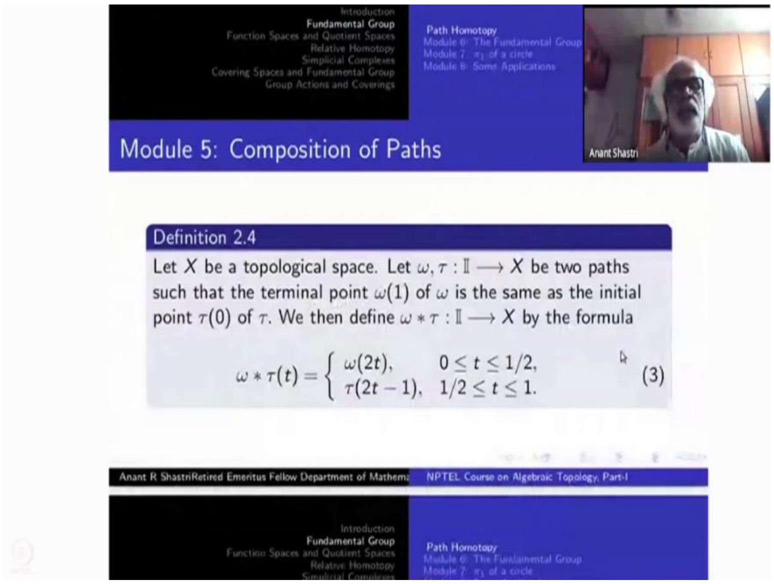
Introduction to Algebraic Topology (Part-I)
Professor. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology Bombay
Lecture 5
Composition of Paths

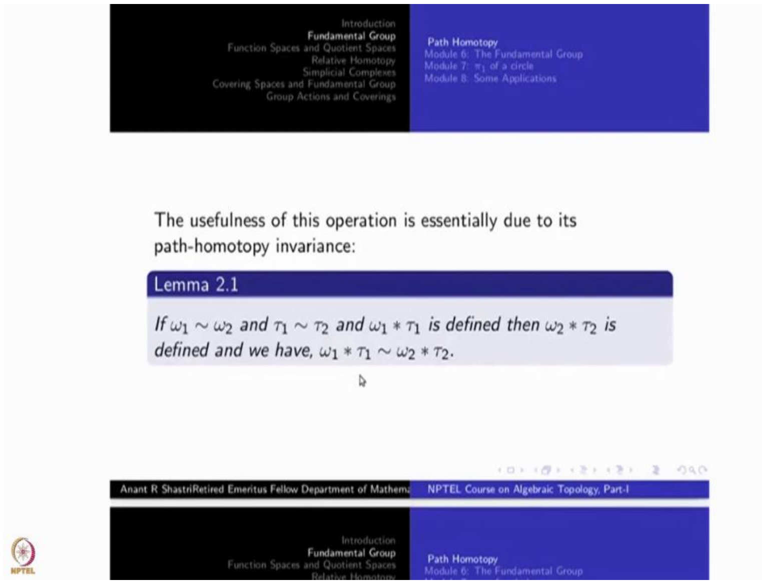
(Refer Slide Time: 00:16)



Let us begin defining composition of paths. Fix any topological space, and take two paths in it. The first path ends at $\omega(1)$, the starting point of the second path is same thing as $\omega(1)$. So, $\tau(0) = \omega(1)$. In that case, we will define $\omega * \tau$---- $\omega\omega$ followed by τ. This is not a composition of functions, it is not as if X to Y, followed by Y to Z. Both ω and τ are from $\mathbb{I}$ to X.

And what we need is $\omega\omega(1)$ must be equal to $\tau(0)$. Then we can define $\omega * \tau$. This ω star τ is defined exactly the same as we have defined the homotopy concatenation of two homotopies so that is precisely what we are going to do here, namely in the first half of the interval 0 less than or equal to t less than or equal to half, we will define it as $\omega\omega$, but double the speed, $\omega(2t)$. In the second half again, we are defining it as τ, double the speed and origin has to change--- start at 1/2 and end at 1. Therefore, it is $\tau\tau(2t - 1)$. When you put t equal to half in the first one, it is $\omega\omega(1)$ and then second one it is $\tau(0)$ and those two points are same. Therefore the right hand side will be a continuous function. What is its starting point? It will be $\omega\omega(0)$. What is its end point? It will be $\tau\tau(1)$. So, this is the meaning of composition of two paths.

(Refer Slide Time: 02:38)

