omega is where V omega is in X bar; X bar is a quotient space of p. How to show something is open in the quotient space? We have to show that ϕ inverse of that set is open in P. This is the definition of quotient topology.

So, I have to show that $\phi^{-1}(V_{[\omega]})$ is open in P. Take any $\lambda \in \phi^{-1}(V_{[\omega]})$. Around that I shall produce an open set contained inside phi inverse of omega. Yes or no? For every point inside this, we should produce a nbd contained $\phi^{-1}(V_{[\omega]})$. What are open subsets in the compact-open-topology? We have to produce that some basic open set namely intersection of Ki, Vi et-cetera has to be found out now. So, watch this look at this lambda, it is a path from x_0 to some point inside V. Because $\phi(\lambda)$ belongs to $V_{[\omega]}$ just means $[\lambda] \in V_{[\omega]})$.

So, this entire curve λ can be covered by finitely many ambiently 1-connected open sets $V_1, V_2 \cdots, V_n = V$; the last Vn I can choose it as V itself. In fact, I could I can come back from that side; start with V and then cover some other portion some other portion. Actually, we can take a infinite covering first and then take a finite covering, because the whole thing is compact, lambda is compact. We get a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the closed interval [0,1] such that $\lambda([t_i, t_{i+1}]) \subset V_i, i = 1, 2 \ldots, n$.

So, cut it cut the entire lambda and cover it; so then then cut it into partition into V sub-interval. If W denotes $\cap_{i=0}^{n-1} \langle [t_i, t_{i+1}], V_i \rangle$, the set of all paths λ' starting at x_0 and such that $\lambda'([t_i, t_{i+1}]) \subset V_i, i = 1, 2 \ldots, n$, then it is a basic open set by definition in the compact open topology for P. for every i1 to i; by then definition of the compact-open-topology, this W is open. So, that W is an open subset of P, and lambda is to start with lambda satisfy this property; so $\lambda \in W$.

Claim is that $W \subset \phi^{-1}(V_{[\omega]})$ = $\phi^{-1}(V_{[\omega]})$. (The last equality follows, because we started with $[\lambda] \in V_{\omega]}$.) So, we have to show the same homotopic property for all such λ' inside W. So, here is the last picture.

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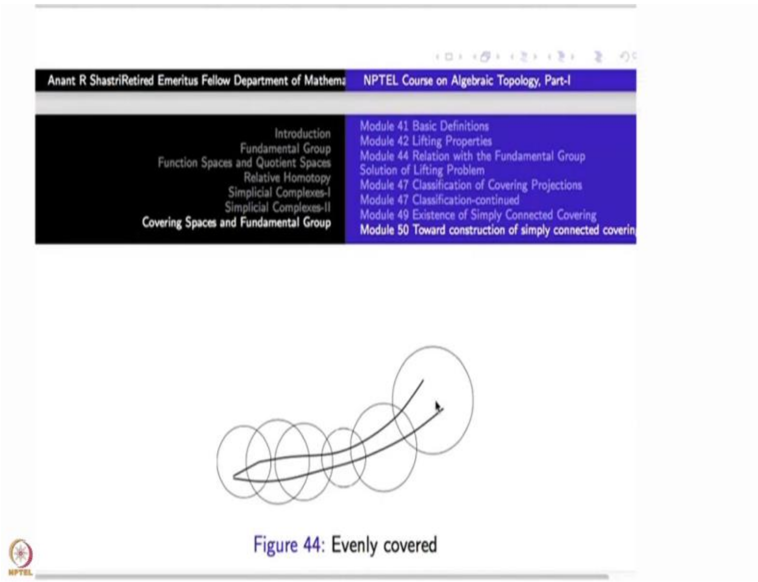

This was your λ which you covered by these evenly covered open subsets $V_1, V_2, V_3, V_4, V_5, V_6, \cdots, V_n = V$. This λ' also has the property that $\lambda'([t_i, t_{i+1}]) \subset V_i$, for all $i = 1, 2, \ldots, n$. Join $\lambda(t_i)$ to $\lambda'(t_i)$ by a path τ_i inside $V_{i-1} \cap V_i$. Denote the restriction of λ (respectively, λ') to the interval $[t_i, t_{i+1}]$ by λ_i (respectively, by λ'_i). It follows that $\lambda_{i-1} * \tau_i * (\lambda'_{i-1})^{-1} * \tau_{i-1}$ is a loop in V_{i-1} and hence is null homotopic inside X.

This way all that you will get is this path is homotopic to this path, composite this back going back, composite going back, composite going back and so on; exactly, similar to the Van Kampen's theorem that we have proved in the beginning. To prove a Van Kampen's theorem is intuitively works. Introducing in between paths here, so it follows that this lambda prime is homotopic to this lambda. And particular lambda prime or lambda are arbitrary it says; it applies to omega also. So, they are all homotopic to omega star or something. So, this completes the completes the proof of that p is a covering projection.