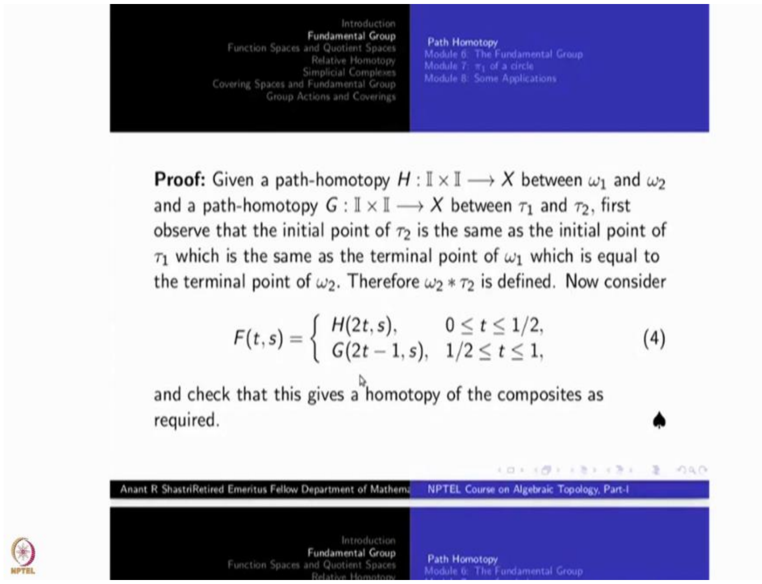


Usefulness of this operation is essentially due to its path-homotopy invariance. What is the meaning of this? Let us see. Suppose you have $\omega\omega_1$ homotopic to $\omega\omega_2$, (here homotopy is actually path homotopy) and $\tau\tau_1$ homotopic to $\tau\tau_2$, then the compositions, corresponding compositions will be also be homotopic, i. e., path homotopic. Remember, $\omega\omega_1$ homotopic to $\omega\omega_2$ always means that the endpoints of $\omega\omega_1$ and $\omega\omega_2$ are the same.

$\omega\omega_1 * \tau\tau_1$ can be defined means that the starting point of $\tau\tau_1$ is the same thing as endpoint of $\omega\omega_1$. The same thing should be true for τ^{ω_2} also. Starting point of τ^{τ_2} will automatically be equal to endpoint of ω^{ω_2}, because the end point of ω^{ω_2} is the same as endpoint of ω^{ω_1}. So, the composition $\omega^{\omega_1} * {}^{\tau_1}$ is defined will imply that $\omega^{\omega_2 * \tau_2}$ is also defined.

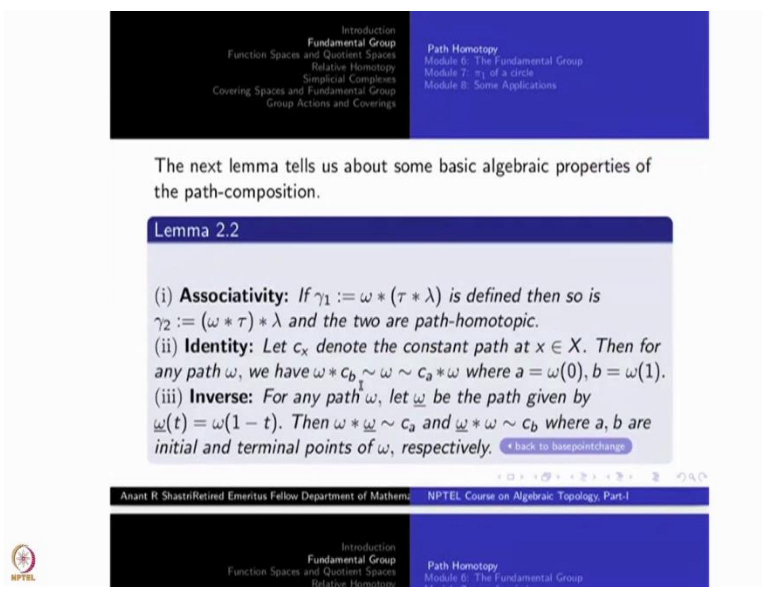
Not only that, this homotopy will imply that the compositions are also homotopic to each other. How does one get this one?---by just putting these two homotopies together just like the way we have done it in the case of the homotopy classes of maps.

(Refer Slide Time: 04:18)



So, H is a homotopy from $\omega\omega_1$ to $\omega\omega_2$, G is homotopy from $\tau\tau_1$ to $\tau\tau_2$. In the first half, you define it as H, H(2t, s). Do not worry about the second coordinate at all. Keep the second coordinate as it is. It is the first coordinate t which defines a path function. The second coordinate keeps changing the paths each time. So, each time you double the speed, H(2t, s) and G (2t – 1, s), exactly the same way as we have defined the compositions. You can easily verify that this capital F (t, s) will be a path homotopy from $\omega\omega_1 * \tau_1$ to $\omega\omega_2 * \tau_2$.

(Refer Slide Time: 05:31)



Next lemma tells us about the basic algebraic properties of path compositions, which I have already summed up in the last module. If you understood this properly this part is automatic, whereas, it may not be that you have understood it. So, let us go through it again. That remark which I have made in the last module is enough to talk about this associativity, identity and inverses, let us see how.

So, let us call the first composition $\omega\omega *$ $(\tau\tau *$ $\lambda\lambda)$. So, I have put the bracket on the second ones. Let us call it γ_1. Suppose that is defined then other one --- $(\omega^{\omega} *$ $^{\tau}\tau)$ and then $*$ $^{\lambda}\lambda\lambda$-- that will be also be defined, call it γ_2. And these two will be path homotopic, that is the associativity of path composition.

Obviously, γ_1 and $\gamma_2.\gamma$ will be completely different, because in defining γ_1, look at what we have to do, the first half of the interval will be occupied by ω and the second half of the interval to be shared between $\tau\tau\tau$ and λ,λ whereas, in the definition of γ_2., the first half of the interval will be occupied by $\tau\,\omega$ as well as $\tau\tau$, - -shared between ω and$\tau\tau$, which means only ¼ of the interval will be for $\omega\omega$.

Therefore, it is obvious that γ_1 as a function, is not equal to γ_2. But a pleasant surprise is that it is path homotopic to γ_2. Similarly, if you compose with the constant path on this side whatever path we get is path homotopic to the original path. Namely, take any path $\omega\,\omega$ starting at, say, at the point x, C_x be the constant path at x you have taken. So, I can take $\omega\,\omega$ $* C_b$, where b is the endpoint, is homotopic to $\omega^{\,\omega}$. Also same with $C_a *$ $^{\omega}$ ω, where a is the initial point of $^{\omega}$ ω; ---a is $^{\omega}$ $\omega(0)$ and b is $\omega^{\,\omega}$ (1).

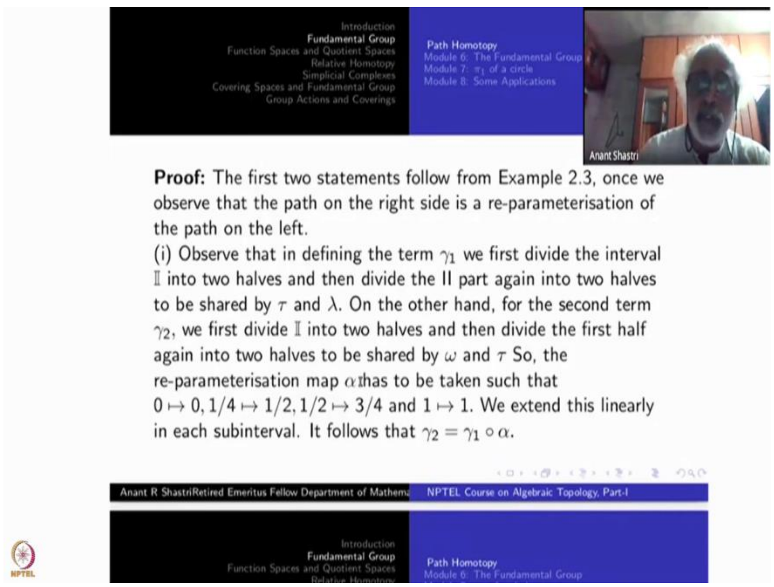
So, one side we have composite with the constant path C_a,--- on the left hand side, on the right hand side constant path C_b. So, these are homotopy-identities for the operation which is path composition. Moreover the inverse path, the so called inverse path, namely the one obtained by tracing the same path in the reverse direction. I have denoted it by omega underline, $\underline{\omega}$, i.e., ω

$\omega\,(1 - t)$, you have to compose it with $\omega\,\omega(t)$, you will get constant path at a. If you take $\underline{\omega}$ first and then take ω , then it will be constant path at b. Wherever you start from and go to the other end and come back by the same path, it will be as if you are all the time at the same original point. So, these are the three statements, all of them can be proved, in fact have been proved by one single remark namely that you can get all of them by different parameterizations.

Different parameterizations of the same path that is why they are path homotopic. Left hand side and right hand side are different parameterizations except the last one- (i) and (ii) are just different parameterizations of the same path. So, let us see why all these homotopies have nothing to do with actual ω and actual space X. It is the property of being defined on a closed interval.

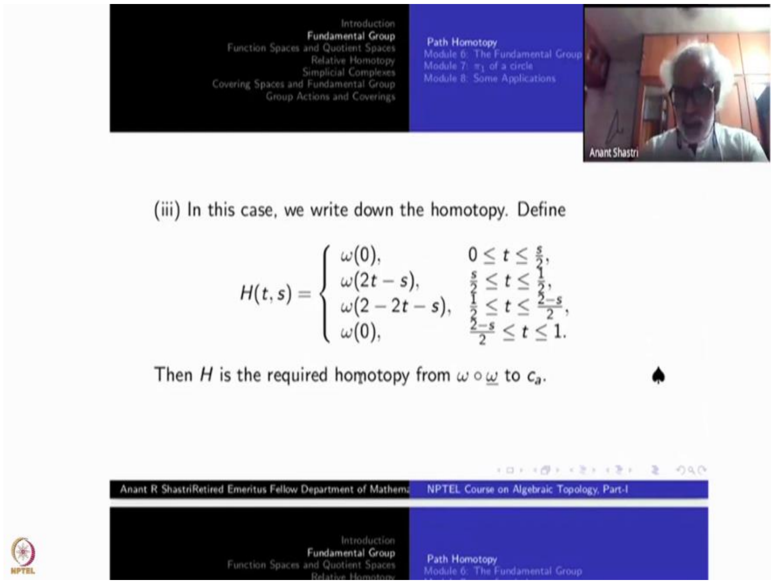
Everything is happening inside of the interval. You should understand that one. So, this is a very simple idea, but it is the best we understood that way. So, instead of making it as if it’s some mystery, we will just write down the formulas for each of them and be done with it. That is the way many books do read, I have done it for your sake.