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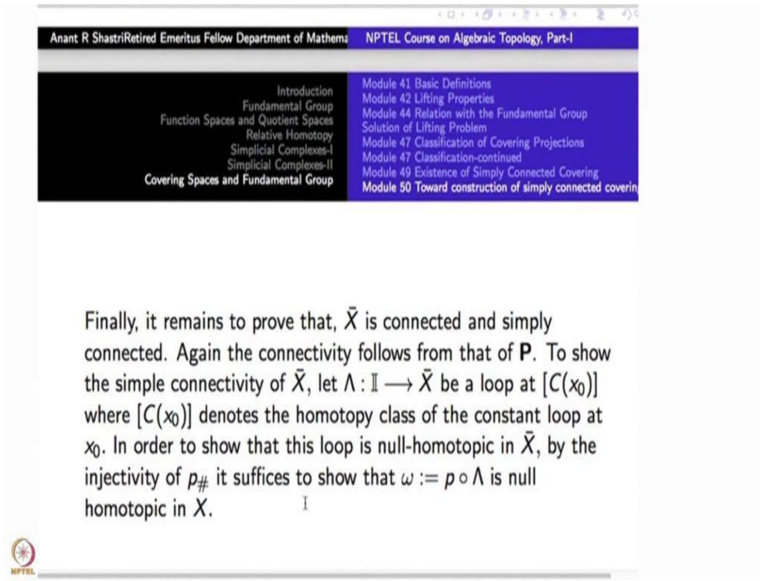

What remains is the proof of simply connectivity of $\bar{X}$. So, let us complete that one. Connectivity follows because P is connected and ϕ is a quotient map, there is no problem. It is path connected because P is path connected and it is quotient map.

To show that $\bar{X}$ is simply connected, let $\Lambda : \mathbb{I} \to \bar{X}$ be a loop at the constant path at x_0. We have to choose the two end-points to be the base point of $\bar{X}$ which is mapped onto $x_0 \in X$. So, I will choose the constant path c_{x_0} of x_0 and its homotopy class to be the base point for $\bar{X}$.

So, Λ is a loop, the loop of loops remember that, in $\bar{X}$ not in X. In order to show that this null homotopic in $\bar{X}$, by the injectivity of $p\#$; (we have proved that p is a covering projection and therefore, $p_\# : \pi_1(\bar{X}, [c_{x_0}]) \to \pi(X, x_0)$ is injective), you go down to X and show that $\omega := p \circ \Lambda$ is null homotopic in X. Let us write $\omega = p \circ \Lambda$. We have to prove that this is null homotopic in X; so we are using the partly proved statements to prove further things here; this is what. So, now what is this lambda? Lambda is map from $\mathbb{I}$ to $\bar{X}$; then we have come here. That Λ is a loop at $[c_{x_0}]$ has to be used.

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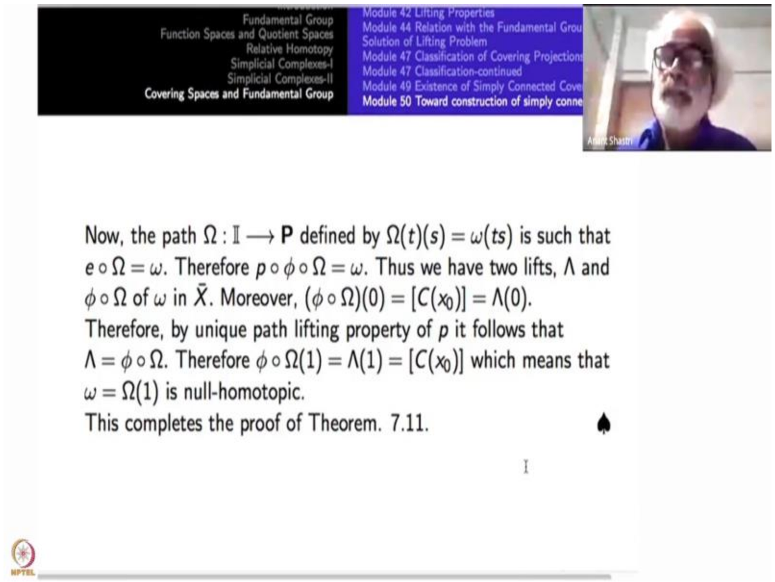

So, let $\Omega : \mathbb{I} \to P$ be defined by the formula $\Omega(t)(s) = \omega(ts)$. This map we have considered earlier to show that P is contractible.

So for each fixed t, and for s=0, we have $\Omega(t)(0) = \omega(0) = x_0 = p \circ \Lambda([c_{x_0}]) = x_0$. That means first of all, each $\Omega(t)$ belongs to P. We check that $e \circ \Omega : \mathbb{I} \to X$ is ω. That is $e(\Omega(t)) = \Omega(t)(1) = \omega(t)$, e being the end-point map. Now we have $p \circ \phi = e$. Therefore, $p \circ \phi \circ \Omega = \omega = p \circ \Lambda$. Thus, we havetwo lifts of ω in $\bar{X}$, one is Λ and another is $\phi \circ \Omega$. Moreover, $\phi \circ \Omega(0) = \phi(c_{x_0}) = [c_{x_0}] = \Lambda(0)$. Therefore, by unique path lifting property of the covering projection p, it follows that this Λ is nothing but $\phi \circ \Omega$, this path here.

So, we took Λ in $\bar{X}$, came down to X via p, call it ω, then we have identified what is this original Λ in terms of ω. It is $\phi \circ \Omega$. This Ω is now all the way inside P. So, you see we have used the property that any arbitrary path inside X can be lifted to the space P very easily. And when we quotient it out, the lifting property is still retained by the covering projection p. This is what it happening.

So, here we have used a lift of ω, so this omega is lifted to Ω in P. And that under quotient map ϕ it goes to Λ.

So, most of it is like tautology; actually, all several construction in mathematics, when you have in a blank where to go for this tautological one. In the beginning of construction of real numbers out of Cauchy sequences of rational numbers. You want every Cauchy sequence to be convergent what did we do? You took equivalence classes of Cauchy sequences, and declared them as real numbers over. Now, Cauchy sequence of Cauchy sequence is convergent is what we have to show; so, this proof is similar to that. We want all kinds of lifts of paths inside the simply connected covering.

So, you declare the covering itself to be set of all paths; but that is too much, so you have to do the equivalence classes. So, it is in simplistic language this is I could have taken the example of metric completion of a metric space; or construction of real numbers. And this similar to that; so, let us stop here. Let us comeback to other things in the next session. Thank you.