(Refer Slide Time: 11: 18)



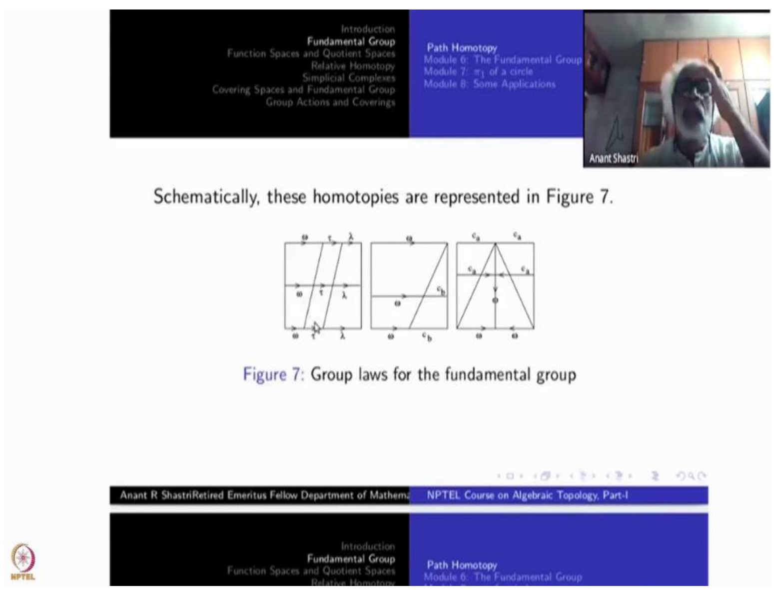
So, having explained this path and how it work, so I am going to tell you, I already told you that the first path occupies ω occupies half the interval first half of the interval and the second half of the interval is shared again between τ and λ. So, these things we will use and write down the homotopies.

(Refer Slide Time: 11:45)



So, here are clear homotopies have been written down. First let me tell you how does one get this one.

(Refer Slide Time: 11:58)



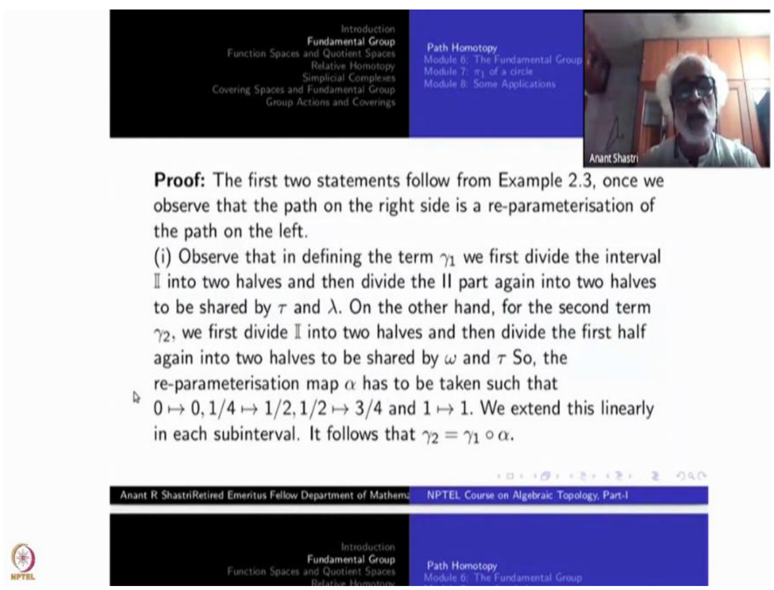
This is the picture. Look at the bottom, in the first there are 3 squares here, $\mathbb{I} \times \mathbb{I}$, $\mathbb{I} \times \mathbb{I}$, $\mathbb{I} \times \mathbb{I}$, for three different cases, we have three different statements there. The first one is associativity $(\omega * \tau) * \lambda$ and then $\omega * (\tau * \lambda)$. So, at the bottom, this is your γ_1 perhaps, in the previous notation.

At the top, it is $\omega * (\tau * \lambda)$. So, that is the definition. τ $(\tau * \lambda)$ λ will be defined first and then compose it with $\omega\omega$ on this side.

So, this is all, this is all you get-- the first half of interval will be $\omega\omega$ and the second path will be shared by τ and λ. So, what do you do? You join them. this 1/4 to 1/2 there, this 1/2 to 3/4 there, and 0 to 0 and 1 to 1 like that. In between homotopies, say at t equal to s, at this point t equal to s sorry, s is equal to s_0, you have to go, not to one fourth here, but you go up to this point by ω , then follow by τ τ and then $\lambda\lambda$. By the time you come here, instead of one fourth, you would go upto ½, i.,e, up to half, you would be taking $\omega\omega\omega$ then τ τ and then $\lambda\lambda$ like this.

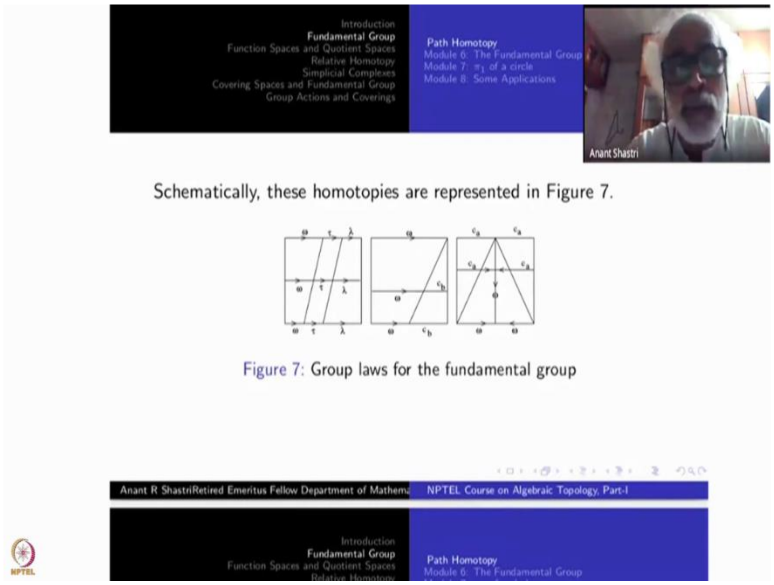
So, this is the picture. The actual values of the coordinates are determined by the intersection of these two lines that is all. So, this line is given by some s equal to s_0 joining some point t_0 here to t_1 here.

(Refer Slide Time: 14:06)



So, what do you get by doing that is the formula here----this one. I have explained (i) here.

(Refer Slide Time: 14:19)



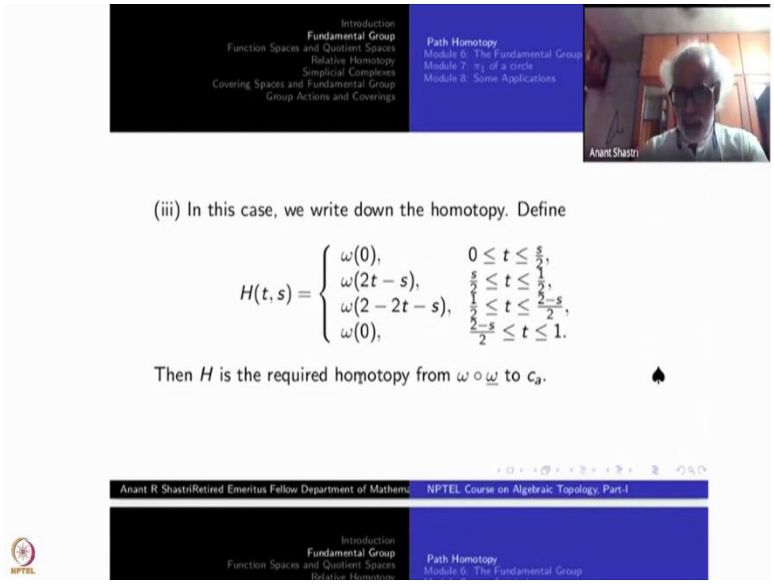
Similarly the (ii) also. So, what I want to do is tell you how this is-- the last one. This is

$\omega * \underline{\omega}$. So, what path you have does not matter--- how it goes up to half and goes back. But you know that any two paths, which have same endpoints inside $\mathbb{I}$ to $\mathbb{I}$, they are homotopic as path homotopies keeping the end point same. So, here what happens if I take $\omega\omega$ and come back by $\underline{\underline{\omega}}$ it will have the end point and the starting point equal to $\omega\omega(0)$.

If I compose $\underline{\omega}$ with ω, starting point and the endpoint both of them will be same the end point of ω. Therefore they are homotopic to the constant paths at the respective points. Over. You do not have to write any formula because α once you have proved, given one single formula remember A(t,s) was defined as (1− s) α(t) +s t.

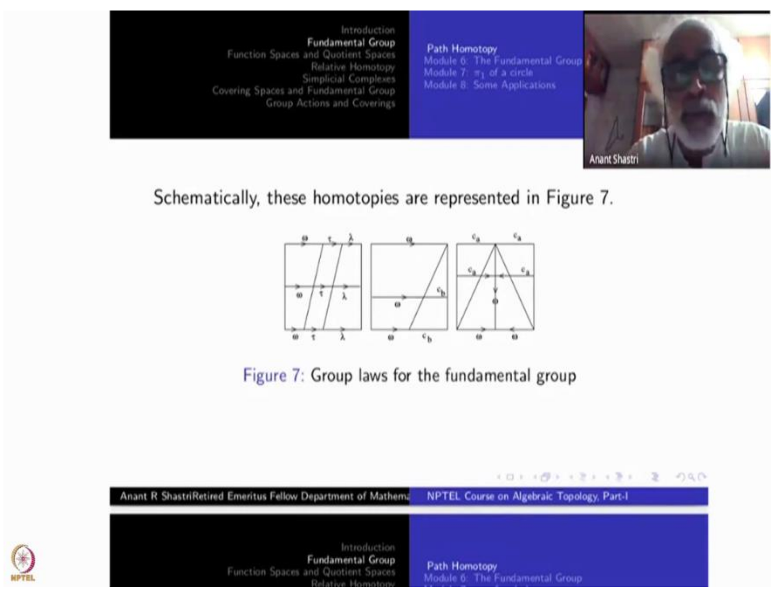
This showed identity map is homotopic to any map . So, any two maps are homotopic. -- the same way, α and β two maps α, β: $\mathbb{I} \to \mathbb{I}$, endpoints are the same whatever endpoints. So, $\alpha(0)$ equal to say $\beta\beta(0)$ and $\alpha\alpha(1)$ equal to $\beta(1)$. Wherever they are in the interval, then you can just take (1− s) α(t) + s β(t) that ensures that α and $\beta\beta$ path homotopic So, that is the thing that is happening in the last one. If you have difficulties in seeing this way, there are just formulas--- you just verify. How the formulas are obtained?

(Refer Slide Time: 16:57)



I told you these are the pictures will tell you how to get the formulas. We will have a session in writing down these formulas later on rigorously at another point not at this point. What we have done here you can see that up to s by 2, this is the function. So, here what is happening? Let us go back look at this one.

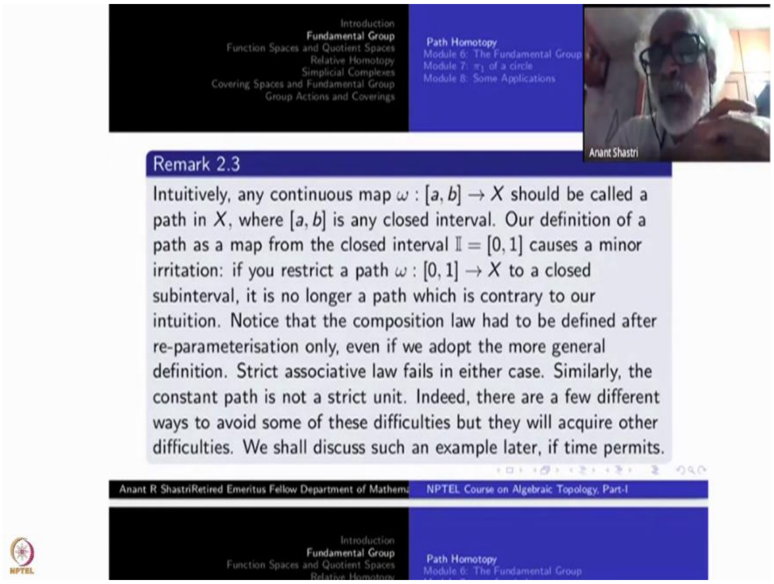
(Refer Slide Time: 17:28)



For s equal to s_0, half of this distance, whatever this distance, half of that will be one thing. After that it will be something else. See up till here, then it is this part, then you have reached. This is constant path at a, this is your C_a. As the homotopy keeps taking, it will not go all the way at all.

The first time you go all the way to the other point and come back. After half the time, you go only halfway and come back. At the final time, you know, you do not worry at all you stay there all the time. That is what this homotopy is doing----- $\omega * \underline{\omega}$ is null homotopic.

(Refer Slide Time: 18:37)

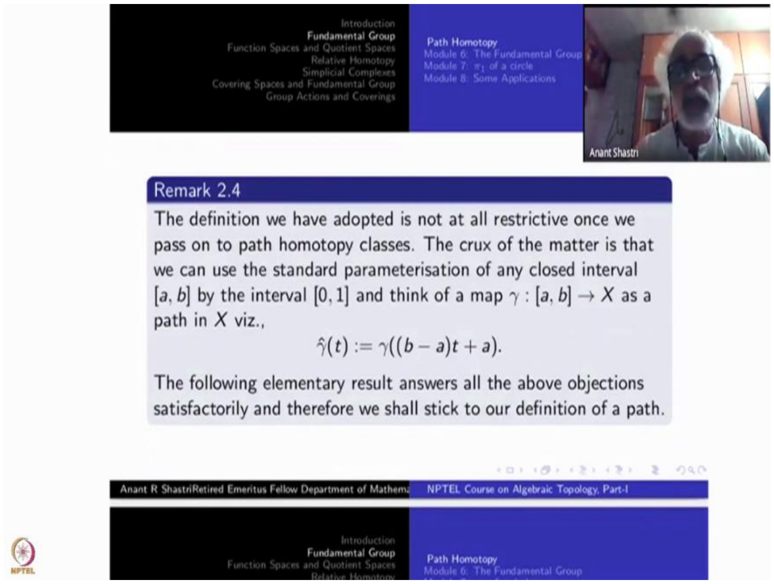


Now, let me tell you a few things about why interval [0, 1] and so on. It is interesting to note that during these homotopy entire action is taking place in the domain itself and so, the proofs that the composition etc, etc, do not depend upon the actual paths, do not depend upon the space X. Because of property three in the above lemma, many authors use the notation $\omega\omega^{-1}$. After verifying this one, we can also do that, but the inverse is the more confusing, it is not the same path, it is up to homotopy.

And that too, not as a functional inverse nor $1/\omega$ ...ω and so on. Those things do not make sense. So, we should also use this notation, but you have to be careful with it. Some people use even

$-\omega$ ω and that is very much valid because in integration theory, if you integrate on the reverse path, what you get is the negative of the original integration. So, many notations are there, they are all, they have their own justification.

(Refer Slide Time: 20:11)



Intuitively, any continuous map, from any interval [a, b] should be called a path. Why restrict to [0, 1]? Why interval the unit interval should be why, why all the time? So, this is what you should have done, some people try to do. Our definition of a path as a map from the closed interval causes a minor irritation. Namely, if you restrict a path $\omega\omega$:[0, 1] →X to a proper closed subinterval it is no longer a path in our definition.

Restriction should be also be a path. You have traced a path, half the path is also a path in that way, but in our definition, we have to reparameterize it to make it a function look like as if from [0, 1] to X. So, some people object. They say this is not a good definition. When people object, you ask them to give a better definition, it is just that. So, it is true that if you restrict to [0, 1], start with $\omega\omega$:[0, 1] →X, just take the restricted function on [0,½], is it a path? that function is not a path in our definition. isn't it?

Note that composition law had to be defined after re-parameterization only. Even if we adopt the more general definition there may be some other problem. So, if you take [a, b] arbitrary and [c, d] arbitrary, so ω(b) is equal to τ(c), then what is the domain interval for $\omega * \tau$ you take? Do you take [a, b] or [c, d] or [a, b] union [c, d]; [a, b] union [c, d] may not be an interval, even if they are, there may be more overlapping than just the endpoints.

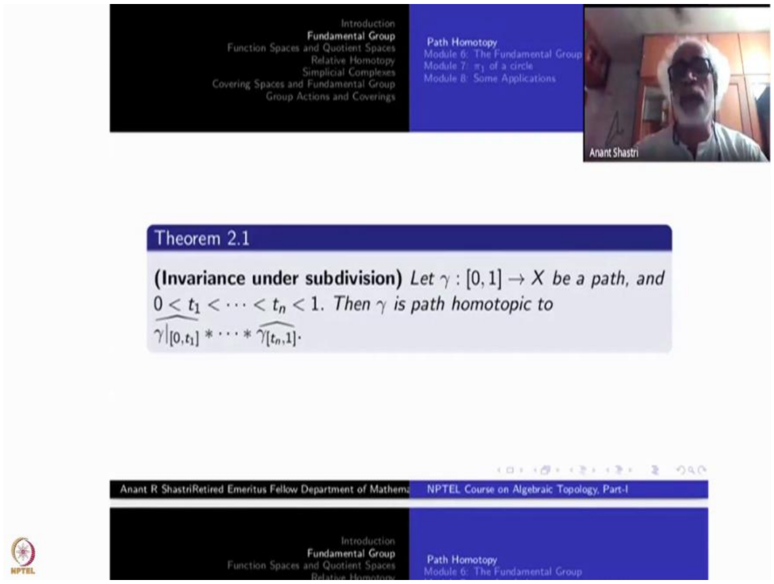
So, arbitrary intervals again need to be converted into some standard intervals and then only you can compose them. That is a point I mean--- it is not as if the interval [0, 1] we have taken is at fault. Strict associativity law always fails. Similarly, the constant path is not a strict unit. Indeed there are a few different ways to avoid some of these difficulties, but some other difficulties you will acquire, we shall discuss such a thing, I have given you an example in a form of exercise, later on.

So, what I want to say is no matter what kind of definition you take, there will be some problem. For example, I will right now tell you instead of the closed [0, 1], you can take the entire real line, but then you want that your path should end somewhere otherwise you know that a non compact thing will not be called a path. You should have a starting point and an ending point therefore what we do? ---take the entire real line, but take only those functions which are constant after a, after some point after say 1 and before 0.

So, everything less than 0 will be mapped to $\omega\omega(0)$ and everything bigger than 1 ωmapped to ω (1). --- $\omega\omega(1)$ is equal to $\omega\omega(t)$, t bigger than 1. So, you put that condition. You understand instead of [0, 1] you put any 2 points a, b and that it is function continuous everywhere. But for all t less than a it is $\omega\omega(a)$, and for all t bigger than b, it is $\omega\omega(b)$.

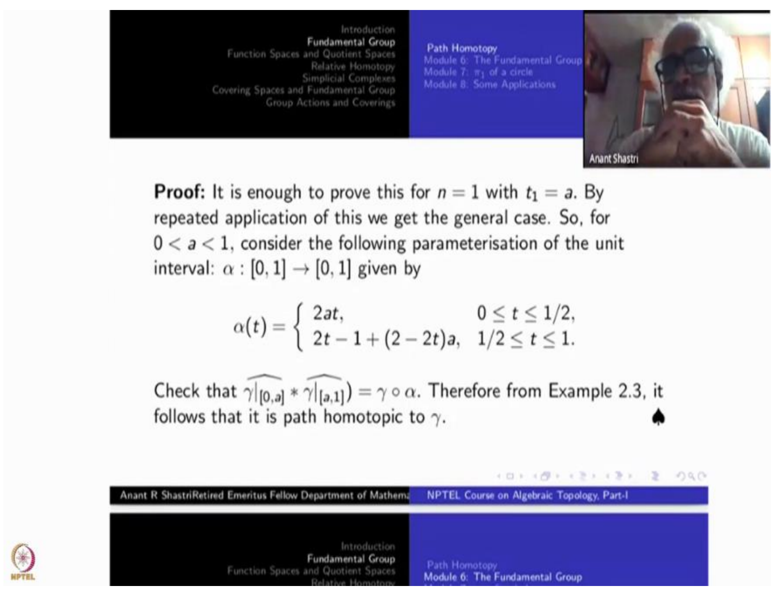
If you take this definition, then restrictions etc have no problems. So, such things can be slightly modified. But whatever you do there will be some other problem--- this is what I want to tell you, okay? Therefore, the definition with [0, 1] as domain is found the best among all other definitions.

(Refer Slide Time: 25:14)



So, how to deal with this when you cut? Take a path. Now cut it into two parts. Then I would like to think of this as composition-- original path as a composition of these two paths, this is called subdivision. The subdivision should be allowed and it is easy to adopt it in our definition and we have a beautiful theorem there which will take care of all objections of this type.

(Refer Slide Time: 25:53)

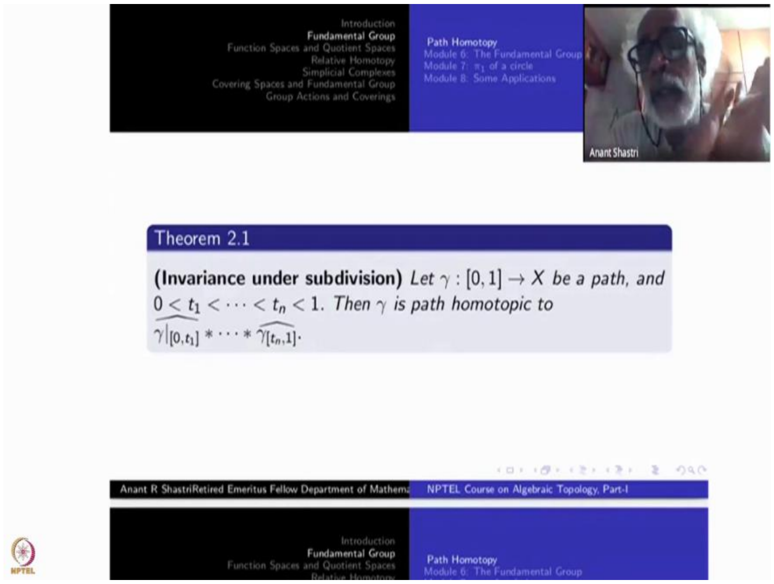


So, let us come to that one. This result is precisely called `invariance under subdivision' up to homotopy of, what you call the concatenation or the composition of paths. Namely, take any path $\gamma : [0, 1] \to X$. Now, you divide the interval into n parts by choosing points $0 < t_1 < \cdots < t_{n-1} < 1$.

That is a subdivision. Now, you restrict the original path to subinterval, 0 to t_1, t_1 to t_2, t_2 to t_3 and so on, call them $\gamma_1, \gamma_2, \ldots, \gamma_n$.

Now if you take the composition in our sense, you will not get the original path γ, but what you will get is homotopic to, path homotopic to γ. To prove this one you have to do it for only one single division namely 0, 1 put a t1 in between cut it down, re-parameterize as these two paths on original [0, 1], both of them. Reparameterize and define the composition. What do you get is the original path up to homotopy, absolutely. Therefore you can safely say that --- you know all objections of this kind -- half the path is a path fine. Only thing is you think of this as again taking place on the interval [0, 1], that is all.

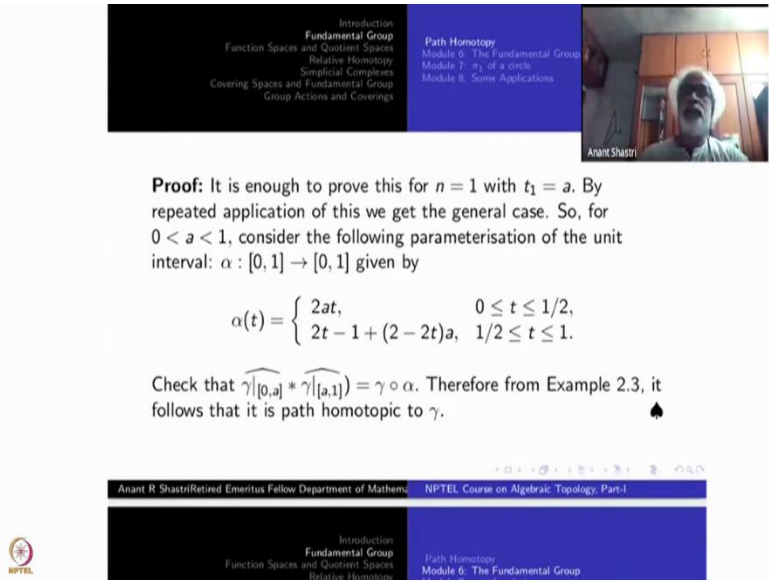
(Refer Slide Time: 27:53)



So, here I have proved it for n = 1 case, namely only t_1 is taken. Then by repetition, the general case will follow. By induction, it will follow. It is enough to prove this for n = 1. So, with t_1, I will call it as a, 0 < a < 1. Consider the following reparameterization of the unit interval from α $\alpha:[0, 1] \to [0, 1]$ given by: $\alpha\alpha(t) = 2ta$, in $0 \leq t \leq ½$; $=(2t - 1) + (2 - 2t)a$, in $1/2 \leq t \leq 1$. So, what I have done? In the first half of interval, look where t = ½ will go? It will go to a. When t is 0, it will go to 0. So, half has gone to a and 0 goes to 0, half goes to a and 1 goes to 1. What do you get here? t = 1 this is 0, this is 2t −1 is 1, this is the path.

So, this is a reparameterization of suppose [0, 1] itself. You may think of this as identity path and I am cutting it instead of at ½ , I am cutting it at a, these two are homotopic, path homotopic, this is what this α(t) says. Therefore, once you take this one, γ restricted to [0, a]. Reparameterize by this method that is the γ hat. Then $\widehat{\gamma_{[0,a]}} * \widehat{\gamma_{[a,1]}}$ is nothing but union of $\gamma^{\gamma_{[0,a]}}$ with $\gamma_{[a,1]}\alpha$. Therefore, it is homotopic to, path homotopic to γ itself. Because α is a reparameterization.

(Refer Slide Time: 30:20)



So, this is the proof of this general theorem-- invariance under subdivision. After this you can do all the algebra of composition of these paths very smoothly. So, now, we shall specialize to the case when the endpoints are the same--- that we will do the next module. Thank you.